Comment on thickness-corrections to Nambu-wall

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Abstract

We comment on some calculations concerning the finite-thickness corrections to the (generalized) Nambu action for a curved domain wall in Minkowski space. Contrary to a recent claim in the literature, we find no first order corrections in the wall-thickness, and only one second order correction proportional to the Ricci curvature of the wall. These results are obtained by consistently expanding the action and the equations of motion for the scalar field.
Cosmic strings and domain walls play an important role in cosmological models concerning phase transitions in the early universe, galaxy formation, etc...[1,2]. Although the domain wall is a topological defect of one dimension higher than the string, it is in some senses easier to describe: From the field theoretic point of view a domain wall can be formed in theories of just one real scalar field whereas the strings involve more complicated field-configurations, and from a (generalized) world-sheet point of view the domain wall, being a hypersurface, has no torsion and only one component of the mean curvature and the Gaussian curvature.

The dynamics of walls is usually described by a generalized Nambu action corresponding to the assumption that the (other) dimensions of the wall are much greater than the thickness. During the last few years, however, there has been some interest in calculating the finite thickness corrections to the Nambu action. The main results are due to Gregory, Haws and Garfinkle [4,5,6], who analyzed the coupled Einstein-scalar equations for a thick gravitating wall, using methods from differential geometry. Much more recently the same problem (although in flat spacetime) was attacked by Silveira and Maia [7] using a somewhat simpler approach. After making a suitable Ansatz for the scalar field in the vicinity of the wall, they expanded the action and the equations of motion around a so-called "locally non-plane solution" in powers of the wall-thickness \(\epsilon\). We find however, that their calculations are inconsistent: First of all the results of Ref. 7 depend crucially on whether the thickness expansions are made directly in the action or in the equations of motion (and this is not because of the Ansatz for the scalar field!). Secondly the final result of Ref. 7 for the effective action of the wall is plagued with the disease that partial integrations completely change the power counting and therefore mix the different terms of different powers in the expansion.

The purpose of this note is to present a consistent derivation of the effective action for the curved thick wall, using the simple approach of Ref. 7.

The starting point is the action for a real scalar field in Minkowski space:

\[
S = \int \left[ -\frac{1}{2} \eta_{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] d^4 x,
\]

where \(\eta_{\mu \nu} = \text{diag}(-1,1,1,1)\) and:

\[
V(\phi) = \lambda (\phi^2 - v^2)^2.
\]
It is well-known that this model enjoys wall-solutions [2,3], and the idea is now to introduce a coordinate system based on the wall-surface. We thus write:
\[ z^\mu(\sigma^A, \xi) = x^\mu(\sigma^A) + \xi n^\mu(\sigma^A), \]
where \( x^\mu(\sigma^A) \) describes the wall-surface embedded in Minkowski space, \( \sigma^A \) \( (A = 0,1,2) \) are the 3 intrinsic coordinates on the wall-surface and \( n^\mu(\sigma^A) \) is the vector normal to the surface:
\[ \eta_{\mu\nu} n^\mu n^\nu = 1, \quad \eta_{\mu\nu} n^\mu x^\nu_A = 0. \]
The metric in the new coordinate system is:
\[ ds^2 = \eta_{\mu\nu} dz^\mu dz^\nu = g_{AB} d\sigma^A d\sigma^B + 2g_{A\xi} d\sigma^A d\xi + g_{\xi\xi} d\xi^2, \]
where:
\[ g_{AB} = \eta_{\mu\nu} x^\mu_A x^\nu_B + 2\xi \eta_{\mu\nu} n^\mu_A x^\nu_B + \xi^2 \eta_{\mu\nu} n^\mu_A n^\nu_B, \]
\[ g_{\xi\xi} = 1, \quad g_{A\xi} = g_{\xi A} = 0. \]
The action (1) now reads:
\[ S = \int \sqrt{-g} [ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)] d^3 \sigma d\xi, \]
where \( g_{\mu\nu} \) is given by (6). The corresponding equations of motion are:
\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - \frac{\partial V}{\partial \phi} = 0. \]
We will consider a domain wall solution \( \phi \) and expand the equations of motion (8) in powers of the thickness \( \epsilon \). We will assume that close to the wall the field \( \phi \) depends only on the transverse coordinate \( \xi \). The field itself therefore is expanded as:
\[ \phi = \phi_{(0)}(\xi) + \phi_{(1)}(\xi) + \phi_{(2)}(\xi) + ..., \]
with \( \phi_{(1)} \) of order \( \epsilon \), \( \phi_{(2)} \) of order \( \epsilon^2 \), etc. Furthermore we will assume that the derivative in the normal direction is one order in \( \epsilon \) lower than the derivatives in the tangential directions. This latter assumption was not included in the approach of Silveira and Maia [7], but it is in fact a necessary assumption, not only for the consistency of the formal calculations; if the expansion in
the thickness is to be meaningful at all, the wall-thickness is supposed to be small (compared to the curvature radius of the wall), which implies fast fall-off properties of $\phi$ [4,5,6]. The various objects are then assigned the following orders in $\epsilon$:

$$
\phi_0 \sim 1, \quad \phi_1 \sim \epsilon, \quad \phi_2 \sim \epsilon^2, \\
\xi \sim \epsilon, \quad \partial_A \sim \epsilon^n, \quad \partial_\xi \sim \epsilon^{n-1}, \\
\partial_\xi \phi_0, \partial_\xi \phi_1 \sim \epsilon^{-2}, \quad V(\phi_0) \sim \epsilon^{-2}.
$$

(10)

Using (6) we can expand $\sqrt{-g}$ in powers of $\xi$, which is then automatically an expansion in $\epsilon$ [7]:

$$
\sqrt{-g} = \sqrt{-G[1 + K_\xi \xi + \frac{1}{2}(K_{\xi\xi} - K_\xi) \xi^2 + \ldots]}.
$$

(11)

Here $G_{AB}$ is the induced metric on the wall-surface:

$$
G_{AB} = \eta_{\mu\nu} x^\mu_A x^\nu_B,
$$

(12)

$K_\xi$ is the mean curvature [8]:

$$
K_\xi = G^{AB} \eta_{\mu\nu} n^\mu_A x^\nu_B = n^\mu_A x^\nu_A,
$$

(13)

and $K_{\xi\xi}$ is the Gaussian curvature [8]:

$$
K_{\xi\xi} = G^{AC} G^{BD} \eta_{\mu\nu} n^\mu_A x^\nu_B \eta_{\sigma\tau} n^\sigma_C x^\tau_D = n^\mu_A n^\nu_A.
$$

(14)

At the 2 lowest orders in $\epsilon$ the expansion of the equations of motion (8) leads to:

$$
\epsilon^{-2} : \quad \partial_\xi^2 \phi_0 - \frac{\partial V}{\partial \phi} |_{(0)} = 0, \quad (15)
$$

$$
\epsilon^{-1} : \quad \partial_\xi^2 \phi_1 + K_\xi \partial_\xi \phi_0 - \frac{\partial^2 V}{\partial \phi^2} |_{(0)} \phi_1(1) = 0. \quad (16)
$$

If we make instead the expansion directly in the action (7) we find at lowest order:

$$
S_0 = \int \sqrt{-G}[\frac{1}{2} \partial_\xi \phi_0 \partial_\xi \phi_0 - V(\phi_0)] d^3 \sigma d\xi,
$$

(17)

which is consistent with (15). At the next order:

$$
S_1 = \int \sqrt{-G}[\partial_\xi \phi_1 \partial_\xi \phi_0 - \frac{\partial V}{\partial \phi} |_{(0)} \phi_1 + K_\xi(\frac{1}{2} \partial_\xi \phi_0 \partial_\xi \phi_0 - V(\phi_0)) \xi] d^3 \sigma d\xi.
$$

(18)
The first 2 terms in the square bracket vanish because of the lowest order
equation of motion (15), and also the $K_\eta$-term vanishes: The general solution
of (15) is a Weierstrass elliptic function [9] which however reduces to a
hyperbolic function when the appropriate boundary conditions are invoked:

$$\phi(0) = v\tanh(\sqrt{2}\lambda v\xi).$$

The $K_\eta$-term of (18) now vanishes after integration over $\xi$ because of anti-
symmetry. It is important to stress here that $\phi(0)$ of (19) is not the plane wall
solution usually discussed in the literature [2,3]; equation (19) only expresses
the local behaviour of the wall.

