Effective Lagrangian for self-interacting scalar field theories in curved spacetime

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Abstract: We consider a self-interacting scalar field theory in a slowly varying gravitational background field. Using zeta-function regularization and heat-kernel techniques, we derive the one-loop effective Lagrangian up to second order in the variation of the background field and up to quadratic terms in the curvature tensors. Specializing to different spacetimes of physical interest, the influence of the curvature on the phase transition is considered.

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1 Introduction

In new inflationary models [1, 2], the effective cosmological constant is obtained from an effective potential, which includes quantum corrections to the classical potential of a scalar field [3]. This potential is usually calculated in Minkowski space, but to be fully consistent, the effective potential or more generally the effective action must be calculated for more general spacetimes, taking into account dynamics, geometry and topology of the spacetime itself. In order to analyze the influence of these properties of the spacetime on the effective action in a self-interacting theory, a variety of methods has been developed in the last years. In the context of general considerations not referring to a special spacetime let us mention the quasilocal approximation scheme for slowly varying background gravitational field at zero temperature [4] and non-zero temperature [5], furthermore the renormalization-group approach especially elaborated in [6, 7, 8]. In addition there are a lot of calculations in specific background spacetimes, paying special attention on the role of constant curvature [9], on topology [10-18], on a combination of both [19, 20, 21] and finally on the anisotropy in different Bianchi type universes [22-29].
In this paper we will use heat-kernel techniques to derive the one-loop effective Lagrangian for varying background fields up to second order in the variation of the background and up to quadratic terms in the curvature tensors (similar techniques have been applied in [30] and more recently in [31]). Only under this restrictive condition when the background field changes very slowly compared to the fluctuation field, it makes sense to adhere to the effective potential formulation of symmetry breaking [32, 33]. In order to obtain this expansion, we make use of the partially summed form of the Schwinger-DeWitt asymptotic development of the effective action introduced by Jack and Parker [34]. The nonrenormalized effective Lagrangian is obtained in Sec. 2 and the renormalization is performed in Sec. 3. Especially we obtain the small background field expansion relevant for the discussion of a phase transition.

In the following sections, these general results are applied to several special cases. In Sec. 4 we treat some maximally symmetric spaces and the effective potential is found up to quadratic order in the scalar curvature \( R \) and the effect of the curvature on the phase transition is examined. In Sec. 5 we consider the static Taub universe in the limit of large anisotropy (for the small anisotropy expansion see [22]). The influence of a possible rotation on the phase transition of the early universe in analyzed using the example of a G"odel spacetime in Sec. 6. Furthermore, in Sec. 7 we easily extract the results for the most general Bianchi-type I model (for several limiting cases of this model see [22, 23, 24, 28, 29, 35]) deriving on the one side more detailed results than the ones given up to now (higher order in the curvature and with the assumption of varying background field) and deriving on the other side new results by including a net expansion of the universe together with nonvanishing shear (generalizing [35, 29]). The conclusions summarize our results. In the appendices we state some necessary tensor identities and the curvature tensors of the Bianchi-type I model.

Throughout the paper we will use units in which \( \hbar = c = G = 1 \).

## 2 Effective action

The aim of this section is to derive a quasilocal approximation for the effective action of a self-interacting massive scalar field coupled to a n-dimensional smooth background spacetime \( \mathcal{M} \) with Lorentzian metric \( g_{\mu\nu} = \text{diag}(-, +, ..., +) \) and scalar curvature \( R \). The classical action describing the theory is given by

\[
S[\phi, g_{\mu\nu}] = -\int_{\mathcal{M}} \left[ \frac{1}{2} \phi L\phi + V(\phi) \right] dvol \mathcal{M}
\]

where \( L = -\Box + m^2 + \xi R \) (\( \Box \) is the D'Alembert operator of the manifold \( \mathcal{M} \)) and \( V(\phi) \) is a potential describing the self-interaction of the scalar field and which contains furthermore local expressions of dimension \( n \), involving curvature tensors and non-quadratic terms in the field, independent up to a total divergence. These latter terms have in general to be included to ensure the renormalizability of the theory [10]. The action (2.1) has a minimum at \( \phi = \hat{\phi} \) which satisfies the classical equation of motion

\[
L\phi + V'(\hat{\phi}) = 0
\]

the prime denoting derivative with respect to \( \phi \). Quantum fluctuations \( \phi = \phi - \hat{\phi} \) around the classical background \( \hat{\phi} \) satisfy an equation of the form (to lowest order in \( \phi \))

\[
A\phi = \left( L + V''(\hat{\phi}) \right) \phi = 0
\]

In the functional integral perturbative approach, the effective action is expanded in powers of \( \hbar \) as

\[
\Gamma[\phi] = S[\phi] + \Gamma^{(1)} + \Gamma'
\]
where $S[\hat{\phi}]$ is the classical action and the one-loop contribution $\Gamma^{(1)}$ to the action is given by

$$\Gamma^{(1)} = \frac{i}{2} \log \det \frac{A}{\mu^2} \quad (2.5)$$

The term $\Gamma'$ represents higher loop corrections which we are not going to discuss. The introduction of the arbitrary mass parameter $\mu$ is necessary in order to keep the action dimensionless.

