Thermal Green’s Functions
from Quantum Mechanical Path Integrals II:
Inclusion of Fermions

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Abstract

In a previous paper we have shown how, for bosonic fields, the generating
functional in both relativistic quantum field theory and thermal field theory
can be evaluated by use of a standard quantum mechanical path integral. In
this paper we extend this method to include fermionic fields. A particular
problem is posed by Green’s functions with external fermionic lines, where
the different boundary conditions of bosons and fermions in imaginary time
have to be accommodated within one path integral expression. The general
procedure is worked out in the example of scalar and spinor self-energies in a
simple model with a Yukawa coupling of a scalar to a Majorana spinor.

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I. INTRODUCTION

Loop corrections in thermal field theory can be calculated using a Feynman diagrammatic approach which is similar to the way in which analogous processes can be computed in ordinary quantum field theory [1]. In the Matsubara formulation of thermal field theory, the space is Euclidean rather than Minkowski with (anti-)periodic boundary conditions in imaginary time with extent $\beta = 1/(kT)$, where $T$ is the temperature. Thus the frequencies of all fields are either integer multiples of $2\pi/\beta$ (for Bose particles) or half-integer multiples (for Fermi particles). Apart from having sums over frequencies instead of integrals, the perturbation theory is the same as at zero temperature.

Recently an alternative to the Feynman diagrammatic approach has been proposed [2] in the context of operator regularization [4]. In the latter, the evaluation of Green’s functions is based on the computation of a matrix element of the form

$$M_{xy} = \langle x | e^{-iHt} | y \rangle$$

(1)

in powers of the background fields upon which the “Hamiltonian” $H$ depends. If this is done by using the “Schwinger expansion” [5]

$$e^{A+B} = \sum_{n=1}^{\infty} \int_0^\infty d\alpha_1 \cdots d\alpha_n \delta(1 - \alpha_1 - \alpha_2 \ldots - \alpha_n)$$

$$\times e^{\alpha_1 A} B e^{\alpha_2 A} \cdots B e^{\alpha_n A}$$

(2)

then the calculation is quite similar in form to what is encountered in the conventional Feynman diagram approach: both loop-momentum and Feynman parameter integrals have to be evaluated. The alternative approach based on quantum mechanical path integrals [2] computes Eq. (1) more directly, and turns out to obviate loop-momentum integrals completely.

This latter method has been applied to calculations in relativistic quantum field theoretical models involving scalars, spinors, and gauge fields to one and two loop order [2]. In a previous paper we have shown how this approach can be used to evaluate Green’s functions in thermal field theory in the purely bosonic case [6]. Again, no loop integrals over spatial momenta are encountered, whereas in place of the sum over Matsubara frequencies one has to perform a sum over winding numbers of paths with respect to the circular imaginary time, the two conceptually different sums being related by Jacobi’s imaginary transformation for theta functions. The representation as a sum over winding numbers was shown to also constitute an interesting alternative for the derivation of high-temperature expansions of Green’s functions.

In this paper we consider the generalization of this alternative approach to include fermions, which differ in their boundary conditions in imaginary time, being antiperiodic rather than periodic. In the case that only fermions are involved, it is not surprising to find that the only difference consists of having to evaluate an alternating sum over winding

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1A similar approach has been developed independently, at one loop order, in Ref. [3].
numbers of paths in imaginary time, which, through the Jacobi transformation, corresponds to the usual sum over half-integer Matsubara frequencies. However, it is not so obvious how the different boundary conditions have to be accounted for if both fermionic and bosonic fields appear in a single loop diagram, which is to be replaced by a certain quantum mechanical path integral. To illustrate how this is done, we calculate the two-point functions at one-loop order in a simple model in which a scalar field is coupled to a Majorana spinor by a Yukawa interaction. The generalization to more complicated (and more interesting) field theories is entirely straightforward, given the methods described in Ref. [2].

