THE MASTER FIELD OF $\text{QCD}_2$ AND THE
'T HOOFT EQUATION

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Abstract

We rewrite the action for $\text{QCD}_2$ in the light cone gauge only in terms of a bilocal mesonic field. In this formalism the $1/N$ expansion can be done in a straightforward way by a saddle point technique that determines the master field to be identified with the vacuum expectation value of the bilocal field. Finally we show that the equation of motion for the fluctuations around the master field is identical with the 't Hooft meson equation.

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1 Introduction

The large $N$ colour expansion [1] is probably the most promising technique for arriving at an analytical understanding of long distance properties of nonabelian gauge theories. The major obstacle to the realization of this program is the fact that it has been impossible up to now to compute the sum over planar diagrams in a matrix model in a closed form unless one works in zero or one dimension [2].

Recently Kazakov and Migdal [3] have introduced a scalar matrix model coupled to a gauge field without its kinetic term that on one hand is exactly solvable [4] in the large $N$ expansion and on the other hand may have the property of inducing the gluon kinetic term. The reason of its solvability must be found in the fact that the functional integration over the angular degrees of freedom can be exactly performed and after that one is left with a "vector" model containing only the $N$ eigenvalues of the scalar matrix that can be solved by a standard saddle point technique determining the master field corresponding to the density of eigenvalues.

Although it is not clear if this model or a modified version of it will induce pure Yang-Mills theory it is, however, remarkable that one has been able to construct matrix models where the sum over planar diagrams can be explicitly performed in an arbitrary space-time dimension and not just in zero and one dimension as in Ref [2].

In this paper we consider quantum chromodynamics in two dimensions ($QCD_2$) and we show that in the light cone gauge it is possible to reformulate it completely in terms of a bilocal mesonic field. The master field corresponding to the vacuum expectation value of the bilocal mesonic field is then fixed in the limit of a large number of colours by a saddle point equation whose solution is equal to the fermion propagator constructed in the original paper by 't Hooft [5].

We consider then the quadratic term containing the fluctuation around the saddle point and we show that the equation of motion constructed from it gives exactly the integral equation found in Ref. [5] for the mesonic spectrum.

Although our results are not new we feel that the method used could be very useful for tackling more complicated problems that are still unsolved.
2 The master field and the spectrum of QCD$_2$.

We consider the action

$$S = \int d^2x \left\{ -\frac{1}{4g_0^2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \bar{\psi}^i (i \not\nabla - m_i^1 \not\tau - m_i^2 \gamma_5) \psi^i \right\} \tag{1}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i[A_{\mu}, A_{\nu}]$, $i \not\nabla_{AB} = \gamma_\mu (i \not\partial_{\mu} - A_{AB} T^a_{AB})$, $a, b = 1...N^2 - 1$ are the indices of the adjoint representation of the colour group, $A, B = 1...N$ run over the fermionic representation of the colour $SU(N)$ and $i, j,...$ are flavour indices. If we choose the gauge $A_{\mu}^a = 0$ and we normalize the trace over the fundamental representation to one, we can rewrite the previous action$^1$ as

$$S = \int d^2x \left\{ \frac{1}{2g_0^2} (\partial_x - A_{\mu}^a)^2 + i \sqrt{2} (\bar{\psi}_+^i \partial_x^+ \psi_+^i + \bar{\psi}_-^i \partial_x^- \psi_-^i) - (m_i^1 + m_i^2) \bar{\psi}_+^i \psi_+^i - (m_i^1 - m_i^2) \bar{\psi}_-^i \psi_-^i - A_{\mu}^a \sqrt{2} \bar{\psi}_+^i T^a_{AB} \psi_+^i \right\}$$

Integrating over $A_{\mu}^a$ we get

$$S = \int d^2x \left\{ i \sqrt{2} (\bar{\psi}_+^i \partial_x^+ \psi_+^i + \bar{\psi}_-^i \partial_x^- \psi_-^i) - (m_i^1 + m_i^2) \bar{\psi}_+^i \psi_+^i - (m_i^1 - m_i^2) \bar{\psi}_-^i \psi_-^i \right\}$$