The action (2.4) may be expanded in terms of derivatives of the background field, that is

$$\Gamma[\hat{\phi}] = \int_M \left[ V(\hat{\phi}) + \frac{1}{2} Z(\hat{\phi}) \hat{\phi}_\mu \hat{\phi}^\mu + ... \right] dvol(M) \quad (2.6)$$

The zero derivative term is called the effective potential. In a static homogeneous spacetime, $\hat{\phi}$ is a constant field and the effective action reduces to $\Gamma[\hat{\phi}] = vol(M) V(\hat{\phi})$. Under these conditions the concept of the effective potential is well defined, otherwise one has to work with the effective action [4].

As mentioned in the introduction, our first tool will be to compute the quantities $V(\hat{\phi})$ and $Z(\hat{\phi})$ in eq. (2.6) up to quadratic powers in curvature tensors and up to second derivatives of curvature terms. The goal to achieve this result will be the use of zeta-function regularization in combination with the use of heat-kernel techniques. In the zeta-function regularization scheme the functional determinant in eq. (2.5) is defined by [36, 37]

$$\Gamma^{(1)}[\hat{\phi}] = -\frac{i}{2} \zeta'(\frac{A}{\mu^2}) (0) \quad (2.7)$$

where $\zeta_A(s)$ is the zeta-function associated with the operator $A$, eq. (2.3), and the prime denotes differentiation with respect to $s$. This means with $\lambda_j$ as the eigenvalues of $A$, the zeta-function $\zeta_A(s)$ is defined by

$$\zeta_A(s) = \sum_j \lambda_j^{-s} = \frac{i^s}{\Gamma(s)} \sum_j \int_0^\infty dt \ t^{s-1} e^{-i\lambda_j t}$$

$$= \frac{i^s}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} K(x, x, t) \quad (2.8)$$

where the kernel $K(x, x', t)$ satisfies the equation

$$i \frac{\partial}{\partial t} K(x, x', t) = AK(x, x', t)$$

$$\lim_{t \to 0} K(x, x', t) = |g|^{-\frac{1}{2}} \delta(x, x') \quad (2.9)$$

In order to obtain the derivative expansion, eq. (2.6), of the effective action, the following ansatz by Jack and Parker is suggested [34]

$$K(x, x', t) = -i \frac{\Delta V_M(x, x')}{(4\pi it)^{n/2}} \Omega(x, x', t) \exp\{i \left( \frac{\sigma^2(x, x')}{4t} - tM^2 \right)\} \quad (2.10)$$

where $\sigma(x, x')$ is the proper arc length along the geodesic $x'$ to $x$, $\Delta V_M(x, x')$ is the Van Vleck-Morette determinant and $M^2 = m^2 + (\xi - 1/6)R + V''(\hat{\phi})$. For $t \to 0$ the function $\Omega(x, x', t)$ may be expanded in an asymptotic series

$$\Omega(x, x', t) = \sum_{l=0}^\infty a_l(x, x')(it)^l \quad (2.11)$$

where the coefficients $a_l$ have to fulfill some recurrence relations. Using the ansatz (2.10) it has been shown in ref. [34], that the dependence of $a_l$, $l = 1, ..., \infty$, on the field $\hat{\phi}$ is only through
derivatives of the field. As a result, the use of eq. (2.11) for the zeta-function (2.8) leads to the expansion

\[ \zeta_A(s) = -\frac{i}{(4\pi)^{n/2}} \left( \frac{1}{\Gamma(s)} \right) \sum_{l=0}^{\infty} \Gamma(s + l - \frac{n}{2}) \int_{\mathcal{M}} a_l(x,x) M^{n-2l-2s} dvol(\mathcal{M}) \]  

(2.12)

which represents already the derivative expansion of the zeta-function \( \zeta_A(s) \), which enables us to find expansion (2.6) without problems, once the coefficients \( a_l(x,x) \) are known. On the calculation of the Minakshisundaran-Seeley-DeWitt coefficients there is a rapidly increasing amount of literature and nowadays the first four coefficients are known explicitly for manifolds without boundary [38, 39] and algebraic programs for the computation of arbitrary coefficients are available [40] (for manifold with boundary see [41-46]).

The expansion is sensible if the effective mass \( M^2 \) of the theory is large compared to the magnitude of some typical curvature radius \( |R| \) of the spacetime, \( M^2 \gg |R| \).

Due to the different pole structure of \( \Gamma(s + l - n/2) \) around \( s = 0 \) for odd \( n \), we need to consider these two cases separately.

Let us first treat odd \( n \). Then obviously \( \zeta_A(0) = 0 \) and for the unrenormalized one-loop effective action, eq. (2.7), one easily finds

\[ \Gamma_o^{(1)}[\hat{\phi}] = -\frac{1}{2(4\pi)^{n/2}} \sum_{l=0}^{\infty} \Gamma(l - \frac{n}{2}) \int_{\mathcal{M}} a_l(x,x) M^{n-2l} dvol(\mathcal{M}) \]  

(2.13)

So for odd dimensions, the one-loop quantum corrections do not depend on the arbitrary renormalization scale \( \mu \).