II. THERMAL GREEN’S FUNCTIONS INVOLVING FERMIONS

We consider a four-dimensional model for a scalar field $A$ and a Majorana spinor $\chi$ with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}Ap^2A - \frac{i}{2}\bar{\chi}\gamma^0\partial_0\chi - \frac{1}{2}\lambda\bar{\chi}A\chi \quad (p = -i\partial, \bar{\chi} = \chi^T C).$$

In the imaginary-time formulation of thermal field theory [1], the four-dimensional space is Euclidean, $x^0$ having the finite range $0 \leq x^0 \leq \beta$, with Bose (Fermi) fields subject to periodic (anti-periodic) boundary conditions.

We shall consider scalar and fermionic Green’s functions in turn, which — in the diction of the background field method [7] — means that we have to introduce background fields for $A$ and $\chi$, respectively, while the other field appears only as a quantum field.

A. Scalar Self-Energy

With the spinor field $\chi$ being a purely quantum field and $A$ having a background contribution $B$, the generating functional for scalar Green’s functions at one-loop order reads

$$Z^{(1)}[B] = \text{Sdet}^{-\frac{1}{2}}(\begin{pmatrix} -p^2 & 0 \\ 0 & -(i\dot{\phi} + \lambda B) \end{pmatrix})$$

$$= \text{const.} \times \text{Det}^{\frac{1}{2}}(i\dot{\phi} + \lambda B),$$

where “Sdet” denotes the functional superdeterminant [8] with the appropriate boundary conditions, which, for the final, ordinary functional determinant in Eq. (4) are antiperiodic ones, because the latter is the fermionic part of the former.

Introducing the matrix operator

$$H = -(i\dot{\phi} + \lambda B)^2 = p^2 - i\lambda(\dot{\phi}B + B\dot{\phi}) - \lambda^2B^2$$

$$= (p - i\lambda\gamma B)^2 - 3\lambda^2 B^2,$$

we regulate Eq. 4 by the $\zeta$ function [4,9]

$$\zeta(s) = \text{Tr} \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{-Ht}$$

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according to \( \ln Z^{(1)}[B] = \frac{1}{4} \zeta'(0) \). We therefore have to evaluate

\[
\text{Tr} \exp(-Ht) = \int d^4x \: d^4y \: \delta(x - y) \langle x | \text{tr} \exp(-Ht) | y \rangle. \tag{7}
\]

While at one-loop order only the diagonal matrix elements enter, at higher loop orders the general matrix elements are the basic building blocks of the generating functional, if defined by the method of operator regularization [4].

In order to obtain the one-loop contribution to the (thermal) scalar self-energy, only the term bilinear in the background field \( B \) needs to be extracted from Eq. (7).

1. Evaluation by Schwinger expansion

We first evaluate Eq. (7) by employing the Schwinger expansion, Eq. (2), which reproduces the expressions derived from the more customary Feynman rules together with Feynman parametrization of denominators. This will serve as a reference for the results to be derived from the use of quantum mechanical path integrals.

To second order in \( B \), Eq. (2) yields

\[
\text{Tr} \exp(-Ht) \bigg|_{B^2} = \lambda^2 \text{Tr} \left[ t^2 B^2 e^{-\alpha t^2} - \frac{t^2}{2} \int_0^t d\tau e^{-(1-\alpha)\tau^2} (\dot{B} \dot{B} + B \ddot{B}) e^{-\alpha \tau^2 (\dot{B} + B \ddot{B})} \right]. \tag{8}
\]

Evaluating further by insertion momentum states \( |p\rangle \), we have, remembering the antiperiodic boundary conditions for the functional determinant in Eq. (4) and hence for the functional trace in the subsequent expressions,

\[
\langle x | p \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{\beta}} \exp \left[ \frac{p}{\beta} + \frac{2\pi}{\beta} \left( n_p + \frac{1}{2} \right) x_4 \right], \tag{9a}
\]

and

\[
\langle p | B | q \rangle = \frac{1}{(2\pi)^{3/2} \sqrt{\beta}} B(p - q, \frac{2\pi}{\beta} (n_p - n_q)), \tag{9b}
\]

with \( n_p \) and \( n_q \) integers. After appropriate shifts of the integration and summation variables, and then using \( \int e^{-\alpha t} \beta q / (2\pi)^3 = (4\pi t)^{-3/2} \), we end up with

