Abelian current algebra and the Virasoro algebra on the lattice

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Abstract. We describe how a natural lattice analogue of the abelian current algebra combined with free discrete time dynamics gives rise to the lattice Virasoro algebra and corresponding hierarchy of conservation laws.

Introduction

Two modern directions in 1+1-dimensional quantum field theory – we mean conformal field theory (CFT) and quantum inverse scattering method – that developed at first quite independently have eventually proved to be closely related. The following observation \cite{1}, for instance, provides a clear evidence of their connection. The central place in CFT belongs to the Virasoro algebra whose generators $L_m$ are the Fourier modes of the improved stress-energy tensor having before quantization the Poisson bracket

$$\{l(x), l(y)\} = 2\gamma(l(x) + l(y))\delta(x - y) + \gamma\delta'''(x - y)$$.

This very bracket is well known in the inverse scattering method as one of the brackets in the KdV hierarchy of Poisson structures. Combined with the similar observation concerning $W$-algebras it provides a good reason to think that integrable models are associated somehow with conformally invariant ones or, if you wish, are their deformations.

While this idea is quite transparent for the classical equations of motion and Poisson brackets, the corresponding quantum picture is rather patchy. On one hand, an approach proposed in \cite{2} allows in principle to construct quantum commuting conservation laws for the Virasoro algebra. On the other hand, neither \cite{2} nor the quantum inverse scattering method deliver reasonably effective construction of that laws. We hope that recent expansion of the latter method to the models in discrete space–time \cite{3,4,5} will possibly throw new light on the nature of quantum integrability in general and the structure of conservation laws in particular.

At the same time, since the discrete space–time picture happened to be quite effective for the sine–Gordon model it would be natural to try to discretize also CFT. This idea is not completely new. Our approach here differs from the previous efforts \cite{1,6,7,8,9} in discretizing not only the spatial but also the time variable.

\textsuperscript{\dagger}Submitted for publication in Phys. Lett. B
\textsuperscript{\dagger}Supported by the Russian Academy of Sciences and the Academy of Finland
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We will consider the simplest model where the above mentioned stress-energy tensor emerges as the Miura transformation
\[ t = p^2 + p' \]
of the periodic free field (abelian current) \( p(x) \) having the Poisson bracket
\[ \{p(x), p(y)\} = \gamma \delta'(x - y) \]
and the equation of motion
\[ p_1(x, t) + p_x(x, t) = 0 \]
\[ p(x, 0) = p(x) . \]

In section 1 we introduce the corresponding lattice model. In section 2 we examine its evolution operator. In section 3 we find lattice counterparts of the Miura transformation and the improved stress-energy tensor. In section 4 we describe a family of commuting operators which contains this evolution operator. In section 5 we introduce a suitable exchange algebra and modify accordingly the evolution operator.

1 Current algebra on the lattice and the free field model in the discrete space-time

Consider the operators \( w_n, n = 1, \ldots, 2N \) with the commutation relations
\[ w_{n-1}w_n = q^2w_nw_{n-1} , \quad n = 2, 3, \ldots, 2N \]
\[ w_{2N}w_1 = q^2w_1w_{2N} \]
\[ w_m w_n = w_n w_m \quad if \quad 1 < |m - n| < 2N - 1 , \]
where \( q \) is a complex constant. Adopting the notations
\[ w_{n+2kN} = w_n , \quad n = 1, 2, \ldots, 2N \quad k \in \mathbb{Z} \]
and introducing the periodic Kroneker symbol
\[ \delta_n \equiv \sum_{k \in \mathbb{Z}} \delta_{n,2kN} \]
one can collect all commutation relations into a single formula
\[ w_m w_n = q^{2(\delta_{m-n+1} - \delta_{m-n-1})}w_n w_m , \quad m, n \in \mathbb{Z} . \]

In the classical limit \( q = e^{i\pi}, \hbar \to 0 \) this leads to the Poisson bracket
\[ \{w_m, w_n\} = -2\gamma(\delta_{m-n+1} - \delta_{m-n-1})w_n w_m . \]

In the continuous limit, setting \( x = n\Delta, N\Delta = 2\pi, w_n \sim e^{2\Delta \varphi(x)} \) and \( \Delta \to 0 \) one gets precisely the periodic free field mentioned in the introduction.