The calculation for even \( n = 2k \) is slightly more difficult. After some algebra one finds

\[ \Gamma_e^{(1)}[\hat{\phi}] = -\frac{1}{2(4\pi)^k} \int_{\mathcal{M}} \left\{ \sum_{l=0}^{k} \frac{(-1)^l}{l!} M^{2l} a_{k-l}(x,x) \left( C_l - \log \frac{M^2}{\mu^2} \right) ight. \\
\left. + \sum_{l=1}^{\infty} \frac{(l-1)! a_{k+l}(x,x)}{M^{2l}} \right\} dvol(\mathcal{M}) \]  

(2.14)

where we introduced \( C_l = \sum_{k=1}^{l} (1/k) \).

Let us stress, that these results, eqs. (2.13) and (2.14), are valid for arbitrary dimensions of spacetime and for arbitrary self-interaction potential \( V(\hat{\phi}) \). In the next chapter we shall proceed with the renormalization of the theory and from now on we shall restrict ourselves to \( n = 4 \) and to a quartic self-interaction, that is \( V(\hat{\phi}) = \lambda \hat{\phi}^4/24 \).

3 Renormalization

For \( n = 4 \), eq. (2.14) reduces to

\[ \Gamma^{(1)}[\hat{\phi}] = \frac{1}{32\pi^2} \int_{\mathcal{M}} \left\{ \frac{M^4}{2} \left( \log \frac{M^2}{\mu^2} - \frac{3}{2} \right) + a_2(x,x) \log \frac{M^2}{\mu^2} \\
- \sum_{l=0}^{\infty} \frac{2l a_{3+l}}{M^{2l+1}} \right\} dvol(\mathcal{M}) \]  

(3.1)

We recall that our aim is to obtain the effective action up to second order in the curvature and in the derivative of the scalar field. Then we will need the coefficients \( a_l \) up to \( a_3 \) for the operator \( A \). They are given in appendix A. It has to be noted that in \( a_3 \) (see eq. (A.1)), all terms but
one \( (\lambda \tilde{\phi}^2 \phi / 2M)^2 \) are of higher order with respect to our requirements. Then we disregard all those terms and write

\[
\Gamma^{(1)}[\tilde{\phi}] = \frac{1}{64\pi^2} \int_M \left\{ M^4 \left( \log \frac{M^2}{\mu^2} - \frac{3}{2} \right) + 2a_2 \log \frac{M^2}{\mu^2} - \frac{\lambda^2 \tilde{\phi}^2 \phi^2}{6M^2} \right\} \text{dvol}(M) + (3.2)
\]

The first term in the integral, eq. (3.2), obviously corresponds to the Coleman-Weinberg result in flat space [3].

The quantum correction \( \Gamma^{(1)}[\tilde{\phi}] \) depends on the arbitrary renormalization scale \( \mu \), this dependence being removed by the renormalization procedure which we are going to describe now. As is well known by now, one is forced to take into consideration the most general quadratic gravitational Lagrangian [47] and so one has to consider the classical Lagrangian

\[
L_{\text{cl}} = \eta \Box \tilde{\phi}^2 - \frac{1}{2} \phi \Box \phi + \Lambda + \frac{1}{24} \lambda \tilde{\phi}^4 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{2} \xi R \tilde{\phi}^2 + \varepsilon_0 R + \frac{1}{2} \varepsilon_1 R^2 + \varepsilon_2 C + \varepsilon_3 G + \varepsilon_4 \Box R
\]

with the corresponding counterterm contributions

\[
\delta L_{\text{cl}} = \delta \eta \Box \tilde{\phi}^2 + \delta \Lambda + \frac{1}{24} \delta \lambda \tilde{\phi}^4 + \frac{1}{2} \delta m^2 \tilde{\phi}^2 + \frac{1}{2} \delta \xi R \tilde{\phi}^2 + \delta \varepsilon_0 R + \frac{1}{2} \delta \varepsilon_1 R^2 + \delta \varepsilon_2 C + \delta \varepsilon_3 G + \delta \varepsilon_4 \Box R
\]

necessary to renormalize all coupling constants. Of course \( \Lambda \sim 0 \) and \( \eta = 1/4 \). By \( C \) and \( G \) we indicate respectively the square of the Weyl tensor and the Gauss-Bonnet density. They read

\[
C = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 2 R_{\mu \nu} R^{\mu \nu} + \frac{1}{3} R^2
\]

\[
G = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + R^2
\]

The renormalization conditions are given by [4, 29]

\[
\Lambda = L \bigg|_{\phi = \varphi_0, R = 0}
\]

\[
\lambda = \frac{\partial^4 L}{\partial \tilde{\phi}^4} \bigg|_{\phi = \varphi_1, R = 0}
\]

\[
m^2 = \frac{\partial^2 L}{\partial \tilde{\phi}^2} \bigg|_{\phi = 0, R = 0}
\]

\[
\xi = \frac{\partial L}{\partial R \partial \tilde{\phi}^2} \bigg|_{\phi = \varphi_3, R = R_3}
\]

\[
\varepsilon_0 = \frac{\partial L}{\partial R} \bigg|_{\phi = 0, R = 0}
\]

\[
\varepsilon_1 = \frac{\partial^2 L}{\partial R^2} \bigg|_{\phi = 0, R = R_5}
\]

\[
\varepsilon_2 = \frac{\partial L}{\partial C} \bigg|_{\phi = 0, R = R_6}
\]

\[
\varepsilon_3 = \frac{\partial L}{\partial G} \bigg|_{\phi = 0, R = R_7}
\]

\[
\varepsilon_4 = \frac{\partial L}{\partial \Box R} \bigg|_{\phi = 0, R = R_8}
\]

\[
\eta = \frac{\partial L}{\partial \Box \phi^2} \bigg|_{\phi = \varphi_0, R = 0}
\]