\[
\text{Tr} \exp(-Ht) \bigg|_{B^2} = -4\lambda^2 \int \frac{d^3p}{(4\pi t)^{3/2} \beta} \sum_{n_p n_q} \left\{ -te^{-\left(\frac{2\pi}{\beta} \left( n_p + \frac{1}{2} \right) \right)^2 t} \right. \\
+ t^2 \int_0^t d\tau e^{-\alpha (1-\alpha) \tau^2] \left[ 3 \alpha (1 - 2\alpha) p^2 + \left( \frac{2\pi}{\beta} \right)^2 (n_p (n_q + \frac{1}{2})^2 + 2(n_q + \frac{1}{2})^2) \right] B(-p, -\frac{2\pi}{\beta} n_p). \tag{10}
\]

In accordance with the Feynman rules in the Matsubara formalism, we encounter imaginary frequencies for the scalar background fields which are integer multiples of \( 2\pi / \beta \), and there is a further summation over half-integer \( (n_q + \frac{1}{2}) \) multiples corresponding to the frequencies of the internal fermion lines.
2. Evaluation by a quantum mechanical path integral

We now compute the matrix element appearing in Eq. (7) by using the representation through the standard quantum mechanical path integral [10]

\[
\langle x | \exp \left\{ -\frac{1}{2} (p - A)^2 + V \right\} | y \rangle = \mathbb{P} \int Dq \exp \int_0^t d\tau \left[ -\frac{1}{2} \dot{q}^2(\tau) - \dot{q}(\tau) \cdot A(q(\tau)) - V(q(\tau)) \right],
\]

with \( q(0) = y \) and \( q(t) = x \). \( \mathbb{P} \) stands for path ordering in the case that \( A \) and \( V \) are matrix valued.

Comparing with Eq. (5) and rescaling \( t \to t/2 \) for convenience, we identify \( A = i \lambda \gamma B \) and \( V = \frac{3}{2} \lambda^2 B^2 \). Consequently we have

\[
\text{Tr} \exp(-Ht) = \text{Tr} \mathbb{P} \int Dq \exp \int_0^t d\tau \left[ -\frac{1}{2} \dot{q}^2(\tau) - \dot{q}(\tau) \cdot \gamma B - \frac{3}{2} \lambda^2 B^2 \right].
\]

To second order in \( B \) this becomes

\[
\left. \text{Tr} \exp(-Ht) \right|_{B^2} = \lambda^2 \text{Tr} \int Dq \ e^{-\frac{1}{2} \int_0^t d\tau \dot{q}^2(\tau)} \left\{ -\frac{3}{2} \int_0^t d\tau_1 B^2(q(\tau_1)) + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dot{q}(\tau_1) \cdot \gamma B(q(\tau_1)) \dot{q}(\tau_2) \cdot \gamma B(q(\tau_2)) \right\},
\]

where in the last term the integration over \( \tau_2 \) is restricted by the necessity of path ordering of the \( \gamma \) matrices.\(^2\)

If we now take \( B \) to be a plane wave field,

\[
B(q(\tau_1)) = \frac{bi_1}{(2\pi)^{3/2} \sqrt{\beta}} e^{i k_1 \cdot q(\tau_1)} \equiv \frac{bi_1}{(2\pi)^{3/2} \sqrt{\beta}} e^{i [k_1 q(\tau_1) + (\overline{k}) \cdot \gamma q(\tau_1)]},
\]

and assign labels 1 and 2 to the \( \gamma \) matrices contracted with \( \dot{q}(\tau_1) \) and \( \dot{q}(\tau_2) \), respectively, then we can write

\[
\left. \text{Tr} \exp(-Ht) \right|_{B^2} = \frac{\lambda^2 b_i b_j}{(2\pi)^{3/2} \beta} \text{Tr} \int Dq \ e^{-\frac{1}{2} \int_0^t d\tau \dot{q}^2(\tau)} \left\{ -\frac{3}{2} \int_0^t d\tau_1 e^{i \int_0^\tau d\tau \dot{q}(\tau)(k_1 + k_2) \delta(\tau - \tau_1)} + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{i \int_0^\tau d\tau \dot{q}(\tau)(i k_1 \delta(\tau - \tau_1) + i k_2 \delta(\tau - \tau_2) - \gamma_1 \delta(\tau - \tau_1) - \gamma_2 \delta(\tau - \tau_2))} \right\},
\]

provided we keep only the linear terms in \( \gamma_1 \) and \( \gamma_2 \).

