The Super $W_3$ Conformal algebra and the Boussinesq hierarchy

by

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Abstract: The bihamiltonian structure of the $N = 2$ Supersymmetric Boussinesq equation is found. It is not reduced to the corresponding classical structure and hence it describes the pure supersymmetric effect. For the supersymmetric Boussinesq equation which contains the classical partner the Lax pair is given explicitly. Thus we prove the integrability of this equation.

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1 Introduction

The Boussinesq equation

\[ u_{tt} = -\gamma^2 (u_{xxx} + 8uu_x)_x \]  

with \( \gamma \) an arbitrary constant, was first derived to describe shallow-water waves propagating in both directions [1]. Its equivalent form is

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}_t = \gamma \begin{pmatrix}
  v_x \\
  -u_{xxx} - 8uu_x
\end{pmatrix},
\]

which appears to be more comfortable for the Lax formulation. Indeed, the Lax operator

\[ L = \partial_{xxx} + 2u\partial_x + v, \]  

characterizes the \( Bs_q \) hierarchy via the Lax equations

\[ L_{tn} = \gamma \left( L^{n/3} \right)_+, L \]; \quad n = 1, 2, ... \]  

where \( \partial_x = \partial/\partial_x \) and + denotes the projection of the fractional power \( L^{n/3} \) onto its purely differential part. The operators \( L \) and \( L^{n/3} \) in the formula (4) we shall call the Lax pair.

The Boussinesq equation itself corresponds to the first non-trivial flow given by \( n = 2 \). The system described by the Boussinesq equation is the bi-hamiltonian system. The first hamiltonian structure was found by Gelfand and Dickey [2] while the second one was conjectured and constructed by Adler [3]. The proof of the validity of this conjecture can be found in [4].

In the 1988 several scientists [5-7] recognized that the second hamiltonian structure of the Boussinesq hierarchy, the Gelfand-Dickey algebra [8] associated to the Boussinesq Lax operator, is a classical representation of the so called \( W \)-algebra.

The \( W \)-algebras were first introduced by Zamolodchikov [9] in order to extend polynomially the Virasoro algebra by higher spin fields. Since the introduction of \( W \)-algebras they have been the subject of intensive investigation by both physicists and mathematicians. The unexpected relationship between the \( W \)-algebras and the noncritical string [10] and Toda lattice models [11] has been found.

On the other hand, in the last three years there has been a considerable interest in the supersymmetric extension of Zamolodchikov’s \( W_n \) algebras. Recently the super \( W \)-algebra has been constructed in the extended and non-extended case [12-16]. The tempting problem with possibly important implication in the super \( W_3 \) gravity and the related matrix models, is to find the generalized super Boussinesq-type hierarchy via the Lax pair approach. The similar program has been successfully applied to the supersymmetric Korteweg-de Vries equation which is connected with the supersymmetric extension of the \( W_2 \) algebra (e.g. with the super Virasoro algebra) [17-19].

There has been some attempt to utilize the Lax operator for the construction of the nonextend super \( W_3 \) algebra as well as for the construction of the supersymmetric extension of the Boussinesq equation. For example Nam [20] examined a generalization of the Miura transformation and accompanying factorization of the even Lax operator. However, he concludes that the nontrivial factorization of the Lax operator is impossible.
and thus a nontrivial supersymmetric version of the extended conformal algebra cannot be constructed. On the other hand Figueroa et al [13] constructed the non-extended supersymmetric version of the $W_3$ algebra using the odd Lax operator. However, his odd Lax operator does not generate the equation of motion and hence there is no the supersymmetric version of the Boussinesq equation in his approach. The negative results in this framework have been reported also in [21,22].

Recently Ivanov and Krinovos [16] have constructed the nontrivial Hamiltonian flow on the extended super $W_3$ algebra yielding $N = 2$ superextension of the Boussinesq equation. Their superextension turns out to involve a free parameter and is reducible in the bosonic sector to the Boussinesq equation only at a special value of this parameter. The super $W_3$ algebra has been obtained by the supersymmetrization of the Miura transformation and the existence of such a transformation is the important property of the given system. Indeed, such a transformation relates the second Poisson bracket with the generalized Gardner-Zakharov-Faddeev bracket [8]. Moreover such a transformation allows us to construct the representation of the $N = 2$ super $W_3$ algebra out of the superfield of the lower conformed dimension than the one characterizing the original fields.