The conditions (3.7) determine the counterterms to be

\[
64\pi^2 \delta \Lambda = \lambda m^2 \varphi_0^2 \log \frac{m^2}{\mu^2} - \frac{m^2 \varphi_0^2}{2} - \frac{\lambda \varphi_0^4}{24} - \lambda m^2 \varphi_0^4 + \frac{\lambda \varphi_0^4}{4} \log \frac{M^2}{\mu^2}
\]
where we introduced $M_i^2 = m^2 + (\xi - \frac{1}{6}) R_i + \frac{1}{2} \varphi_0^2$. For the sake of generality, we chose different values $\varphi_i, R_i$ for the definition of the physical coupling constants. This is due to the fact that in general they are measured at different scales, the behaviour with respect to a change of scale being determined by the renormalization group equations.

After some calculations one finds the renormalized effective Lagrangian in the form

$$64\pi^2 L_r^{(1)} = \frac{C}{60} \log \frac{M_r^2}{M_6^2} - \frac{G}{180} \log \frac{M_r^2}{M_7^2} - \left( \xi - \frac{1}{6} \right) \frac{\Box R}{3} \log \frac{M_r^2}{M_8^2}
- \lambda \varphi_0^4 \left[ \log \frac{M_6^2}{M_7^2} - \frac{3}{2} - \frac{4(M_1^2 - m^2)(2M_1^2 + m^2)}{3M_1^2} \right]
+ \left\{ \left( \xi - \frac{1}{6} \right) R \left[ \log \frac{M^2}{M_5^2} - \frac{3}{2} - \frac{2\varphi_0^2}{M_5^2} \right] + m^2 \left[ \log \frac{M^2}{M_6^2} - \frac{1}{2} \right] \right\} \lambda \dot{\phi}^2
+ \left\{ \left( \log \frac{M_2^2}{M_1^2} - \frac{25}{6} \right) + \frac{1}{4} \lambda \dot{\phi}^4 \right\} \lambda \Box \phi^4
+ \left( \log \frac{M^2}{M_1^2} - \frac{25}{6} \right) + \frac{16m^2}{3M_1^4} \frac{\lambda^2 \dot{\phi}^2}{6} \log \frac{M^2}{M_5^2} - \lambda \ddot{\phi}_\mu \dot{\phi}^\mu
$$

(3.9)

For the discussion of the phase transition of the system, the expansion of $L_r^{(1)}$ for small values of the background field is of relevance. It is easily found to be

$$L_r^{(1)} = \Lambda_{\text{eff}} + \frac{\lambda \dot{\phi}^2}{64\pi^2} \left\{ m^2 \log \left( 1 + \left( \xi - \frac{1}{6} \right) \frac{R}{m^2} \right) - \frac{\lambda \ddot{\phi}_\mu \dot{\phi}^\mu}{6[m^2 + (\xi - \frac{1}{6}) R]} \right\}$$
\[ + R \left( \xi - \frac{1}{6} \right) \log \frac{m^2 + \left( \xi - \frac{1}{6} \right) R}{M_3^2} - \lambda \varphi_3^2 M_3^2 - 1 \]  

(3.10)

\[- \frac{\lambda}{12(m^2 + \left( \xi - \frac{1}{6} \right) R)} \left\{ \Box \phi^2 + 2 \left( \xi - \frac{1}{6} \right) \Box R - \frac{C}{10} + \frac{G}{30} \right\} + O(\phi^4) \]

where \( \Lambda_{\text{eff}} \) (the cosmological constant) represents a complicated expression not depending on the background field \( \hat{\phi} \). Realizing that we are consistently working only in the small curvature and slowly varying background field approximation, the latter equation is equivalent to

\[
L_{(1)}^r = \Lambda_{\text{eff}} + \frac{\lambda \dot{\phi}^2}{64 \pi^2} \left\{ \left( \xi - \frac{1}{6} \right) R \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi_3^2}{M_3^2} \right] 
\right.
\]

\[
+ \left( \xi - \frac{1}{6} \right)^2 \frac{R^2}{2m^2} + \frac{C}{120m^2} - \frac{G}{360m^2}
\]

\[- \frac{\lambda \hat{\phi}_{\mu \nu} \hat{\phi}^\mu}{6m^2} - \frac{\lambda \Box \phi^2}{12m^2} - \left( \xi - \frac{1}{6} \right) \frac{\Box R}{6m^2} \right\} + \text{higher order terms} \]

(3.11)

From this, the known result [48]

\[
Z(\hat{\phi}) = \frac{\lambda^2 \hat{\phi}^2}{192 \pi^2 m^2} \]

(3.12)

immediately follows.