The path integral is now of the form

\(^2\)An alternative to explicit path ordering of \( \gamma \) matrices, which moreover removes their explicit appearance altogether, could be given by the introduction of fermionic path integration variables in addition to \( q \) [11]. However, this has no bearing on the problem of boundary conditions in imaginary time, which is our main concern here.
\[ P_{(x,y)}^{\beta}[\Gamma] = \int Dq \exp \int_0^t d\tau \left[ -\frac{1}{2} \dot{q}(\tau) + q(\tau) \cdot \Gamma(\tau) \right], \]  

(16)

where the superscript \( \beta \) reminds of the fact that \( q_4 \) lives on a circle with circumference \( \beta \), i.e. \( q_4 \equiv q_4 + n\beta \), with \( n \) integer. This topological constraint prevents us from a direct evaluation by a shift of variable

\[ \tilde{q}(\tau) = q(\tau) - \int_0^\tau d\tau' G(\tau, \tau') \Gamma(\tau') \]

(17)
as in zero-temperature field theory [2]. In the latter,

\[ G(\tau, \tau') = \frac{1}{2} |\tau - \tau'| - \frac{1}{2} (\tau + \tau') + \frac{\tau \tau'}{t} \]

(18)

would render the path integral purely Gaussian.

The problem is that of a path integral quantization of a particle on a circle, and the general solution has been given in Ref. [12] in terms of a superposition of unconstrained path integrals,

\[ P_{(x,y)}^{\beta}[\Gamma] = \sum_{n=-\infty}^{\infty} e^{i\delta_n} P_{(x,x_{4+n\beta})}^{\infty} \]

(19)

which corresponds to summing over all paths from \( y \) to \( x \) which have different winding number in the fourth variable.

In the purely bosonic case treated in Ref. [6], the phases \( \delta_n \) could be set to zero. The present path integral, however, is supposed to give the matrix elements for taking a functional trace with antiperiodic boundary conditions in imaginary time. We therefore have to require that \( P_{(x,y)}^{\beta} \) picks up a phase equal to \(-1\) when making one turn in the circular imaginary time. This determines \( \delta_n = n\pi \), and \( P^{\beta}_{(x,y)} \) is given by an alternating sum over winding numbers.

On the left hand side of Eq. (19), we can indeed perform the shift of Eq. (17), yielding

\[ P_{(x,y)}^{\infty}[\Gamma] = \frac{1}{(2\pi t)^2} \exp \left[ -\frac{(x - y)^2}{2t} + \frac{1}{t} \int_0^t d\tau [x\tau + y(t - \tau)] \cdot \Gamma(\tau) \right. \]

\[ \left. -\frac{1}{2} \int_0^t d\tau d\tau' G(\tau, \tau') \Gamma(\tau) \cdot \Gamma(\tau') \right] \]

(20)

with \( G \) given by Eq. (18).

Evaluating the above path integral in this manner and computing the trace by integrating over \( x \) after setting \( x = y \), which has the effect of equating \( k_1 = -k_2 \equiv k \), we obtain

\[ \text{Tr} \left. \exp(-Ht) \right|_{B^2} = \frac{2\lambda^2 b_1 b_2}{\beta(2\pi t)^2} \sum_{n} (-1)^n e^{-n^2 \beta^2 / 2t} \left\{ \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-k^2 \alpha(1-\alpha) - i\beta k_4 \alpha} \left[ -\frac{4}{t} - k^2(\alpha - \frac{1}{2})^2 + (\frac{n\beta}{t} - ik_4(\alpha - \frac{1}{2}))^2 \right] \right\}, \]