Unfortunately, the most desirable case \(|q| = 1\) is rather difficult and we prefer to start with less realistic but simpler one with \( q \) inside the unit circle. We will also temporarily adopt one more simplifying assumption. Up to the final section the two central elements of the algebra of \( w \)’s
\[ C_1 = \prod w_{odd} \]
\[ C_2 = \prod w_{even} \]
will always assumed be equal

\[ C_1 = C_2 \]

We have completed the description of the “phase space” and are now to choose the “equations of motion”. As the spatial variable is already discretized, it is only natural to make the time to be discrete as well. Then the inevitable replacement for the continuous free equation \( p_t = -p_x \) will be just

\[ w_n(t + 1) = w_{n-1}(t) , \quad t \in \mathbb{Z} \]

\[ w_n(0) = w_n . \]

As such a dynamics preserves commutation relations between \( w \)'s, there should exist an “evolution” operator \( U \) such that

\[ w_n(t + 1) = U^{-1}w_n(t)U . \]

Comparing the last two formulas one sees that \( U \) is the shift operator

\[ w_n U = U w_{n-1} \]

which will be the subject of our interest in the rest of this paper.

## 2 Shift operator and the braid group

In this section we will show that the operator

\[ U = h_1 h_2 \ldots h_{2N-1} \]

(notice that the product is one site shorter than the lattice) is indeed the shift operator if

\[ h_n = \theta(w_n) \]

\[ \theta(w) = \sum_{k \in \mathbb{Z}} q^{k^2} w^k = \theta_0(q, w) , \]

where \( \theta_0(q, w) \) is the theta–function.

To verify this statement let us note first that the theta–function satisfies the functional equation

\[ \frac{\theta(q w)}{\theta(q^{-1} w)} = \frac{1}{w} . \]

This allows to obtain commutation relations for the operators \( w_n \) and \( h_n \), for instance,

\[ w_n h_{n-1} = w_n \theta(w_{n-1}) = \theta(q^{-2} w_{n-1}) w_n = q^{-1} \theta(w_{n-1}) w_{n-1} w_n = q^{-1} h_{n-1} w_{n-1} w_n . \]

Now one easily obtains the crucial commutation relation

\[ w_n h_{n-1} h_n = h_{n-1} h_n w_{n-1} \]

which for \( n = 2, 3, \ldots, 2N - 1 \) yields immediately

\[ w_n U = \ldots w_n h_{n-1} h_n \ldots = h_{n-1} h_n w_{n-1} \ldots = U w_{n-1} . \]

Due to the existence of the two central elements in the algebra of \( w \)'s the remaining two relations are determined by those \( 2N - 2 \) which we have already got:

\[ w_1 U = \frac{C_1}{w_3 w_5 \ldots w_{2N-1}} U = \frac{C_1}{w_2 w_4 \ldots w_{2N-2}} = \frac{C_1}{C_2} w_{2N} \]
$$w_{2N}U = U \frac{C_2}{C_1} w_{2N-1} .$$

Since $C_1 = C_2$, this completes the proof that $U$ is the shift operator, i.e. that for any $n$ from 1 to $2N$

$$w_n U = U w_{n-1} .$$

Several remarks are in order now. First, the commutation relation between $w$ and the pair of $h$'s is valid not only for bare $w$ but for any function of it and for instance for $\theta(w)$ itself. This means that the operators $h_n$ generate the braid group of the $A_{2N-1}^{(1)}$ type:

$$h_n h_{n-1} h_n = h_{n-1} h_n h_{n-1} , \quad n = 2, 3, \ldots, 2N$$

$$h_1 h_{2N} h_1 = h_{2N} h_1 h_{2N} .$$

Second, the proof and the definition of the operator $U$ may create the impression that $2N$-th site of the lattice is somehow distinguished. That is not so just because

$$U = h_n h_{n+1} \ldots h_{n+2N-1}$$

for any $n \in \mathbb{Z}$.

Third, the possibility to express the shift operator as a product of local factors certainly reflects the fact that the corresponding continuous dynamics $p_t = -p_t$ is generated by the local hamiltonian

$$H = \frac{1}{2\gamma} \int_0^{2\tau} p^2(x)dx .$$

And however different the theta–function and the density of this hamiltonian $p^2$ may look, the well known formula

$$\theta_0(q, q^{2\tau}) = q^{-n^2} \theta_0(q, 1)$$

shows that there is no contradiction between them.