Let us first restrict to static homogeneous spacetimes, so that \( \hat{\phi} \) is a constant field. Restricting to the linear curvature approximation it is seen, that for \( R < 0 \) and \( \xi < 1/6 \) the one-loop term will help to break symmetry, whereas for \( \xi > 1/6 \) the quantum contribution acts as a positive mass and helps to restore symmetry. For \( R > 0 \) the conclusions are obviously reversed. This behaviour has already been found for several examples (see [49, 29, 20]), here it is seen to be valid for any smooth manifold. For non-smooth manifolds such as orbifolds, conical singularities lead to additional contributions in eq. (A.1) and eq. (3.11) remains no longer valid (for a recent example see [21]).

The influence of the higher order terms as well as the derivative terms in eq. (3.11) can in general not be stated. But in the next sections we will, starting from the general results (3.9)-(3.11), analyze several theories and so determine in more detail these contributions.

4 Self-interacting \( \phi^4 \) theory in maximally symmetric spaces

Let us first consider maximally symmetric spaces and direct products of them. More explicitly we treat the 4-dimensional manifolds \( \mathcal{M}^4, \mathbb{R} \times \mathcal{M}^3, S^1 \times \mathcal{M}^3, \mathcal{M}^2 \times \mathcal{M}^2 \), with \( \mathcal{M}^i = \tilde{\mathcal{M}}^i / \Gamma \), where \( \tilde{\mathcal{M}}^i \), which is equal to \( \mathbb{R}^i, S^o \) or \( H^i \) (\( S^o \) and \( H^i \) being the n-sphere and the n-dimensional Lobachevsky space respectively), is the covering manifold and \( \Gamma \) is a discrete group of isometries of \( \tilde{\mathcal{M}}^i \), which acts freely and without fixed points. In appendix B we summarized useful formulas about the geometric tensors in these kind of manifolds.

In the present section the metric is taken to be Euclidean.

For the given examples we may assume a constant background field \( \hat{\phi} \) and so the relevant quantity is the effective potential \( V(\hat{\phi}) = \text{vol}^{-1}(\mathcal{M}) \Gamma \hat{\phi} \). We will concentrate on the small \( \hat{\phi} \)-expansion, eq. (3.11). The quantum corrections to the mass of the field are then defined by

\[
L_{(1)}^r = \Lambda_{\text{eff}} + \frac{1}{2} m^2 \hat{\phi}^2 + O(\hat{\phi}^4)
\]

(4.1)
and using eq. (3.11) it is given by

\[
m^2_T = \frac{\lambda}{32\pi^2} \left\{ \left( \xi - \frac{1}{6} \right) R \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi^2}{M_3^2} \right] - \frac{R^2}{2m^2} \left( \xi - \frac{1}{6} \right)^2 \right\} + \left( \xi - \frac{1}{6} \right)^2 \frac{R^2}{2m^2} + \frac{C}{120m^2} - \frac{G}{360m^2} \right\}
\]

(4.2)

Depending on the manifold \( \tilde{M}_i \), the linear curvature helps to break or to restore symmetry as described below eq. (3.12).

Let us now consider the influence of the higher order terms. For \( M_4 \) we have \( C = 0 \) and \( G = \frac{1}{6} R^2 \), see (B.3), so

\[
m^2_T = \frac{\lambda}{32\pi^2} \left\{ \left( \xi - \frac{1}{6} \right) R \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi^2}{M_3^2} \right] - \frac{R^2}{2m^2} \right\} + \left( \xi - \frac{1}{6} \right)^2 \frac{R^2}{2m^2}
\]

(4.3)

For \( 6 \xi \in \left[ (1 - \sqrt{1/30}), (1 + \sqrt{1/30}) \right] \) the quadratic terms help to break symmetry, otherwise they help to restore it.

In the case \( \mathbb{R} \times M^3 \) we find \( C = 0, G = 0 \), so as a result

\[
m^2_T = \frac{\lambda}{32\pi^2} \left\{ \left( \xi - \frac{1}{6} \right) R \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi^2}{M_3^2} \right] + \left( \xi - \frac{1}{6} \right)^2 \frac{R^2}{2m^2} \right\}
\]

(4.4)

For these examples the \( R^2 \)-contributions are always positive and help to restore symmetry.

Finally, for \( M = M^2 \times N^2 \), (B.7) and (B.8) yield \( C = \frac{2}{3} R(M^2)R(N^2) \) and \( G = 2R(M^2)R(N^2) \), which also lead to eq. (4.4).

As is well known and as is seen also in these examples, in the small curvature limit the topology has no visible influence on the presented results.

This concludes the examples of spaces of constant curvature, and we will now consider the static Taub universe.

5 Self-interacting \( \phi^4 \) theory in a static Taub universe

The metric of a diagonal mixmaster universe is given by

\[
ds^2 = -dt^2 + \sum_{a=1}^{3} l_a^2 (\sigma^o)^2
\]

(5.1)

where \( \sigma^o \) are the basis one-forms

\[
\sigma^1 = \cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\phi \\
\sigma^2 = -\sin \psi \, d\theta + \cos \phi \, \sin \theta \, d\phi \\
\sigma^3 = \cos \theta \, d\phi + d\psi
\]

(5.2)

The \( l_a \)'s are the three principal curvature radii of the homogeneous space and are constants for a static universe. The cases when any two of the \( l_a \)'s are equal are the Taub universes. The case when all three \( l_a \)'s are equal is the closed Friedmann-Robertson-Walker universe.