The appearance of the Miura transformation is a strong indication that the given system could possess the bihamiltonian structure. In this paper we show that indeed for the special values of a free parameter of the supersymmetric Boussinesq equation this equation has bihamiltonian structure. Interestingly for this value of a free parameter the supersymmetric Boussinesq equation does not contain the classical Boussinesq equation.

For the equation which contains the classical partner we have found the Lax formulation using the symbolic manipulation package REDUCE [23]. It appeared that the roots of the Lax operator are not uniquely determined, hence we check all possibilities allowing for the existence of the Lax pair and finally we arrive to the three different Lax formulations.

2 The Bi-hamiltonian structure of the Super symmetric Boussinesq equation

The classical Boussinesq equation (1.2) can be written down as a bi-Hamiltonian system

$$
\begin{pmatrix}
u
\end{pmatrix}_t = P_1 \nabla \cdot \int \left( \frac{1}{2} u_x^2 - \frac{4}{3} u^3 + \frac{1}{2} v^2 \right) dx = P_2 \nabla \cdot \int \frac{1}{2} v dx
$$

(2.1)

Here $\nabla = \left( \frac{\delta}{\delta u} \right)$ and the two Hamiltonian operators $P_1$ and $P_2$ are given by

$$
P_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}
$$

(2.2)

$$
P_2 = \begin{pmatrix} \partial^3 + u \partial + \partial u \\ \partial v + 2 v \partial \\ -(\partial^5 + 2(\partial^3 u + u \partial^3) + 3(\partial^2 u \partial + \partial u \partial^2)) + 8(\partial u^2 + u^2 \partial) \end{pmatrix}
$$

(2.3)
The Poisson bracket defined by the Hamiltonian operator $P_2$ corresponds to the classical version of the $W_3$ algebra. In order to obtain the $P_2$ one can use the Miura transformation in the form

$$ u = P_{1x} - \frac{1}{2}(P_1^2 + P_2^2) $$

$$ v = -P_{2xx} + 3P_1P_{2x} + P_{1x}P_2 + \frac{2}{3}P_2^3 - 2P_1^2P_2 $$

and assume the following Poisson bracket (free field representation) on the fields $P_1$ and $P_2$

$$ \{P_i, P_j\} = -\delta_{ij}\delta(x-y) . $$

The compatibility of the two Hamiltonian structures is granted by the observation that the simply shift $v \rightarrow v + \lambda$ produces the Hamiltonian operator $P_2 + 3\lambda P_1$.

Now we would like to check whether the similar properties occur in the supersymmetric level. The basic object in the supersymmetric analysis is the superfield and the supersymmetric derivative. The superfields are the superfermions or the superbosons depending, in addition to $x$ and $t$, upon two anticommuting variables, $\theta$, and $\theta_{2}(\theta_{1}\theta_{2} = -\theta_{2}\theta_{1} , \theta_{1}^2 = 0 , \theta_{2}^2 = 0)$. Its Taylor expansion with respect to the $\theta$’s is

$$ \phi(x, \theta_1 \theta_2) = \omega(x) + \theta_1 \xi_1(x) + \theta_2 \xi_2(x) + \theta_2 \theta_1 u(x) $$

where in below the fields $\omega, v$ are to be interpreted as the boson (fermion) fields while $\xi_1, \xi_2$ as the fermion (boson) fields for the superboson (superfermions) field $\phi$ respectively. We choose the following representaion on the superderivatives

$$ D_1 = \partial_{\theta_1} - \frac{1}{2} \theta_2 \partial_x , \quad D_2 = \partial_{\theta_2} - \frac{1}{2} \theta_1 \partial_x $$

and as the consequence we obtain

$$ D_1 D_1 = D_2 D_2 = 0 $$

$$ \{D_1 D_2\} = -\partial_x . $$

In the paper [16] author construct the supersymmetric operator product expansion (SOPE) of the $N = 2$ super $W_3$ algebra in terms of the spin 1 and spin 2 supercurrent. This representaion has been obtained by the free superfield realization or in other terms via the super Miura transformation. In order to transform SOPE to the Poisson bracket we use the supersymmetric version of the Cauchy theorem

$$ \frac{1}{2\pi i} \int dz_1(z_{12})^{-n-1}(\theta_{12}\bar{\theta}_{12}, \theta_{12}, \bar{\theta}_{12}, 1) = $$