### 3 Miura transformation and the Virasoro algebra

In the continuous theory the Miura transformation $t = p^2 + p'$ turned the free field into the improved stress–energy tensor or, in the hamiltonian language, into the “improved” hamiltonian density

$$H = \frac{1}{2\gamma} \int_0^{2\tau} p^2(x)dx = \frac{1}{2\gamma} \int_0^{2\tau} (p^2(x) + p'(x))dx = \frac{1}{2\gamma} \int_0^{2\tau} t(x)dx .$$

We will now describe a similar effect on the lattice. More precisely, we shall show that

$$U = h_1 h_2 \ldots h_{2N-1}$$

$$\approx s(w_1) (s(w_1^{-1})s(w_2) s(w_2^{-1}) \ldots s(w_{2N-1}) s(w_{2N-1}^{-1}))$$

$$= (s(w_1^{-1}) s(w_2) s(w_2^{-1}) s(w_3) \ldots s(w_{2N-1}^{-1})) s(w_{2N})$$

$$= s(w_1^{-1} + w_2 + qw_1^{-1} w_2) s(w_2^{-1} + w_3 + qw_2^{-1} w_3) \ldots s(w_{2N-1}^{-1} + w_{2N} + qw_{2N-1}^{-1} w_{2N}) ,$$

where the function $s(w)$ has the form

$$s(w) = 1 + \sum_{k=1}^{\infty} \frac{q^{s(k-1)}}{(q^{-1} - q)(q^{-2} - q^2) \ldots (q^{-k} - q^k)} w^k$$

and the sign $\approx$ means the equality up to a constant factor.

One clearly sees that for this chain of transformations to be possible the function $s(w)$ should have two properties:
\begin{enumerate}
\item \( s(w)s(w^{-1}) \approx \theta(w) \); \\
\item \( s(v)s(u) = s(v + u + qvu) \) if \( u, v \) is a Weyl pair, i.e. \( uv = q^2vu \). \\
It indeed possesses both ones and the list of its notable features can be continued: \\
\item \( s(u)s(v) = s(u + v) \) if again \( uv = q^2vu \).
\end{enumerate}

All three equalities may be verified by the straightforward but elaborate calculation using the explicit formulas for the both functions involved. We prefer however to do without all that combinatorics. The technics which we are going to outline below will be based on the observation that the function \( s(w) \) satisfies the functional equation

\[
\frac{s(qw)}{s(q^{-1}w)} = \frac{1}{1 + w} .
\]

It is now easy to see why the first of the three equalities is true. Indeed, both sides of it satisfy the same functional equation

\[
\frac{s(qw)s((qw)^{-1})}{s(q^{-1}w)s((q^{-1}w)^{-1})} = \frac{1}{w} = \frac{\theta(qw)}{\theta(q^{-1}w)}
\]

and thus the functions \( s(w)s(w^{-1}) \) and \( \theta(w) \) coincide (up to the constant \( s^2(1)/\theta(1) \)) at the points \( w = q^{2k}, k \in \mathbb{Z} \). This in fact means that they coincide everywhere. To make such a conclusion one certainly has to examine first the analitical properties of that functions. We leave this in many ways important question to be discussed in detail elsewhere.

To verify the second equality let us first note that the operators

\[
\mathcal{U} = u + quv^{-1}
\]

\[
\mathcal{V} = u + v + qvu
\]

form just another Weyl pair

\[
\mathcal{U}\mathcal{V} = q^2\mathcal{V}\mathcal{U}
\]

which is “dual” to the original one in the sense that the inverse map

\[
u^{-1} = \mathcal{U}^{-1} + \mathcal{V}^{-1} + q\mathcal{V}^{-1}\mathcal{U}^{-1}
\]

\[
v^{-1} = \mathcal{V}^{-1} + q\mathcal{U}^{-1}\mathcal{V}^{-1}
\]

is similar to the direct one. The r.h.s. \( s(\mathcal{V}) \) of the equality which we are verifying satisfies the commutation relations

\[
[s(\mathcal{V}), \mathcal{V}] = 0
\]

\[
\mathcal{U}s(\mathcal{V})\mathcal{U}^{-1} = \frac{s(\mathcal{V})}{1 + q\mathcal{V}} .
\]

One may check that the same is true for the l.h.s. \( s(\mathcal{V})s(\mathcal{U}) \). This means (at least on the formal level adopted here) that \( s(\mathcal{V}) \) and \( s(\mathcal{V})s(\mathcal{U}) \) coincide up to a constant which is equal to \( 1 \) just because \( s(0) = 1 \).