$^1$Conventions.

$$x^\pm = x^0 \pm x^1 \quad A^\mu B_\mu = A^\mu B^\mu - A^\mu B_- + A_- B^\mu$$

$$\gamma_+ = \gamma_- = \left( \begin{array}{cc} 0 & \sqrt{2} \\ 0 & 0 \end{array} \right) \quad \gamma_+ = \gamma_- = \left( \begin{array}{cc} 0 & 0 \\ \sqrt{2} & 0 \end{array} \right) \quad \gamma_5 = -\gamma_0 \gamma_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \quad P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$\psi = \left( \begin{array}{c} \psi_+^i \\ \psi_-^i \end{array} \right) \quad \bar{\psi} = \left( \begin{array}{cc} \psi_+^i \\ \psi_-^i \end{array} \right) \quad \chi \bar{\psi} = -\frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{2} \psi P_{RX} \\ \bar{\psi}_{\gamma_+ \chi} \\ \sqrt{2} \psi P_{LX} \end{array} \right)$$
\[ + y_R^2 x_R \int d^2 x \, d^2 y \, G(x-y) \left\{ \bar{\psi}_+^{A_i}(x) \psi_+^{A_j}(y) \bar{\psi}_+^{B_j}(y) \psi_+^{B_i}(x) - \frac{R}{N} \bar{\psi}_+^{A_i}(x) \psi_+^{A_i}(x) \right\} \]

where we used \[ \sum_x T_{AB}^x T_{CD}^x = x_R (\delta_{BC} \delta_{AD} + \frac{R}{N} \delta_{AB} \delta_{CD}) \]
and where \[ G(x) = -\frac{1}{2} \delta(x^+) |x^-| = \int \frac{d^2 k}{(2\pi)^2} e^{ik \cdot x} \frac{1}{k^+}. \]

The interaction term suggests to introduce the field

\[ \rho_{ij}^-(x, y) = \sum_A \bar{\psi}_+^{A_j}(y) \gamma_+ \psi_+^{A_i}(x) \]

and its partners

\[ \rho_{ij}^+(x, y) = \sum_A \bar{\psi}_+^{A_j}(y) \gamma_+ \psi_+^{A_i}(x) \]

\[ \sigma_{ij}^+(x, y) = \sum_A \bar{\psi}_+^{A_j}(y) \gamma_5 \psi_+^{A_i}(x) \]

\[ \sigma_{ij}^-(x, y) = \sum_A \bar{\psi}_+^{A_j}(y) \gamma_5 \psi_+^{A_i}(x) \]

\[ \sigma_{R,L}^+(x, y) = \frac{\sigma_{ij}^+(x, y) + \sigma_{ij}^-(x, y)}{\sqrt{2}} \]

Now we want compute the jacobian of the transformation (see for instance

\[ ^* \text{This relation is only valid for the fundamental representation of SU(N), with } x_R = 1 \]
and \[ R = -1. \text{ We keep, however, } x_R \text{ and } R \text{ arbitrary in order to be able to trace back easily the origin of the various terms that we get in the final expression.} \]

Such a relation is essential for performing explicitly the large \( N \) expansion, as it will appear clearly in the following. Its key property is that it allows one to rewrite the product of the two colour currents as a product where the colour indices are saturated within each current (such currents are colour singlets under the residual gauge transformations for \( x^+ = y^+ \)).
from the $\tilde{\psi}, \psi$ to the $\rho, \sigma$.