$$ = \frac{1}{n!} (1, D_1, -D_2, \frac{1}{2}[D_1 D_2]) (\partial^n \Phi) $$

where

$$ \theta_{12} = \theta_1 - \theta_2 \quad , \quad \bar{\theta}_{12} = \theta_1' - \theta_2' \quad , \quad z_{12} = z_1 - z_2 + \frac{1}{2}(\theta_1 \theta_2' - \theta_2 \theta_1') . $$
As the result we obtain the super analog of the $P_2$ operator with the following entries

\[ P_{211} = \frac{c}{4} [\mathcal{D}_1, \mathcal{D}_2] \partial_x + \mathcal{J}_x + \mathcal{J} \partial_x + \mathcal{J} \partial_x + (\mathcal{D}_1 \mathcal{J}) \mathcal{D}_2 + (\mathcal{D}_2 \mathcal{J}) \mathcal{D}_1 \]

\[ P_{212} = 2 \cdot \partial_x \cdot T + (D_2 T) D_1 + (D_1 T) D_2 \]

\[ P_{221} = \partial \cdot T + T \partial_x + (D_2 T) D_1 + (D_1 T) D_2 \]

\[ P_{222} = -\frac{c}{8} [\mathcal{D}_1, \mathcal{D}_2] \partial_{xxx} - 2 \mathcal{J} \partial_{xxx} - 6 (\mathcal{D}_2 \mathcal{J}) \mathcal{D}_1 \partial_{xx} - 6 (\mathcal{D}_1 \mathcal{J}) \mathcal{D}_2 \partial_{xx} - 6 \mathcal{J}_x \partial_{xx} + \left( 5 T - 2 ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J}) + \frac{8}{c} \mathcal{J}^2 \right) [\mathcal{D}_1, \mathcal{D}_2] \partial_x - \left( 8 (\mathcal{D}_2 \mathcal{J}_x) + \frac{16}{c} \mathcal{J} (\mathcal{D}_2 \mathcal{J}) + 5 (\mathcal{D}_2 T) \right) \mathcal{D}_1 \partial_x + \left( -8 (\mathcal{D}_1 \mathcal{J}_x) + \frac{16}{c} \mathcal{J} (\mathcal{D}_1 \mathcal{J}) + 5 (\mathcal{D}_1 T) \right) \mathcal{D}_2 \partial_x + \left( \frac{3}{2} ([\mathcal{D}_1, \mathcal{D}_2] T) - 6 \mathcal{J}_xx + u_3 \right) \partial_x - (3 \mathcal{D}_2 T_x) + 3 (\mathcal{D}_2 \mathcal{J}_xx) + \mathcal{Ψ} \mathcal{D}_1 + (3 \mathcal{D}_1 T_x) - 3 (\mathcal{D}_1 \mathcal{J}_xx) - \bar{\mathcal{Ψ}} \mathcal{D}_2 \right) \]

\[ + \left( -2 \mathcal{J}_xxx + ([\mathcal{D}_1, \mathcal{D}_2] T_x) + \frac{1}{2} u_3 x + \frac{1}{2} (\mathcal{D}_1 \Psi) + \frac{1}{2} (\mathcal{D}_2 \Psi) - \frac{4}{c} ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J} \mathcal{J}_x) \right) \]

where $c$ is an arbitrary constant (central extension term) while

\[ \Psi = \frac{8}{c} \partial_x (\mathcal{J} \mathcal{D}_2 \mathcal{J}) - \frac{72}{c} T \mathcal{D}_2 \mathcal{J} + \frac{36}{c} ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J})(\mathcal{D}_2 \mathcal{J}) + \frac{8}{c} \mathcal{J} (\mathcal{D}_2 T) - \frac{128}{c^2} \mathcal{J}^2 (\mathcal{D}_2 \mathcal{J}) + \frac{4}{c} \mathcal{J}_x (\mathcal{D}_2 \mathcal{J}) \]

\[ \bar{\Psi} = -\frac{8}{c} \partial_x (\mathcal{J} \mathcal{D}_1 \mathcal{J}) - \frac{72}{c} T \mathcal{D}_1 \mathcal{J} + \frac{36}{c} ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J})(\mathcal{D}_1 \mathcal{J}) + \frac{8}{c} \mathcal{J} (\mathcal{D}_1 T) - \frac{128}{c^2} \mathcal{J}^2 (\mathcal{D}_1 \mathcal{J}) - \frac{4}{c} \mathcal{J}_x (\mathcal{D}_1 \mathcal{J}) \]

\[ u_3 = \frac{56}{c} \mathcal{J} T - \frac{32}{c} \mathcal{J} ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J}) + \frac{128}{c^2} \mathcal{J}^3 + \frac{120}{c} (\mathcal{D}_1 \mathcal{J})(\mathcal{D}_2 \mathcal{J}) . \]