The third property of \( s \) we actually do not need and will not verify (this could be done in roughly the same manner as for the second one). Note that it is the standard functional relation for the so called \( q \)-exponential.

So, we have at last established that

\[
\mathcal{U} = s_1s_2\ldots s_{2N-1}
\]

where

\[
s_n = s(w_n^{-1} + w_{n+1} + qw_n^{-1}w_{n+1}) .
\]
The appearance of the combinations of the operators \( w_n \) involved here is hardly surprising. As was gradually understood during the investigation of the Liouville equation on the lattice \([6, 7, 9, 8, 10]\) the discrete Miura transformation is likely to have the form
\[
t_n = \frac{1}{4}(1 + w_n^{-1} + w_{n+1}^{-1} + q w_n^{-1} w_{n+1}) .
\]
The “stress–energy” tensor \( t_n \) defined like this has the correct continuous classical limit
\[
t_n \sim 1 - \Delta^2(x)
\]
as well as closed commutation relations and some other promising properties \([9, 8, 10]\). That gives the right to say that \( t \)'s generate the lattice Virasoro algebra. It was not however easy to get real benefits of this construction because on the lattice neither a suitable Fourier transform nor the “diffeomorphisms” are available yet and thus one cannot obtain lattice counterparts of the generators \( L_m \) of the Virasoro algebra directly from \( t \)'s. We have just made the first step in this direction and established in what sense the “generator” of the only evident lattice “diffeomorphism” is the zero’th “Fourier mode” of the discrete stress–energy tensor. At the same time we seem to get the key to the understanding of the whole Fourier transform relevant for our discrete free field theory.

4 Yang–Baxter equation

We now change the direction and turn from the “conformal” side of our discrete–discrete free field model to its link with the quantum inverse scattering method.

Another name for the braid group commutation relation (see sect.2) is the Yang–Baxter equation. In this context it reads:
\[
\theta(v)\theta(u)\theta(v) = \theta(u)\theta(v)\theta(u) \quad if \quad uv = q^2 vu .
\]
The argument of the theta–function is not to be mistaken here for a spectral parameter. Genuine Yang–Baxter equation with (multiplicative) spectral parameter should have the form:
\[
r(\lambda, v)r(\lambda u, u)r(\mu, v) = r(\mu, u)r(\lambda u, v)r(\lambda, u) \quad if \quad uv = q^2 vu ,
\]
the “\( R \)-matrix” \( r(\cdot, \cdot) \) here being a function of two variables first of which is a complex number and the second one is an operator. A suitable for us solution to this equation has been found in \([11]\):
\[
r(\lambda, w) = 1 + \sum_{k=1}^{\infty} \frac{(1 - \lambda)(q - \lambda q^{-1}) \ldots (q^{k-1} - \lambda q^{-k+1})}{(q^{-1} - \lambda q)(q^{-2} - \lambda q^2) \ldots (q^{-k} - \lambda q^k)} (w^k + w^{-k}) .
\]
It is easy to notice that
\[
r(0, w) = \theta(w)
\]
and thus expect the function \( r(\lambda, w) \) to satisfy some functional equation similar to the one satisfied by the theta–function but with the deformed r.h.s.. Such an equation was proposed in \([9]\):
\[
\frac{r(\lambda, q w)}{r(\lambda, q^{-1} w)} = \frac{1 + \lambda w}{\lambda + w} .
\]
The natural order of things is probably opposite to the historical one presented above. Both the explicit form of the function \( r \) and the fact that it satisfies the Yang–Baxter equation are actually the corollaries of this functional equation. It may be also useful to know that with respect to its another argument the function \( r(\lambda, w) \) satisfies the equation
\[
\frac{r(q \lambda, w)}{r(q^{-1} \lambda, w)} = (1 + \lambda w)(1 + \lambda w^{-1}) .
\]
Let us forget for a while about our finite periodic lattice and consider instead an infinite one with a suitable rapidly decreasing boundary conditions for the operators $w_n, n \in \mathbb{Z}$. Multiplying the R-matrices $r(\lambda, w_n)$ along the lattice