$$J_{[\rho_+ \rho_-, \sigma_R, \sigma_L]} = \int [D\tilde{\psi}_+^A \, D\psi_+^A \, D\tilde{\psi}_-^A \, D\psi_-^A]$$

$$\Pi_{ij} \delta[\rho_+^i(x, y) - \sqrt{2} \sum_A \tilde{\psi}_+^A(y) \psi_+^A(x)] \Pi_{ij} \delta[\rho_-^j(x, y) - \sqrt{2} \sum_A \tilde{\psi}_-^A(y) \psi_-^A(x)]$$

$$\Pi_{ij} \delta[\sigma_R^i(x, y) - \sqrt{2} \sum_A \tilde{\psi}_-^A(y) \psi_+^A(x)] \Pi_{ij} \delta[\sigma_L^j(x, y) - \sqrt{2} \sum_A \tilde{\psi}_+^A(y) \psi_-^A(x)]$$

$$= \int [D\tilde{\psi}_+^A \, D\psi_+^A \, D\tilde{\psi}_-^A \, D\psi_-^A][D\alpha_+^i \, D\alpha_-^j \, D\beta_R^i \, D\beta_L^j]$$

$$e^{\alpha_+^i(y, x) \delta(x, y) \sum_A \tilde{\psi}_+^A(y) \psi_+^A(x)} + e^{\alpha_-^j(y, x) \delta(x, y) \sum_A \tilde{\psi}_-^A(y) \psi_-^A(x)}$$

$$e^{\beta_R^i(y, x) \delta(x, y) \sum_A \tilde{\psi}_-^A(y) \psi_+^A(x)} + e^{\beta_L^j(y, x) \delta(x, y) \sum_A \tilde{\psi}_+^A(y) \psi_-^A(x)}$$

where the sum over the flavour and space-time indices is understood.

If we introduce the matrices

$$M = ||M_{PQ}|| = \begin{pmatrix} \beta_R^i(y, x) & \alpha_+^i(y, x) \\ \alpha_-^j(y, x) & \beta_R^j(y, x) \end{pmatrix}$$

$$U = ||U_{PQ}|| = \begin{pmatrix} \sigma_R^i(y, x) & \rho_+^i(y, x) \\ \rho_-^j(y, x) & \sigma_R^j(y, x) \end{pmatrix}$$

$$\tilde{\Phi}_Q = (\tilde{\psi}_-^A(y) \, \tilde{\psi}_+^A(x) \, ) \quad \Phi_P = (\psi_+^A(y) \, \psi_-^A(x) \, )$$

where $P \equiv (x|\alpha)$ and $Q \equiv (y|\beta)$, we can rewrite the exponent of the integrand (6) as

$$J[U] = \int [d\tilde{\Phi}^A d\Phi^A] ||dM|| \exp [Tr(MU) - \sqrt{2} \Psi^A M \Psi^A]$$

$$\propto \int [||dM|| \exp [Tr(MU) + NTr \log M]]$$

where $N$ is the dimension of the fermionic representation and $Tr \equiv tr_x tr_i tr_\sigma$.

Evaluating this integral with the saddle point method we get

$$J[U] \propto \exp[-NTr \log U]$$

where we have neglected non leading contribution in $N$.

If we define the matrix

$$D = ||D_{PQ}|| = \begin{pmatrix} \sqrt{2}(m_1^i + m_2^i) \delta^i \delta^i(x - y) & i \delta^i \partial_x \delta^i(x - y) \\ i \delta^i \partial_x \delta^i(x - y) & -\frac{1}{\sqrt{2}}(m_1^i - m_2^i) \delta^i \delta^i(x - y) \end{pmatrix} =$$

$$e^{\sqrt{2}(m_1^i + m_2^i) \delta^i \delta^i(x - y)} e^{i \delta^i \partial_x \delta^i(x - y)} e^{-\frac{1}{\sqrt{2}}(m_1^i - m_2^i) \delta^i \delta^i(x - y)}$$

$$= \begin{pmatrix} -m_L^i \delta^i(x - y) & i \delta^i \partial_x \delta^i(x - y) \\ i \delta^i \partial_x \delta^i(x - y) & -m_R^i \delta^i(x - y) \end{pmatrix}$$