(21)

where \( \alpha \equiv (\tau_1 - \tau_2)/t \). Because \( k_4 = \frac{2\pi \beta}{n_k} \) with integer \( n_k \), we find that the integrand \( f(\alpha) \) in Eq. (21) is symmetric under \( \alpha \leftrightarrow 1-\alpha \) together with \( n \leftrightarrow -n \), which allows us to simplify
\[ \int_0^t d\tau_1 \int_0^\tau_2 d\tau_2 f(\alpha) \rightarrow t^2 \int_0^1 d\alpha (1 - \alpha)f(\alpha) = \frac{t^2}{2} \int_0^1 d\alpha f(\alpha). \] (22)

Eq. (21) involves an infinite sum different from the one encountered in the result of the more conventional Schwinger expansion, Eq. (10). Introducing Jacobi’s θ functions [13]

\[ \theta_2(z|\tau) = \sum_{n=-\infty}^{\infty} \exp \left( i\pi [\tau(n + \frac{1}{2})^2 + 2zn + \frac{1}{2}] \right), \] (23a)
\[ \theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} \exp \left( i\pi [\tau n^2 + 2zn] \right), \] (23b)
\[ \theta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \left( i\pi [\tau n^2 + 2zn] \right), \] (23c)

the path integral result Eq. (21) can be expressed in terms of \( \theta_4 \) and its derivatives with respect to its two arguments, whereas the conventional result Eq. (10) requires \( \theta_2 \). Jacobi’s so-called imaginary transformation

\[ \theta_{3+i}(z|\tau) = (-i\tau)^{\frac{1}{2}} e^{-iz^2/\tau} \theta_{3-i}(\frac{z}{\tau} - \frac{1}{\tau}), \] (24)

where \( i = -1, 0, \) or 1, relates \( \theta_4 \) to \( \theta_2 \), and indeed, after some algebra, establishes the equivalence of the two results.

B. Spinor Self-Energy

We shall now consider the case of Green’s functions with external fermionic lines, which receive radiative corrections from both (quantum) scalar and fermionic degrees of freedom at the same time. By taking now \( A \) to be a purely quantum field and \( \chi \) having a background part \( \eta \), the generating functional for fermionic Green’s functions at one-loop order reads

\[ Z^{(1)}[\eta] = S\text{det}^{-\frac{1}{2}} Y[\eta] = S\text{det}^{-\frac{1}{2}} \left( \begin{array}{cc} \lambda^2 & \lambda \bar{\eta} \\ \lambda \eta & i\phi \end{array} \right). \] (25)

Dropping an irrelevant constant over-all factor, we may rescale the supermatrix \( Y \) in Eq. (25) with the field-independent supermatrix

\[ X = \begin{pmatrix} 1 & 0 \\ 0 & -i\phi \end{pmatrix}, \] (26)

and regulate the superdeterminant of \( H = XY \) using the \( \zeta \) function

\[ \zeta(s) = \text{Str} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-Ht} \] (27)

according to \( \ln Z^{(1)}[\eta] = -\frac{1}{2} \zeta'(0) \). We thus are confronted with having to evaluate

\[ \text{Str} \exp(-Ht) = \int d^4x \int d^4y \delta(x - y) \langle x | \text{str} \exp \left[ -\left( \begin{array}{cc} p^2 & \lambda \bar{\eta} \\ -i\lambda \eta & p^2 \end{array} \right) t \right] | y \rangle, \] (28)

where “Str” denotes the supertrace including a functional trace with the appropriate boundary conditions and “str” the supertrace in the purely algebraic sense.