The Taub universe has been first considered in [22] with the aim of analyzing the effect of curvature anisotropy on symmetry breaking. This was a continuation of investigations into the symmetry behaviour of a self-interacting field in curved spacetime using the Einstein universe.
as an example. The Taub universe was then treated as a perturbative expansion around the Einstein universe in powers of the anisotropy \( \alpha = (l_1^2/l_3^2) - 1 \), thus the results obtained are valid for small anisotropy.

In terms of the anisotropy the scalar curvature \( R \) is given by

\[
R = \frac{4l_1^2 - l_3^2}{l_1^4} = \frac{(3 + 4\alpha)}{2l_1^2(1 + \alpha)}
\]

\[
= \frac{3}{2l_1^2} \left( 1 + \frac{\alpha}{3} + \mathcal{O}(\alpha^2) \right)
\]

where the expansion for small values of \( \alpha \) makes explicit the expansion around the Einstein universe.

As already mentioned, our expansion is consistent for small values of the curvature (compared to the mass \( m \) of the field). Defining \( l_1 = \sqrt{N + 1} \), that is \( \alpha = N \), we obtain

\[
R = \frac{4N + 3}{2(N + 1)^2l_3^2}
\]

Thus for fixed values of \( l_3 \), we see that our expansion is sensible for large values of \( N \), which means large anisotropy \( \alpha \), so we deal with the range not considered in [22]. The relevant quantities for the mass \( m_T^2 \) are [22]

\[
G = 0, \quad C = \frac{4N^2}{3(N + 1)^4l_3^4},
\]

So we find

\[
m_T^2 = \frac{\lambda}{32\pi^2} \left\{ \left( \xi - \frac{1}{6}\right) \frac{4N + 3}{2(N + 1)^2l_3^2} \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi_3^2}{M_3^2} \right] + \frac{1}{2m^2(N + 1)^4l_3^4} \left[ \left( \xi - \frac{1}{6}\right) \left( \frac{4N + 3}{4} + \frac{N^2}{45} \right) \right] \right\}
\]

where once more higher order terms in the curvature help to restore the symmetry.

6 Self-interacting \( \phi^4 \) theory in a Gödel spacetime

Our next example will be the Gödel spacetime in order to consider the question of how a possible rotation of the early universe influences the phase transition. In the Gödel case, as in the static metrics, the classical field can be chosen to be constant. This model has been recently considered for a massless scalar field theory [49] making use of the operative continuation renormalisation method [50]. We generalize these results to the massive case.

The Gödel metric is defined by

\[
ds^2 = -dt^2 + dx^2 - \frac{1}{\sqrt{2}} \exp(2\sqrt{2}\omega x)dy^2 - 2 \exp(\sqrt{2}\omega x) dt \ dy + dz^2
\]

where \( \omega \) is the so called vorticity, a measure of the constant rotation of the matter flow. As a function of the vorticity, the relevant geometric tensors read

\[
R = -2\omega^2, \quad G = 0, \quad C = \frac{16}{3} \omega^4,
\]

which leads to the mass

\[
m_T^2 = \frac{\lambda}{16\pi^2} \left\{ \left( \xi - \frac{1}{6}\right) \omega^2 \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda \varphi_3^2}{M_3^2} \right] + \omega^4 \left[ \left( \xi - \frac{1}{6}\right)^2 + \frac{1}{45} \right] \right\}
\]

Also in this case, the corrections to the linear curvature term help to restore the symmetry.
7 Self-interacting $\phi^4$ theory in a Bianchi type-I universe

Let us now consider the Coleman-Weinberg symmetry breaking mechanism in a Bianchi type-I universe with the metric

$$ds^2 = -dt^2 + \alpha(t) \sum_{i=1}^{3} e^{2\beta_i(t)} (dx^i)^2$$ (7.1)

with the $\beta_i$ taken to be traceless, $\sum_{i=1}^{3} \beta_i = 0$, so that $g = \alpha^3(t)$. $\alpha(t)$ is a positive parameter called $a^2$ in ref. [29], where this model was very recently considered with the aim of investigating the influence of the shear on the symmetry breaking mechanism. In this reference $\alpha(t) = 1$ was considered and for small anisotropy $\beta_i(t)$ the influence of curvature up to linear orders in $R$ was found. The expansion of ref. [29] is consistent for small scalar curvature $R$ which is identical to say for small shear. But this is exactly the range our results are valid. So using our general expansion, we will extend the results of [29] to nonconstant $\alpha(t)$ and arbitrary anisotropy up to quadratic orders in the curvature, that is equivalent to fourth order derivative terms in the scale factors $\alpha(t)$ and $\beta_i(t)$.