Using this operator to the equation

\[ \left( \begin{array}{c} \mathcal{J} \\ T \end{array} \right)_t = P_2 \cdot \nabla \int (T + \alpha \mathcal{J}^2) dx , \]

where $\alpha$ is an arbitrary constant, we obtain the $N = 2$ supersymmetric extension of the Boussinesq equation

\[ \mathcal{J}_t = 2 T_x + \alpha \left( \frac{c}{4} ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J}_x) + 4 \mathcal{J} \mathcal{J}_x \right) \]

\[ T_t = -2 \mathcal{J}_xxx + ([\mathcal{D}_1, \mathcal{D}_2] T_x) + \frac{80}{c} (\mathcal{D}_1 \mathcal{J} \mathcal{D}_2 \mathcal{J})_x - \frac{16}{c} (\mathcal{J} [\mathcal{D}_1, \mathcal{D}_2] \mathcal{J}_x) \]

\[ - \frac{16}{c} \mathcal{J}_x ([\mathcal{D}_1, \mathcal{D}_2] \mathcal{J}) + \frac{256}{c^2} \mathcal{J}^2 \mathcal{J}_x + \left( \frac{40}{c} - 2 \alpha \right) (\mathcal{D}_1 \mathcal{J} \mathcal{D}_2 T) + \left( \frac{64}{c} + 4 \alpha \right) \mathcal{J}_x T + \left( \frac{24}{c} + 2 \alpha \right) \mathcal{J} T x + \left( \frac{40}{c} - 2 \alpha \right) (\mathcal{D}_2 \mathcal{J})(\mathcal{D}_1 T) . \]
Now we would like to check whether the Hamiltonian operator $P_2$ produces, similarly as in the classical case, the first Hamiltonian operator. Therefore let us shift the $T \rightarrow T + \lambda$ in the $P_2$ operator, obtaining

$$P_1 = \begin{pmatrix} 0 & \frac{2\partial_x}{\partial^2} \\ 2\partial_x & 5[D_1, D_2]\partial_x + \frac{1}{c}\{64\mathcal{J}_x + 56\mathcal{J}\partial_x \\ + 72(D_2\mathcal{J})D_1 + 72(D_1\mathcal{J})D_2\} \end{pmatrix}$$ \hspace{1cm} (2.15)$$

It is easy to check that $P_1$ is an antisymmetric operator with the vanishing Shouten bracket [24] and hence $P_1$ defines a proper Hamiltonian operator.

In order to find the Hamiltonian which produces (via $P_1$) the equation (2.14) we shall first simplify the equation 2.14 by shifting the $T$ superfunction to $T \rightarrow T + 2[D_1, D_2]\mathcal{J} + 16\mathcal{J}^2$ \hspace{1cm} (2.16)

Then the super-Boussinesq equation for $\alpha = -\frac{16}{c}$ reduces to

$$\mathcal{J}_t = 2T_x$$ \hspace{1cm} (2.17)

$$T_t = -3([D_1, D_2]T_x) - \frac{72}{c}\mathcal{J}T_x +$$

$$+ \frac{72}{c}[(D\mathcal{J})(D_2T) + (D_2\mathcal{J})(D_1\mathcal{J})]$$ \hspace{1cm} (2.18)

while the $P_1$ operator transform to the following form

$$P_1 = \begin{pmatrix} 0 & \frac{2\partial_x}{\partial^2} \\ 2\partial_x & -3[D_1, D_2]\partial_x - \frac{72}{c}\mathcal{J}\partial_x \\ & + \frac{72}{c}[(D_1\mathcal{J})D_2 + (D_2\mathcal{J})D_1] \end{pmatrix}$$ \hspace{1cm} (2.19)$$

Now it is not so difficult to prove that the following quantity

$$H_1 = \int \frac{1}{2} T^2 dx$$

is the conserved quantity and produces via (2.19) the equation (2.17-2.18).