$$U(\lambda) = \ldots r(\lambda, w_{n-1}) r(\lambda, w_n) r(\lambda, w_{n+1}) \ldots$$

we obtain the commuting family of operators

$$[U(\lambda), U(\mu)] = 0$$

containing the shift operator

$$U = U(0) \ .$$

Following the guidelines worked out in the previous section we factorize the function $r(\lambda, w)$

$$r(\lambda, w) \approx s(\lambda, w) s(\lambda, w^{-1})$$

$$s(\lambda, w) = \frac{s(w)}{s(\lambda w)} = 1 + \sum_{k=1}^{\infty} \frac{(1 - \lambda)(q - \lambda q^{-1}) \ldots (q^{k-1} - \lambda q^{-k+1})}{(q^1 - q)(q^2 - q^2) \ldots (q^k - q^k)} w^k$$

$$\frac{s(\lambda, qw)}{s(\lambda, q^{-1}w)} = \frac{1 + \lambda w}{1 + w} , \quad s(q\lambda, w) = s(q^{-1}\lambda, w) = 1 + \lambda w \ ,$$

then discover that the multiplication rule still holds

$$s(\lambda, v) s(\lambda, u) = s(\lambda, v + u + qvu) \ , \quad uv = q^2vu \ ,$$

then rearrange accordingly the decomposition of $U(\lambda)$

$$U(\lambda) \approx \ldots s_{n-1}(\lambda) s_n(\lambda) s_{n+1}(\lambda) \ldots$$

$$s_n(\lambda) = s(\lambda, w_n^{-1} + w_{n+1} + qw_n^{-1}w_{n+1})$$

and eventually find out that $U(\lambda)$ is the quantum lattice counterpart of the generating function of the KdV conservation laws hierarchy.

Unfortunately, the periodic case is harder to control and the natural and basically correct scheme outlined above is not easy to apply there (we will address this problem in the forthcoming paper).

Nevertheless, the emergence of a fully fledged $R$-matrix in the free lattice theory indicates that the link between conformal invariance and complete integrability has not disappeared on the way to the lattice. It also comes as no surprise that the same $R$-matrix happens to be heavily involved in the construction of the evolution operator and the conservation laws of the quantized Hirota (discrete–discrete sine–Gordon) model [5]. All that just firms our belief that good lattice models inherit some really essential properties of their continuous counterparts being at the same time much simpler to live with and thus providing a good starting point for the investigation of 1+1-dimensional quantum field theories.

5 Exchange algebra

Let us come back to the more general case when $C_1 \neq C_2$ (see sect.1). Notice first of all that the shift operator which we discussed up to now would permute $C_1$ and $C_2$:

$$C_1 U = U C_2 \ , \quad C_2 U = U C_1 \ .$$

It only means that if they are not equal this operator does not exist. What does exist in that case is the operator shifting by two sites:

$$w_n V = V w_{n-2} \ .$$

Its local decomposition has certainly the same density as that of the operator $U$ but is two times longer:

$$V = h_2 h_3 \ldots h_{2N} h_1 h_2 \ldots h_{2N-1} = h_n h_{n+1} \ldots h_{n+4N-2} \ .$$
All that we did with the operator $U$ can be now almost literally repeated for $W$. At the same time we gain the possibility to realize the algebra of the operators $w_n$ as a subalgebra of the so called exchange algebra.

To make it clear let us first turn to the classical continuous case where that algebra has the form

$$
\{\psi(x), \psi(y)\} = -\frac{1}{2\gamma} \epsilon(x - y) \psi(x) \psi(y) \quad , \quad 0 < x, y < 2\pi
$$

where $\epsilon(x)$ is the sign function. This is certainly only a half of the definition. It remains to specify the boundary condition. Let us introduce the quasimomentum $C$

$$
\{C, \psi(x)\} = -\gamma C \psi(x)
$$

and continue the field $\psi(x)$ outside the fundamental domain by means of the quasiperiodicity condition

$$
\psi(x + 2\pi) = C \psi(x)
$$

As a result we get the Poisson bracket on the whole real axis

$$
\{\psi(x), \psi(y)\} = -\frac{1}{2\gamma} \epsilon_{2\pi}(x - y) \psi(x) \psi(y) \quad ,
$$

where the function $\epsilon_{2\pi}(x)$ is a kind of quasiperiodic sign function:

$$
\epsilon_{2\pi}(x) = 2k + 1 \quad if \quad 2\pi k < x < 2\pi(k + 1) \quad , \quad k \in Z
$$