The scalar curvature of the Bianchi type-I model (7.1) together with the Gauss-Bonnet density and the Weyl tensor are stated in appendix C (see eqs. (C.1)-(C.3)). It is seen, that small curvature expansion means slowly varying scale factor $\alpha(t)$ and anisotropy $\beta_i(t)$, with no restriction on the magnitude of $\alpha(t)$ and $\beta_i(t)$ themselves.

For nonconstant $\alpha(t)$ for the sake of simplicity, we will only state the linear curvature term. However, in principle also quadratic order terms in the curvature are easily obtained from the tensors given in appendix C, but this result would be difficult to survey and we will give it only for constant $\alpha$.

Concentrating on the effective potential part, the mass for nonconstant $\alpha(t)$ in the linear curvature approximation reads

$$m_T^2 = \frac{\lambda}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left[ \frac{3\alpha''}{\alpha} + Q_1 \left( \log \frac{m^2}{M_3^2} - \frac{\lambda\varphi^2}{M_3^2} \right) \right]$$ (7.2)

In eq. (7.2), $Q_1 = \sum_{i=1}^{3} (\beta_i'')^2$ is a measure of the shear and the prime denotes differentiation with respect to $t$. Comparing with the result of Berkin [29], there are of course additional contributions due to the nonconstant scale factor $\alpha$, but the general feature in terms of the scalar curvature as stated in eq. (3.11) remains true.

For constant $\alpha$ up to fourth order derivative terms the result reads

$$m_T^2 = \frac{\lambda}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda\varphi^2}{M_3^2} \right] + Q_2 \left( \xi - \frac{1}{6} \right)^2 + \frac{1}{45} + \frac{Q_2}{60m^2} + \frac{2}{45m^2} \frac{d}{dt} (\beta_1'\beta_2'\beta_3')$$ (7.3)

where $Q_2 = \sum_{i=1}^{3} (\beta_i'')^2$ was set and explicit use of the $\beta_i$ to be traceless was made. Depending on the $\beta_i$'s, the fourth order derivative terms may help to restore or to break symmetry. When the $\beta_i$'s are constants, the fourth order derivative terms always help to restore symmetry.

Finally, let us concentrate on $\beta_i = 0$, $i = 1, 2, 3$ (This is the case of the spatially flat Robertson-Walker metric; see [35]). For this case we find

$$m_T^2 = \frac{\lambda}{32\pi^2} \left( \xi - \frac{1}{6} \right) \left[ \log \frac{m^2}{M_3^2} - \frac{\lambda\varphi^2}{M_3^2} \right] + \frac{9}{2m^2} (\xi - \frac{1}{6})^2 \left( \frac{\alpha''}{\alpha} \right)^2 - \frac{1}{240m^2} \left( \frac{\alpha'}{\alpha} \right)^2 \left[ \frac{2\alpha''}{\alpha} - \left( \frac{\alpha'}{\alpha} \right)^2 \right]$$ (7.4)
Once more, depending on the scaling function $\alpha$, higher curvature terms may help to restore or to break symmetry. More precisely, if $\alpha''$ is negative or greater than $(\alpha')^2/15\alpha$, the symmetry may be restored with the help of higher curvature terms.

8 Conclusions

In new inflationary models, the effective cosmological constant is obtained from an effective potential, which includes quantum corrections to the classical potential of a scalar field \[3\]. This potential is usually calculated in Minkowski spacetime, whereas to be fully consistent the effective potential must be calculated for more general spacetimes. For that reason an intensive research has been dedicated to the analysis of the one-loop effective potential of a self-interacting scalar field in curved spacetime and in spacetimes with nontrivial topology \[4-29\].

In this paper we provided a simple approach for the calculation of the derivative expansion of the effective Lagrangian in the background field, the expansion being consistent if the effective mass $M^2 = m^2 + (\xi - \frac{1}{6}) R + \frac{\hat{\phi}^2}{2\alpha}$ of the theory is large compared to a typical magnitude of the curvature. The result has been used for several examples of physical interest to analyze the influence of the gravitational field on the phase transition.

In the linear curvature approximation it is seen, that for $R < 0$ and $\xi < \frac{1}{6}$ the one-loop term will help to break symmetry, whereas for $\xi > \frac{1}{6}$ the quantum contribution acts as a positive mass and help to restore symmetry. The conclusions are reversed if we choose $R > 0$. The influence of the higher curvature terms depends on the spacetime under consideration and is stated explicitly in the respective sections.

Using the provided approach it was very simple to extract the relevant information for several spacetimes, thus extending recent results.

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A Appendix: heat coefficients

As mentioned in Sec. 2, in this appendix we report on some results concerning the heat-kernel expansion of a second order elliptic differential operator. The operator of interest in the given considerations is $A = -\Box + X(x)$, defined on a smooth n-dimensional Riemannian manifold without boundary.