Interestingly, for $\alpha = -\frac{16}{c}$ the supersymmetric equation (2.14) is not reduced to the usual classical Boussinesq equation. This limit exists only for $\alpha = -\frac{4}{c}$ [16]. Moreover our first Hamiltonian is not reduced in this limit to the classical partner either. Thus our bi-hamiltonian structure describes a pure supersymmetric effect.

Usually in the soliton’s theory, the bi-hamiltonian structure is established via the celebrated Adler-Konstant-Symes scheme [3,8]. In this construction we have to know the Lax pair. In the next section we present several different supersymmetric extensions of the Lax pair and we show that one of them produces the supersymmetric extension of the Boussinesq equation and coincides with the equation (2.14) only for $\alpha = -\frac{4}{c}$, e.g. the one which contains the classical case.
3 The supersymmetric Lax operator for the supersymmetric Boussinesq equation.

In order to find the supersymmetric extension of the Lax pair (1.3), first let us consider the super-pseudo-differential elements of the supersymmetric algebra

\[ G \in q = \left\{ \sum_{n=-\infty}^{\infty} (b_n + f_n D_1 + ff_n D_2 + bb_n D_1 D_2) \partial^n \right\}, \quad (3.1) \]

where \( b_n, bb_n \) are the superbosons while \( f_n, ff_n \) are the superfermions. The action of the operator \( \partial^{-1} \) is the formal integration defined by the basic rule

\[ \partial^{-1} b = b \partial^{-1} - b_x \partial^{-2} + b_{xx} \partial^{-3}, \quad (3.2) \]
\[ b \partial^{-1} = \partial^{-1} b + \partial^{-2} b_x + \partial^{-3} b_{xx}, \quad (3.3) \]

where \( b \) is a some function.

Let us postulate the following most general form of the Lax operator to be

\[ L = a \partial_{xxx} + \beta D_1 D_2 \partial_{xx} + Z_1 \partial_x + Z_0 \]

where \( a \) and \( \beta \) are arbitrary constants and \( Z_1 \) and \( Z_0 \) are the elements of the super-algebra \( G \). The elements \( Z_1 \) and \( Z_0 \) are constructed from all possible combinations of the elements \( D_1, D_2, \partial_x \) and two superboson fields \( J \) and \( T \). These objects are too complicated to be presented here. By using the symbolic manipulation package REDUCE, we have verified and simplified this ansatz and finally have found several different supersymmetric Boussinesq equations. The most interesting is the one which is reduced to the equation (2.14). For this case the Lax operator (3.4) take the following form

\[ L = D_1[D_{xx} + \eta J \partial_x + T]D_2 \]

where \( \eta \) is an arbitrary constant.

Substituting \( L \) to the (1.4) we obtained the following equation

\[ J_t = \frac{\gamma}{3\eta} [3\eta J_x + \eta^2 J^2 - 6T]_x \]

\[ T_t = \gamma \left[ \frac{2}{3} \eta J_{xxx} + \frac{2}{3} \eta^2 J J_{xx} + \frac{2}{3} \eta (T J_x - JT_x) \\
+ \frac{2}{3} \eta^2 (D_1 J)(D_2 J_x) - T_{xx} + \frac{2}{3} \eta [((D_1 T)(D_2 J) + (D_2 T)(D_1 J))] \right] . \]

This equation coincides with the equation (2.14) only for \( \alpha = -4/c, \ \gamma = 3, \ \eta = 24/c \).

To see it let us shift the function \( T \) in (3.6-3.7) to

\[ T \to -\frac{8}{c} T + \frac{12}{c} J_x z + \frac{96}{c^2} J^2 , \quad (3.8) \]

and next change

\[ D_1 \to D_2 \ , \ D_2 \to D_1 . \quad (3.9) \]
As the result we obtained the following equation

\[ J_t = 2T_x \]  

\[ T_t = -\frac{3}{2} J_{xxx} + \frac{72}{c} ((D_1 J)(D_2 J))_x + \frac{48}{c} J_x T + \frac{48}{c} [(D_1 J)(D_2 T) + (D_2 J)(D_1 T)] + \frac{576}{c^2} J^2 J_x \]  

Now we make the similar transformation for the equation (2.14) e.q. shift \( T \) to

\[ T \rightarrow T + \frac{1}{2} ([D_1, D_2] J) + \frac{4}{c} J^2 \]  

and obtain that (2.14) is equivalent with (3.10-3.11).