Taking the logarithmic derivative of $\psi(x)$ we obtain the familiar periodic free field

$$
p = \frac{\psi'}{\psi} \quad , \quad p(x + 2\pi) = p(x)
$$

$$
\{p(x), p(y)\} = \gamma \delta'(x - y)
$$

The quasimomentum $C$ becomes the central element

$$
C = \exp(\int_0^{2\pi} p(x) dx)
$$

$$
\{C, p(x)\} = 0
$$

It is needless to say that the real reason for the introduction of the exchange algebra is its remarkable connection with the Virasoro algebra via the Sturm–Liouville equation:

$$
-\psi'' + t\psi = 0
$$

where $t$ is the stress-energy tensor $t = p^2 + p'$ (see Introduction).

Analogous construction on the lattice is a bit more sophisticated. We introduce $2N + 2$ operators

$$
\psi_1, \psi_2, \ldots, \psi_{2N} \quad and \quad C_1, C_2
$$

with the commutation relations

$$
\psi_{2m} \psi_{2n+1} = q \psi_{2n+1} \psi_{2m} \quad if \quad 1 < 2m < 2n + 1 < 2N
$$

$$
\psi_{2m+1} \psi_{2n} = q \psi_{2n} \psi_{2m+1} \quad if \quad 1 \leq 2m + 1 < 2n \leq 2N
$$

$$
\psi_{2n+1} C_1 = q^2 C_1 \psi_{2n+1}
$$

$$
\psi_{2n} C_2 = q^2 C_2 \psi_{2n}
$$

$$
[\psi_{2m+1}, \psi_{2n+1}] = [\psi_{2m}, \psi_{2n}] = [\psi_{2m+1}, C_2] = [\psi_{2m}, C_1] = [C_1, C_2] = 0
$$

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Defining then \( \psi_n \) for the values of \( n \) outside the 1 to 2\( N \) range by

\[
\psi_{2n+1+2N} = C_2 \psi_{2n+1}, \quad \psi_{2n+2N} = C_1 \psi_{2n},
\]
we collect all commutation relations into a single formula

\[
\psi_m \psi_n = q^{-\epsilon_{2n}} \psi_n \psi_m,
\]
where

\[
\epsilon_{2n} = 0 \quad \text{and} \quad \epsilon_{2n+1} = 2k + 1 \quad \text{if} \quad 2kN \leq 2n + 1 < 2(k + 1)N.
\]
Taking the “derivatives” of \( \psi \)'s

\[
w_n = \frac{\psi_{n+1}}{\psi_{n-1}}
\]
we reproduce precisely the lattice current algebra introduced in sect.1.

“In between” of these exchange algebra of \( \psi \)'s and current algebra of \( w \)'s producing the models with \( N + 1 \) and \( N - 1 \) degrees of freedom lies a couple of useful algebras both giving models with \( N \) degrees of freedom. The first one is generated by the operators

\[
\phi_1, \phi_2, \ldots, \phi_{2N} \quad \text{and} \quad C,
\]
where

\[
\phi_n = \psi_n \psi_{n+1}, \quad C = C_1 C_2
\]
and thus

\[
\phi_{n+2N} = C \phi_n.
\]
The commutation relations are

\[
\phi_m \phi_n = q^{-\epsilon_{2n}} \phi_n \phi_m,
\]
where

\[
\epsilon_{2kN} = 2k, \quad \epsilon_{2n} = 2k + 1 \quad \text{if} \quad 2kN < n < 2(k + 1)N.
\]
and there is a central element

\[
c = \frac{C_1}{C_2} = \frac{\prod_{n=1}^{2N} \phi_{2n+1}^{-1} \phi_{2n}}{\prod_{n=1}^{2N} \phi_{2n}^{-1} \phi_{2n+1}}.
\]
The second subalgebra is generated by the operators

\[
\psi_2, \psi_4, \ldots, \psi_{2N} \quad \text{and} \quad w_2, w_4, \ldots, w_{2N}
\]
with extremely simple commutation relations

\[
[\psi_{2m}, w_{2n}] = 0 \quad \text{if} \quad m \neq n
\]
and

\[
[\psi_{2m}, \psi_{2n}] = [w_{2m}, w_{2n}] = 0.
\]
First of them looks quite natural and is not bad at all on the infinite lattice. The second one consists just of the independent Weyl pairs and proves to be rather helpful in some applications [5].