(10)
and we rescale the master field $U \rightarrow NU$, we can rewrite the effective action as
\[
\frac{1}{N} S_{\text{eff}} = Tr(DU + i \log U) + \frac{1}{2} g^2 \int d^2 x \, d^2 y \, G(x-y) \, U_{(x,1),(y,2)} \, U_{(y,1),(x,2)} \\
- \frac{1}{2N} g^2 R \int d^2 x \, d^2 y \, G(x-y) \, U_{(x,1),(y,2)} \, U_{(y,1),(x,2)}
\]
where $g^2 = g_0^2 x R / N$. Varying the effective action with respect to $U_{PQ}$, we get the equation for the master field, that reads to the leading order in $N$
\[
D_{PQ} + i (U^{-1})_{PQ} + g^2 \delta_{\alpha,2} \delta_{\beta,1} G(x-y) \, U_{(z,2),(y,1)} = 0
\]
Multiplying by $U$, we get immediately
\[
DU_{PQ} + i \|_{PQ} + g^2 \delta_{\alpha,2} \int d^2 z \, G(x-z) \, U_{(x,2),(z,1)} \, U_{(z,1),Q} = 0
\]
Writing explicitly these equations we find
\[
\begin{align*}
\{ i \partial_x + \rho^{ij}_{+}(x,y) - m^2_{ij} \sigma^{ij}_{L}(x,y) + g^2 \int d^2 z \, G(x-z) \rho^{ij}_{+}(x,z) \rho^{ij}_{+}(z,y) \\
+ i \delta^{ij} \delta^2(x-y) = 0 \\
i \sigma^{ij}_{L}(x,y) - m^2_{ij} \rho^{ij}_{+}(x,y) = 0
\end{align*}
\]
\[
\begin{align*}
\{ i \partial_x - \rho^{ij}_{+}(x,y) - m^2_{ij} \sigma^{ij}_{R}(x,y) + i \delta^{ij} \delta^2(x-y) = 0 \\
i \partial_x + \sigma^{ij}_{R}(x,y) - m^2_{ij} \rho^{ij}_{+}(x,y) + g^2 \, \int d^2 z \, G(x-z) \rho^{ij}_{+}(x,z) \sigma^{ij}_{R}(z,y) = 0
\end{align*}
\]
In particular if we eliminate $\sigma^{ij}_{L}(x,y)$ from the first equation using the second one, we get the fundamental equation
\[
\begin{align*}
\{ i \partial_x + \rho^{ij}_{+}(x,y) + i (m_{ij} m_{L})_{ij} \int d^2 z \, \delta(x^+ - z^+) \, \delta(x^- - z^-) \rho^{ij}_{+}(z,y) \\
+ g^2 \int d^2 z \, G(x-z) \rho^{ij}_{+}(x,z) \rho^{ij}_{+}(z,y) + i \delta^{ij} \delta^2(x-y) = 0
\end{align*}
\]
In order to solve this equation it is better to pass to momentum space. Since $\rho^{ij}_{+}(x,y) = \sqrt{2} < 0 | \sum_A \psi_{+}^{A+j}(y) \psi_{+}^{A+i}(x) | 0 >$ and the vacuum is translationally invariant, we need only one momentum for the Fourier transform of $\rho^{ij}_{+}(x,y)$. The previous equation (15) becomes
\[
\left[ -p_+ \delta^{ik} + \frac{(m_{R} m_{L})_{ik}}{p_-} + g^2 \int dk \, G(k) \rho^{ik}_{+}(p-k) \right] \rho^{kj}_{+}(p) + i \delta^{ij} = 0
\]
and it suggests to set

$$\rho_{\gamma}^{ij}(x,y) = \int \frac{d^2p}{(2\pi)^2} \mathcal{e}^{ip(x-y)} \rho_{\gamma}^{ij}(p) =$$

$$= \delta^{ij} \int \frac{d^2p}{(2\pi)^2} \mathcal{e}^{ip(x-y)} \frac{2i \nu_{-}}{2p_+p_- - 2(m_R m_L)^2 - p_- \Gamma(p) + i\epsilon}$$ (17)

With this substitution eq. (16) becomes eq. (10) of ref. [5]:

$$\Gamma(p) = \frac{4g^2}{(2\pi)^2} \int \frac{d^2k}{k^2} \frac{i(p_+ + k_-)}{2(p + k)^+(p + k)_- - 2(m_R m_L)^2 - (p + k)_- \Gamma(p + k) + i\epsilon}$$ (18)

The explicit solution yields

$$\Gamma(p) = \Gamma(p_-) = \frac{g^2}{\pi} \left( \frac{2g n(p_-)}{\lambda} - \frac{1}{p_-} \right)$$ (19)

where \( \lambda \) is an infra-red cutoff as discussed in ref. [5].