In order to obtain the one-loop contribution to the (thermal) spinor self-energy, we now have to extract the term bilinear in \( \eta \) from Eq. (28).
1. Evaluation by Schwinger expansion

Again, we first lay out the derivation of the thermal spinor self-energy employing the Schwinger expansion, Eq. (2), by which we formally have

\[
\langle x| \text{str} \exp(-\mathbf{H}t)|y \rangle \bigg|_{\eta} = -i\lambda^2 t^2 \int_0^1 d\alpha \langle x| e^{-(1-\alpha)p^2 t} \tilde{\eta} e^{-\alpha p^2 t} \eta |y \rangle.
\]

(29)

In order to correctly take into account the (anti-)periodic boundary conditions of the supermatrix operator $\mathbf{H}$ when computing the functional trace in Eq. (28) in momentum-space, one has to remember that $\tilde{\eta}$ appears in its Bose-Fermi block, and $\eta$ in the Fermi-Bose one. Therefore,

\[
\langle p|\eta|q \rangle = \frac{1}{\sqrt{(2\pi)^4}} \tilde{\eta}(p - q, \frac{2\pi}{\beta} n_p - \frac{2\pi}{\beta} (n_q + \frac{1}{2})),
\]

(30)

with $n_p$ and $n_q$ integers, which leaves us with

\[
\text{Str} \exp(-\mathbf{H}t) \bigg|_{\eta} = -i\lambda^2 t^2 \int_0^1 d\alpha \int \frac{d^3 p \, d^3 q}{(2\pi)^3 \beta} \sum_{n_p n_q} e^{-(1-\alpha)p^2 + (2\pi n_p/\beta)^2 + \alpha(q^2 + (2\pi n_q + 1/2)^2)/\beta)} t \\
\times \tilde{\eta}(p - q, \frac{2\pi}{\beta} (n_p - n_q - \frac{1}{2})) \left[ \frac{2\pi}{\beta} (n_q + \frac{1}{2}) \gamma_4 \right] \eta(q - p, \frac{2\pi}{\beta} (n_q + \frac{1}{2} - n_p)).
\]

(31)

With the shifts $p \rightarrow p + q$, $n_p \rightarrow n_p + n_q$, followed by $q \rightarrow q - \alpha p$, and then using the integral $\int e^{-\alpha^2 t} d^3 q/(2\pi)^3 = (4\pi t)^{-3/2}$, Eq. (31) reduces to

\[
\text{Str} \exp(-\mathbf{H}t) \bigg|_{\eta} = -i\lambda^2 t^2 \int_0^1 d\alpha \int \frac{d^3 p}{(4\pi t)^{3/2} \beta} \sum_{n_p n_q} e^{-(1-\alpha)p^2 + (2\pi n_p/\beta)^2 + \alpha(n_q + 1/2)^2)/\beta)} t \\
\times \tilde{\eta}(p, \frac{2\pi}{\beta} (n_p - \frac{1}{2})) \left[ -(1-\alpha) p + \frac{2\pi}{\beta} (n_q + \frac{1}{2}) \gamma_4 \right] \eta(-p, \frac{2\pi}{\beta} (n_p - \frac{1}{2})).
\]

(32)

2. Evaluation by a quantum mechanical path integral

In order to represent Eq. (28) by the standard quantum mechanical path integral Eq. (11), we rewrite

\[
\mathbf{H} = \begin{pmatrix} p^2 & \lambda \tilde{\eta} \\ -i\lambda \tilde{\eta} & \lambda \eta \end{pmatrix} = \begin{pmatrix} p^2_\mu & 0 \\ -\frac{\lambda}{2} \gamma_\mu \eta \end{pmatrix} + \begin{pmatrix} 0 & \lambda \tilde{\eta} \\ -\frac{\lambda}{2} \gamma_\mu \eta, \nu \end{pmatrix},
\]

(33)

leading to

\[
\langle x| \exp(-\mathbf{H}t/2)|y \rangle = \mathcal{P} \int Dq \exp \int_0^t d\tau \left[ -\frac{1}{2} \dot{q}^2 + i \dot{q}_\mu \left( \frac{\lambda}{2} \gamma_\mu \eta \right) \right] - \left( \frac{\lambda}{4} \gamma_\mu \eta, \nu \right) - \left( \frac{\lambda}{4} \gamma_\mu \eta, \nu \right)
\]

(34)
where we have again rescaled \( t \to t/2 \) in Eq. (28) in order to employ the standard quantum mechanical result Eq. (11) directly.