To cover all emerging versions of the lattice free field model we need to find the shift operator which is good for the largest of these algebras:

\[
\psi_n W = W \psi_{n-2} \quad \text{for any} \quad n \in \mathbb{Z}.
\]
The previous shift operator \( V \) is certainly almost correct and just one more factor turns it into \( W \):

\[
W = \frac{V}{\theta\left(\frac{C_2}{C_1}\right)} = \frac{\theta\left(\frac{\psi_{2n+1}}{\psi_2}\right) \theta\left(\frac{\psi_{2n+2}}{\psi_3} \ldots \theta\left(\frac{\psi_{2n+2N-1}}{\psi_{2n+2N-2}} \right)}{\theta\left(\frac{\psi_2}{\psi_1}\right)}.
\]
Conclusion

The $U(1)$ current algebra obviously corresponds to the case when $|q| = 1$ and the operators $w_n$ are unitary. The series which defined the functions $\theta(w), r(\lambda, w), s(\lambda, w)$ do not make much sense when $|q| = 1$, especially when $q$ is not a root of unity. Fortunately, when $q$ is a root of unity

$$q^i = 1,$$

and the operators $w_n$ are normalized by the same condition

$$w_n^i = 1$$

the equations

$$\frac{\theta(q w)}{\theta(q^{-1} w)} = \frac{1}{w}$$

$$\frac{r(\lambda, q w)}{r(\lambda, q^{-1} w)} = \frac{1 + \lambda w}{\lambda + w}$$

still can be solved:

$$\theta(w) = \sum_{k=0}^{i-1} q^{k^2} w^k$$

$$r(\lambda, w) = 1 + \sum_{k=1}^{i-1} \frac{(1 - \lambda)(q - \lambda q^{-1}) \ldots (q^{k-1} - \lambda q^{k+1})}{(q^{-1} - \lambda q)(q^{-2} - \lambda q^2) \ldots (q^{-k} - \lambda q^k)} w^k .$$

Unfortunately, the same cannot be said about the equation

$$\frac{s(\lambda, q w)}{s(\lambda, q^{-1} w)} = \frac{1 + \lambda w}{1 + w}$$

which has the solution only at the points

$$\lambda = q^{2k} .$$

The case thus deserves separate investigation which is now in progress.

One could compile quite long list of other questions left here without answers. How to turn formal operator functions and calculations into something real, what is the precise definition of the commuting family $U(1)$ in the periodic case, how to extract from there not only the shift operator but something like a hierarchy of local conservation laws, what are the "Fourier modes" of the discrete stress-energy and "diffeomorphisms" of the discrete circle? We will address some of these questions in forthcoming papers. However, we still did not mention what is probably the most interesting problem in this context. The current algebra considered in the paper is abelian. It is natural to ask whether one can apply similar technics to a model where the phase space is a nonabelian current algebra and the equations of motion are again free, i.e. to the chiral WZNW model. A good example of the lattice $SL(N)$ current algebra has been constructed in [12, 13]. Since in the $SL(2)$ case we know a quite convenient free field parametrization of the lattice currents, the corresponding discrete–discrete WZNW model can be treated more or less in the same way as the abelian free field [14]. It would be however more interesting to develop an essentially nonabelian formalism not appealing to free fields. We can’t help feeling that some of the constructions in this paper are more general than they look and thus their nonabelian generalisation is quite possible.

Acknowledgement

We are grateful to B. Feigin, A. Kirillov, E. Sklyanin and M. Semenov-Tian-Shansky for discussions. L. Faddeev is grateful to professors M. Flato and P. Mitter for their hospitality during his visits to Université de Bourgogne and Université Paris VI where this work has been completed.
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