Inserting eq. (17) in eqs. (14), we get the Fourier transform of the master field

$$U^{ij}_0(p) = \frac{i \delta^{ij}}{2p_+p_- - 2(m_R m_L)^2 - p_- \Gamma(p) + i\epsilon} \left( \begin{array}{cc} -2m_R^i & 2p_- \\ 2p_+ - \Gamma(p) & -2m_L^j \end{array} \right)$$ (20)

with

Eq. (18) determines the master field of QCD2 that must be identified with the vacuum expectation value of the quark propagator.

Now we consider the mass spectrum of the theory, i.e. the fluctuations around the master field. To this purpose we write \( U = U_0 + \frac{1}{\sqrt{N}} \delta U \), and we consider the terms in the effective action that are \( O(1) \) in \( N \). They are given by the quadratic terms in the fluctuation \( \delta U \):

$$S^{(2)}_{\text{eff}} = -\frac{i}{2} Tr(U_0^{-1} \delta U U_0^{-1} \delta U) + \frac{1}{2} g^2 \int d^2x \ d^2y \ G(x-y) \ \delta U_{(x_1), (y_1)} \ \delta U_{(x_2), (y_2)}$$

$$- \frac{g^2 R}{2} \int d^2x \ d^2y \ G(x-y) U_{0(x_1), (x_2)} U_{0(y_1), (y_2)}$$ (21)

The term \(- \frac{g^2 R}{2} \int d^2x \ d^2y \ G(x-y) U_{0(x_1), (x_2)} U_{0(y_1), (y_2)} \) is a constant and will be neglected. If we compute \( \langle \delta U_{pq} \delta U_{RS} \rangle \) from eq. (21) we find eq. (7)
of ref. [7], that describes the full quark-antiquark scattering amplitude. The spectrum of the theory is determined by the equation of motion for the fields \( \delta U \) that is given by

\[
i \delta U_{a\beta}^{ij}(x, y) = g^2 \int d^2 u \ d^2 v \ U_{0, \alpha}^{ij}(x - u) \ G(u - v) \ \delta U_{12}^{ik}(u, v) \ U_{0, \gamma}^{kj}(v - y)
\]  

(22)

and that in Fourier space leads to\(^3\)

\[
\delta U_{a\beta}^{ij}(r, s) = -i \ g^2 T_{a\beta}^{(ij)}(s_+ + \frac{s_-}{2}, s_- - \frac{s_-}{2}) \ \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k_-^2} \ \delta U_{12}^{ij}(r, s - k)
\]  

(23)

(no sum over \(i\) and \(j\)), where

\[
\Delta^i(p) = 2p_+ p_- - 2(m_R m_L)^i - p_- \Gamma(p) + i\epsilon
\]  

(24)

and

\[
T_{a\beta}^{(ij)}(p_-, q_-) = \begin{pmatrix} 4p_- m_R^i & -4p_- q_- \\ -4m_R^i m_L^j & 4m_L^j q_- \end{pmatrix}
\]  

(25)

Following ’t Hooft [5], we integrate on both sides on \( s_+ \) and defining the gauge invariant field\(^4\)

\[
\varphi_{a\beta}^{ij}(r, s) = \int \frac{ds_+}{2\pi} \ \delta U_{a\beta}^{ij}(r, s)
\]  