Going one order higher in $\epsilon$ we find:

$$S_2 = \int \sqrt{-G} \left[ -\frac{1}{2} \partial_{(1)} \partial_{(1)} \phi(1) - \frac{1}{2} \frac{\partial^2 V}{\partial^2 \phi} |_{(0)} \phi^2(1) + K_\eta \phi_{(1)} \partial_{(0)} \phi(0) \\
+ (K_\xi K_\xi - K_{\xi\xi})(-\frac{1}{2} \partial_{(0)} \phi_{(0)} \partial_{(0)} \phi(0) - V(\phi_{(0)}))\xi^2 | d^3 \sigma d\xi, \right]$$

where the lowest order equation of motion (15) has been used. Variation
with respect to $\phi(1)$ leads to (16) so in contrary to Silveira and Maia [7] our
actions $S_0$ and $S_2$ are consistent with the equations of motion. What we are
interested in however, is an effective action describing the dynamics of the
wall. Therefore, collecting $S_0$ and $S_2$ and integrating over $\xi$:

$$S = \int \sqrt{-G} [\mu_0 + \tilde{\mu}_2 K_\xi + \mu_2 (K_\xi K_\xi - K_{\xi\xi}) + ...] d^3 \sigma,$$

where:

$$\mu_0 = \int [-\frac{1}{2} \partial_{(0)} \phi(0) \partial_{(0)} \phi(0) - V(\phi(0))] d\xi,$$

$$\tilde{\mu}_2 = \frac{1}{2} \int \phi_{(1)} \partial_{(0)} \phi(0) d\xi,$$

$$\mu_2 = \int \xi^2 [-\frac{1}{2} \partial_{(0)} \phi_{(0)} \partial_{(0)} \phi(0) - V(\phi_{(0)})] d\xi.$$ 

Note that both $\tilde{\mu}_2$ and $\mu_2$ are 2 orders in $\epsilon$ higher than $\mu_0$ (because of (10)).
This is however not the final result. At lowest order in $\epsilon$ we find the equation
of motion for the wall:

$$\frac{1}{\sqrt{-G}} \partial_A (\sqrt{-G} G^{AB} \partial_B x'^{\mu}) = 0,$$
which implies $K_\xi = 0$ at lowest order. In this case equation (16) is easily solved for $\phi_{(1)}$, but the only solution with the appropriate boundary conditions is $\phi_{(1)} = 0$. This means that $\bar{\mu}_2 = 0$ so that the effective action for the wall becomes:

$$S = \int \sqrt{-G} [\mu_0 - \mu_2 R + ...] d^3 \sigma, \tag{26}$$

where, according to the Gauss-Codazzi equation [8], $R = -K_\xi K_\xi + K_\xi$ is the Ricci curvature of the wall. This result is in agreement with Gregory [6], but in disagreement with Silveira and Maia [7].

In conclusion we have calculated the lowest order correction to the Nambu action for a curved domain wall following the simple approach of Silveira and Maia [7]. We found however that when the thickness expansions are made consistently in the action and the equations of motion the extra terms found in Ref. 7 are absent, and therefore the original result of Gregory [6] is correct. We believe that this reflects that when the necessary assumptions for the method to work are made (i.e. $\phi$ depends only on the transverse coordinate and $\phi$ has suitably fast fall-off properties), then there is no such thing as a "locally non-plane solution" as introduced in Ref. 7, and that is why the result of Gregory is correct.
References


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