Interestingly the transformation (3.9) give us also the new form of the Lax operator

\[ L = D_2 \cdot [\partial_{xx} + \eta J \cdot \partial_x + T] D_1 . \]  

Thus the system described by the equations (3.6-3.7) and by (3.10-3.11) for \( \alpha = -4/c \) is integrable and the conservation laws are given by the standard trace form

\[ I_n = tr L^n \]  

where \( tr \) denotes the coefficient standing in the \( \partial^{-1} D_1 D_2 \) term.

For the completeness, let us present other Lax operator which produce also the different supersymmetric extension of the Boussinesq equation. These Lax pairs follows from the observation that the supersymmetric cubic roots of the \( L \) operator are not uniquely defined *. Indeed one can quickly check that

\[ (\varepsilon \partial_x + XD_1 D_2)^3 = a \partial_{xxx} + \beta D_1 D_2 \partial_{xx} \]  

if

\[ a = \epsilon^3 \]  

\[ \beta = X^3 - 3\epsilon X^2 + 3\epsilon X \]  

As we see from the last formulas, there are four different solutions of \( a, \beta, X, \epsilon \).

Namely, the first one characterize \( a = 0, \epsilon = 0 \) and therefore without loosing any generality we can set \( \beta = a = 1 \). This case has been considered previously and produces the equations (3.6-3.7).

For the other cases we can set \( \epsilon = 1 \) without loosing on the generality of discussion as well. Then the equation (3.14) has three different solutions

\[ X_1 = \frac{1}{2} [i(\beta - 1)\frac{1}{3}\sqrt{3} - (\beta - 1)\frac{1}{3} + 2] \]  

\[ X_2 = \frac{1}{2} [i(\beta - 1)\frac{1}{3}\sqrt{3} + (\beta - 1)\frac{1}{3} - 2] \]  

\[ X_3 = (\beta - 1)\frac{1}{3} + 1 \]  

*Notice that the same problem has appeared in the \( N = 2 \) supersymmetric KdV Lax pair where we have two different roots of the Lax pair [19]
Let us denote by $L_i^{1/3} i = 1, 2, 3$ the corresponding roots of the Lax operator. Notice that two of the roots are complex, but it does not disturb us in the construction of the Lax pair in the form (1.4) because we have to construct $(L_2/3)_+^i$ for all $i, j = 1, 2, 3$. Therefore we have to check the six different Lax pairs only. Using once more the symbolic manipulation program REDUCE we found that for two Lax pairs only ($i = j = 3$ and $i = 1, j = 2$) the equation (3.21) produce the same real equation which could be written down as

$$J_t = T_x$$

$$T_1 = a_0(a_1 J_{xx} + a_2 J^3 + a_3(D_2 J)(D_1 J))_x$$

$$+ a_4[T J_x + (D_2 J)(D_1 T) + (D_1 J)(D_2 T)]$$

where

$$a_0 = 3(\beta - 1)^{1/2} \lambda \ , \ a_1 = -3\beta^3 + 6\beta - 3 \ ,$$

$$a_2 = \frac{2}{3} \eta^2 \ , \ a_3 = 6(1 - \beta) \eta$$

$$a_4 = 2\lambda \eta \ , \ \gamma = \lambda^2 \ , \ \beta \neq 1$$

and $\lambda$ is an arbitrary parameter.

The $L$ operator in that case takes the form

$$L = \partial_{xxx} + \beta D_1 D_2 \partial_{xx} + \eta D_1 \cdot J \cdot D_2 + D_1 \cdot T D_2$$

The case $\beta = 1$ should be considered independently. Interestingly for this case we have found the Lax pair in the form (3.13).

### 4 Conclusion.

As we see in this paper it is possible to construct the bihamiltonian structure for the supersymmetric Boussinesq equation. However, our structure is not reduced to the classical bihamiltonian structure and hence it describes a pure supersymmetric effect.

For the superextension which contains the classical Boussinesq equation, we have found the Lax operator. The knowledge of the Lax operator is an important property which allows us to perform the investigation of the given system in more detail on a deeper level. For example, applying the celebrated Adler-Konstant-Symes scheme [3,8], it is possible, in principle, to find the bihamiltonian system and hence the Poisson brackets for the given equation using the Lax pair only. Applying this method to our supersymmetric Lax operator, we can find the Poisson bracket but defined now on a larger subspace than the one on which the Lax operator takes values. Therefore, we expect that we have to apply the Dirac reduction technique in order to find the Poisson
bracket on the same subspace as the Lax operator. This problem we postpone to a further communication.

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