To second order in \( \eta, \tilde{\eta} \) we find from Eq. (34) that

\[
\langle x | \text{str} \exp(-Ht/2) | y \rangle \bigg|_{\eta, \tilde{\eta}} = \int Dq e^{-\frac{1}{2} \int_0^t d\tau \dot{q}^2(\tau)} \text{str} \left[ \frac{1}{2!} \prod_{i=1}^2 \int_0^t d\tau_i \left( -\frac{i}{2} \dot{q}_i \eta_i + \frac{i}{4} \gamma_{\mu i} \eta_i \nu \right) \right],
\]

(35)

where \( q_i \equiv q(\tau_i) \) and \( \eta_i \equiv \eta(q(\tau_i)) \).

Computing the (algebraic) supertrace gives two contributions, the trace over the bosonic part minus the one over the fermionic part, which are almost identical apart from the sign, were it not for the path ordering prescription \( \Pi \), which is required because we are dealing with a matrix valued “Hamiltonian” \( H \). The bosonic part of the supertrace reads

\[
\langle x | \text{tr}_B \exp(-Ht) | y \rangle \bigg|_{\eta, \tilde{\eta}} = \int Dq e^{-\frac{1}{2} \int_0^t d\tau \dot{q}^2(\tau)} \frac{\lambda^2}{4} \int_0^t d\tau_1 \int_0^t d\tau_2 \tilde{\eta}_1 \left( \tilde{q}_2 \eta_2 - \frac{1}{2} \gamma_{\nu \nu} \eta_2 \right),
\]

(36)

whereas the fermionic part equals minus this expression where the indices 1 and 2 are interchanged except in the integration symbols.

Without the topological constraint of a circular imaginary time \( q_4 \), the bosonic and the fermionic trace could be simply combined, resulting in a symmetrized integration domain in \( \tau_1 \) and \( \tau_2 \). However, in the preceding section we have seen that the path integral corresponding to a fermionic trace when expressed in terms of an unconstrained path integral, acquires a nontrivial phase for odd winding numbers in imaginary time, whereas the bosonic part does not. The two pieces have therefore to be treated as two different path integral expressions in the thermal case.

Keeping this difference in mind, we can transform Eq. (36) and its fermionic counterpart into a Gaussian path integral by taking the background wave functions to be plane wave fields whose frequencies are half integral multiples of \( 2\pi / \beta \),

\[
\eta(q(\tau_i)) = \frac{u_i}{(2\pi)^{3/2} \beta^{1/2}} e^{ik_\sigma q(\tau_i)} \equiv \frac{u_i}{(2\pi)^{3/2} \beta^{1/2}} e^{i[k_\sigma q(\tau_i) + (\tilde{q}_\sigma q(\tau_i) + \frac{1}{2} u_i)]}.
\]

(37)

This reduces Eq. (36) to

\[
\langle x | \text{tr}_B \exp(-Ht) | y \rangle \bigg|_{\eta, \tilde{\eta}} = \frac{\lambda^2}{4(2\pi)^3 \beta} \int_0^t d\tau_1 \int_0^t d\tau_2 \tilde{\eta}_1 \int Dq \left\{ e^{\int_0^t d\tau \left[ -\frac{\lambda^2}{2} \dot{q}^2(\tau) + q(\tau) (-i \dot{q}_3 \delta(\tau - \tau_1) + \dot{q}_2 \delta(\tau - \tau_2) - \gamma^\dagger(\tau - \tau_1)) \right]} \right\}_{\text{lin}},
\]

(38)

where the subscript “lin.” indicates that we have exponentiated \( \gamma \) matrices with the provision that only first order terms in \( \gamma \) are to be kept.
Eq. (38) is now of the form of Eq. (16) with linear sources \( \Gamma = -ik_1 \delta(\tau - \tau_1) + ik_2 \delta(\tau - \tau_2) \) in the first term, and \( \Gamma' = \Gamma - \gamma \delta(\tau - \tau_2) \) in the second one. The fermionic part of the supertrace gives rise to a similar expression with source terms in which the labels 1 and 2 are interchanged. The two path integral contributions can now be evaluated by Eq. (20) and Eq. (19), taking into account the alternating signs picked up by the one corresponding to the fermionic part of the supertrace. Performing also the integration over \( x \) after setting \( x = y \), which enforces momentum conservation so that \( k_1 = k_2 = k \), we arrive at the result