(26)

we get choosing \( r_- > 0 \)

\[
\varphi_{a\beta}^{ij}(r, s_-) = g^2 T_{a\beta}^{(ij)}(s_+ + \frac{s_-}{2}, s_- - \frac{s_-}{2}) \ \left\{ \frac{M_i^2}{2|s_+ + \frac{s_-}{2}|} + \frac{M_j^2}{2|s_- - \frac{s_-}{2}|} + \frac{g^2}{\pi \lambda} - r_- \right\}^{-1} 
\]

\[
\theta(s_- + \frac{r_-}{2}) \theta(\frac{r_-}{2} - s_-) \ \int \frac{dk_-}{2\pi k_-^2} \ \varphi_{12}^{ij}(r, s_- - k_-)
\]  

(27)

\(^3\)We define

\[
\delta U(x, y) = \int \frac{d^2 r}{(2\pi)^2} \ \frac{d^2 s}{(2\pi)^2} e^{i r(x - x') + s(x - y)} \delta \hat{U}(r, s)
\]

In the following we suppress the tilde over the Fourier transformed fields.

\(^4\)Notice that this is equivalent to set \( x^+ = y^+ \) in \( \delta U(x, y) \), thus obtaining a gauge invariant object. If \( x^+ \neq y^+ \) then \( U(x, y) \) is not gauge invariant under the residual gauge transformations.
where
\[ M_i^2 = 2(m_R m_L)^i - \frac{g^2}{\pi} \] (28)

In the sector \((\alpha, \beta) = (2, 1)\) it yields the ’t Hooft starting equation (eq. (15) of ref. [5]) when one identifies the Fourier transform of \(\rho_{ij}(x, y)\) with \(\psi(p, r)\).

In the other sectors requiring the cancellation of the IR cutoff \(\lambda\), we get
\[ \varphi_{a\beta}^{ij}(r, s_-) = \frac{T_{a\beta}^{(ij)}(s_- + \frac{r}{2}, s_- - \frac{r}{2})}{4|s_- + \frac{r}{2}||s_- - \frac{r}{2}|} \varphi_{12}^{ij}(r, s_-) \] (29)

Performing the same straightforward manipulations as in ref. [5], one is led to an integral equation for the mass spectrum \((\varphi = \varphi_{12}\); we rescale \(s_- = r_-(x - \frac{1}{2})\) and define \(\mu^2 = 2r_+ r_-\):
\[ \mu^2 \varphi^{ij}(x) = \left[ \frac{M_i^2}{x} + \frac{M_j^2}{1-x} \right] \varphi^{ij}(x) - \frac{g^2}{\pi} P \int_0^1 \frac{\varphi^{ij}(y)}{(y - x)^2} dy \] (30)

that is the famous ’t Hooft equation, with a discrete spectrum of eigenvalues labelled by an integer \(n\) such that \(\mu_n^2 \approx g^2 \pi n, \quad n \to \infty\).

In the other sectors we get the same equation for the mass spectrum, but the mesonic fields change according to
\[ \varphi_{a\beta}^{ij}(x) = C_{a\beta}^{(ij)}(x) \varphi_{ij}(x) \] (31)
with
\[ C_{a\beta}^{(ij)}(x) = \begin{pmatrix} \frac{m_i^2}{(1-x)m_-} & 0 \\ \frac{m_i^2 m_-}{x(1-x)} & \frac{1}{x_m} \end{pmatrix} \] (32)

3 Conclusions

We have shown that it is possible to derive the mass spectrum of QCD\(_2\) in the large \(N\) limit with a functional approach. This approach allows one to generalize the previous results in the presence of chiral masses. The main point that guarantees the successful application of the large \(N\) techniques is the possibility of defining colourless fields (see eqs. (4) and (5)) that allow one to extract the \(N\) dependence both in the action and in the measure of
integration (the mesonic bilocal $U(x,y)$ is a global colour singlet, but it is
variant under the residual gauge transformation unless $x^+ = y^+$).

It would be interesting to discuss the $U(1)$ anomaly, the 3-meson vertex
and other "phenomenological" issues within this formalism.

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References

[4] A.A. Migdal, "Exact solution of induced lattice gauge theory at large