Using the ansatz of Jack and Parker, see eq. (2.10), the explicitly needed coefficients of expansion (2.11) read (for details see ref. \[34\])

\[
\begin{align*}
a_0(x,x) &= 1 \\
a_1(x,x) &= 0 \\
a_2(x,x) &= \frac{1}{6} \Box (X - R/6) + \frac{1}{180} (\Box R + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - R_{\mu\nu} R^{\mu\nu}) \\
a_3(x,x) &= \frac{1}{12} X^\mu X^\mu - \frac{1}{60} \Box^2 X + \frac{1}{90} R^{\mu\nu} X_{;\mu\nu} - \frac{1}{90} R^{\mu\nu} X_{;\mu} \\
&+ \frac{1}{71} (18 \Box^2 R + 17 R_{\mu\nu} R^{\mu\nu} - 2 R_{\mu\nu,\rho} R^{\mu\nu,\rho} - 4 R_{\mu\nu,\rho} R^{\mu\nu,\rho} \\
&+ 9 R_{\mu\nu,\rho,\tau} R^{\mu\nu,\rho,\tau} - 8 R_{\mu\nu,\rho,\tau} R^{\mu\nu,\rho,\tau} + 24 R_{\mu\nu,\rho} R^{\mu\nu,\rho} + 12 R_{\mu\nu,\rho,\tau} R^{\mu\nu,\rho,\tau}) \\
&+ \frac{198}{9} R^{\mu,\nu} R^{\mu,\nu} + \frac{64}{9} R_{\mu\nu} R_{\rho\sigma} R^{\mu\nu,\rho\sigma} + \frac{16}{3} R_{\mu\nu} R_{\rho,\sigma,\tau} R^{\mu,\nu,\rho,\sigma,\tau}.
\end{align*}
\]
\[ + \frac{44}{9} R_{\mu\nu\rho\sigma} R^{\mu\nu\alpha\beta} R_{\alpha\beta}^{\rho\sigma} + \frac{80}{9} R_{\mu\nu\rho\sigma} R^{\mu\alpha\beta} R_{\alpha}^{\nu\beta} \]

The \( a_2 \) coefficient fixes the counterterms in the renormalization procedure, see eq. (3.3), while only the first term in \( a_3 \) is relevant in the approximation in which we are working.

**B Appendix: manifolds with constant curvature**

In this appendix we want to summarize some tensor identities used in section 4.

For maximally symmetric \( n \)-dimensional space \( N \) one has

\[ R_{\mu\nu\rho\sigma}(N) = \frac{R(N)}{n(n-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \]  
\[ R_{\mu\nu} = \frac{R}{n} g_{\mu\nu} \] (B.1)

which leads to

\[ C(N) = 0, \quad G(N) = \frac{(n-2)(n-3)}{n(n-1)} R^2(N) \] (B.3)

For constant curvature spaces \( M \) which are direct products of maximally symmetric spaces, \( M = M_1 \times M_2 \), the following identities are useful [41]

\[ R^2(M) = R^2(M_1) + R^2(M_2) + 2 R(M_1) R(M_2) \] (B.4)
\[ R_{\mu\nu}(M) R_{\mu\nu}(M) = R_{\mu\nu}(M_1) R_{\mu\nu}(M_1) + R_{\mu\nu}(M_2) R_{\mu\nu}(M_2) \] (B.5)
\[ R_{\mu\nu\rho\sigma}(M) R^{\mu\nu\rho\sigma}(M) = R_{\mu\nu\rho\sigma}(M_1) R^{\mu\nu\rho\sigma}(M_1) + R_{\mu\nu\rho\sigma}(M_2) R^{\mu\nu\rho\sigma}(M_2) \] (B.6)

Using (B.4)-(B.6) it is easy to arrive at

\[ C(M) = C(M_1) + C(M_2) + \frac{2}{3} R(M_1) R(M_2) \] (B.7)
\[ G(M) = G(M_1) + G(M_2) + 2 R(M_1) R(M_2) \] (B.8)

which are the relevant results used in Sec. 4.

**C Appendix: some invariants for Bianchi I universe**

Here we give the relevant tensors, for our considerations, up to fourth order derivative in the scale factor \( \alpha(t) \) and the anisotropy \( \beta_i(t) \) of the Bianchi type-I model of Sec. 7.

First, for the scalar curvature one has

\[ R = \frac{3 \alpha''}{\alpha} + Q_1 \] (C.1)

For the Gauss-Bonnet density we find

\[ G = \frac{3}{2} \left( \frac{\alpha'}{\alpha} \right)^2 \left[ \frac{2 \alpha''}{\alpha} - \left( \frac{\alpha'}{\alpha} \right)^2 \right] - Q_1 \left[ \frac{2 \alpha''}{\alpha} + \left( \frac{\alpha'}{\alpha} \right)^2 \right] \]
\[ + \frac{2 \alpha'}{\alpha} \left( 6 \beta'_1 \beta'_2 \beta'_3 - Q_1 \right) + 8 \frac{d}{dt} \left( \beta'_1 \beta'_2 \beta'_3 \right) \] (C.2)

and the square of the Weyl-tensor reads

\[ C = \frac{Q_1}{2} \left( \frac{\alpha'}{\alpha} \right)^2 + \frac{4 Q_1^2}{3} + 2 Q_2 \]
\[ + \frac{\alpha'}{\alpha} \left( 12 \beta'_1 \beta'_2 \beta'_3 + Q_1 \right) + 8 \frac{d}{dt} \left( \beta'_1 \beta'_2 \beta'_3 \right) \] (C.3)

These results are used in Sec. 7 to give the quantum corrections to the mass of the field up to quadratic order in the curvature.
References