\[
\text{Str} \exp(-Ht/2) = \frac{\lambda^2}{4(2\pi t)^2} \sum_n \int_0^1 d\sigma_1 \int_0^{\sigma_2} d\sigma_2 e^{-\frac{x^2}{4t} - \frac{k^2}{2t}(\alpha-\alpha')^2} 
\]

\[
\tilde{u}(-k) \left\{ e^{-i\eta/\beta_k} \left[ e^{i\eta/\beta_k} - \frac{i\beta}{2} \right] \right\}_{\text{lin.}} u(k),
\]

with \( \alpha \equiv \sigma_1 - \sigma_2 \) and \( \sigma_i \equiv \tau_i/t \).

Because \( k_4 = 2\tau(n_k + \frac{1}{2}) \), we find that the integrand of Eq. (39) is symmetric under \( \alpha \leftrightarrow 1-\alpha \), since then \( e^{-i\eta/\beta_k} \leftrightarrow (-1)^n e^{i\eta/\beta_k} \). Therefore we can replace \( \int_0^1 d\sigma_1 \int_0^{\sigma_2} d\sigma_2 \rightarrow \frac{1}{2} \int_0^1 d\alpha \). The same substitution, when applied to only part of the integrand, allows us to write Eq. (39) either as a straight or an alternating sum over winding numbers \( n \), which can be written in terms of Jacobi \( \theta \) functions, Eq. (23), as

\[
\text{Str} \exp(-Ht/2) = \frac{\lambda^2}{4(2\pi t)^2} \int_0^1 d\alpha e^{-\frac{x^2}{4t} + \frac{i\eta}{2t}(1-\alpha)} \tilde{u}(-k) \left\{ \theta_3 \left( -\frac{\beta k_3}{2\pi} + \frac{\beta \gamma_1}{2\pi it} \right) e^{-ik(1-\alpha)} \right\}_{\text{lin.}} u(k),
\]

\[
\text{Str} \exp(-Ht/2) = \frac{\lambda^2}{4(2\pi t)^2} \int_0^1 d\alpha e^{-\frac{x^2}{4t} + \frac{i\eta}{2t}(1-\alpha)} \tilde{u}(-k) \left\{ \theta_4 \left( -\frac{\beta k_3}{2\pi} - \frac{\beta \gamma_1}{2\pi it} \right) e^{-ik\alpha} \right\}_{\text{lin.}} u(k),
\]

where because only the terms linear in \( \gamma \) matrices are to be kept we can treat them as if they were complex numbers. Each of these comparatively compact expressions can easily be shown to be equivalent to the standard result Eq. (32) by means of the Jacobi transformation, Eq. (24). The two representations of Eq. (40) are thus related to sums over integral and half-integral Matsumotoa frequencies, respectively, corresponding to the two possible ways of reading the standard expression primarily as a sum over the internal bosonic or fermionic frequencies.

**III. SUMMARY**

We have shown that the alternative method of evaluation of Green’s functions based on the standard formulæ for quantum mechanical path integrals can be adapted to the case of thermal Green’s functions involving fermions, provided that a separation is made between purely bosonic and purely fermionic contributions. This separation is a natural one when the generating functional is formulated in terms of supermatrices. The resulting quantum mechanical path integrals are topologically constrained, the one associated with fermionic contributions picking up a minus sign for increasing the winding number by one. The evaluation of Green’s functions along these lines leads to Feynman-parametrized results, the
Feynman parameters being related to the points of insertion of external lines, but no loop-momentum integral is encountered. In place of the usual sum over Matsubara frequencies, however, a sum over winding numbers emerges. The latter has been shown in Ref. [6] to be more directly amenable to a high-temperature expansion. The generalization of the results presented here for a simple four-dimensional model of scalars with Yukawa coupling to fermions to more complicated ones can performed with the techniques developed in Ref. [2].

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REFERENCES

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