ASPECTS OF EXACTLY SOLVABLE
QUANTUM-CORRECTED 2D DILATON-GRAVITY THEORIES

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ABSTRACT

After reviewing the basic aspects of the exactly solvable quantum-corrected dilaton-gravity theories in two dimensions, we discuss a (subjective) selection of other aspects: a) supersymmetric extensions, b) canonical formalism, ADM-mass, and the functional integral measure, and c) a positive energy theorem and its application to the ADM- and Bondi-masses.

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1. Basic Aspects

1.1. Introduction

Our starting point is the classical action for dilaton gravity in two dimensions as written by Callan, Giddings, Harvey and Strominger (CGHS) [1]

\[ S_{\text{cl}} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right]. \]  (1.1)

Here \( \phi \) is the dilaton field with \( G \equiv e^\phi \) playing the role of the gravitational coupling constant, \( \lambda^2 \) is referred to as cosmological constant and the \( f_i \) are \( N \) massless conformally coupled matter fields. This action admits classical (non-radiating) static black hole solutions

\[
\begin{align*}
    ds^2 &= -\frac{m}{\lambda} - \lambda^2 x^+ x^- = -d\tau^2 + d\sigma^2 \\
    e^{-2\phi} &= \frac{m}{\lambda} - \lambda^2 x^+ x^- = \frac{m}{\lambda} + e^{2\lambda\sigma}.
\end{align*}
\]  (1.2)

The \( x^\pm \) are Kruskal type coordinates, related to the Schwarzschild type coordinates \( \sigma, \tau \) by \( \lambda x^\pm = \pm e^{\lambda\sigma} \), \( \sigma^\pm = \tau \pm \sigma \). The metric in the latter coordinates is asymptotically Minkowskian as \( \sigma \to \infty \). The parameter \( m \) is the black hole mass, and the \( m = 0 \) solution where \( \phi = -\lambda \sigma \) is called the linear dilaton vacuum (LDV).

The goal then is to quantize the theory described by this action \( S_{\text{cl}} \). If the number of matter fields is different from 24, \( N \neq 24 \), one has to include various contributions to the conformal anomaly. Thus we add

\[ S_{\text{anom}} = -\frac{\kappa}{8\pi} \int d^2 x \sqrt{-g} R \frac{1}{\sqrt{2}} R \]  (1.3)

We will keep \( \kappa \) as a parameter to be determined later on. Note that \( S_{\text{anom}} \) is \( \mathcal{O}(e^{2\phi}) \equiv \mathcal{O}(G^2) \) with respect to the gravitational part of \( S_{\text{cl}} \) and may be thought of as the one-loop contribution of the matter fields. We will refer to \( S_{\text{cl}} + S_{\text{anom}} \) as \( S_{\text{CGHS}} \) (with \( \kappa_{\text{CGHS}} = \frac{N}{12} \)).

\[ \dagger \] A possible term \( \mu^2 \int \sqrt{-g} \) is supposed to be fine-tuned to vanish.
1.2. Conformal invariance and transformation to free fields

**Conformal gauge**

Let us first choose conformal gauge, \(g_{++} = g_{--} = 0\), \(g_{+-} = -\frac{1}{2}e^{2\rho}\). Then

\[
S_{\text{cl}} = \frac{1}{\pi} \int d^2\sigma \left[ e^{-2\phi} \left( 2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho} \right) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right],
\]

\[
S_{\text{anom}} = -\frac{\kappa}{\pi} \int d^2\sigma \partial_+ \rho \partial_- \rho.
\]

The equations of motion derived in conformal gauge must be supplemented by the \(g_{++}\) and \(g_{--}\) equations of motion as constraints:

\[
T_{++} = T_{--} = 0,
\]

\[
T_{\pm\pm} = e^{-2\phi} \left( 4\partial_\pm \phi \partial_\pm \rho - 2\partial_\pm^2 \phi \right) - \kappa \left( \partial_\pm \rho \partial_\pm \rho - \partial_\pm^2 \rho \right)
+ \frac{1}{2} \sum_{i=1}^{N} \left( \partial_\pm f_i \right)^2.
\]

(1.5)

Note that for \(\kappa > 0\) the kinetic term of \(S_{\text{cl}} + S_{\text{anom}}\) is degenerate,

\[
\det \begin{pmatrix} -4e^{-2\phi} & 2e^{-2\phi} \\ 2e^{-2\phi} & -\kappa \end{pmatrix} = 4e^{-2\phi} (\kappa - e^{-2\phi}) = 0 \quad \text{at} \quad e^{-2\phi} = e^{-2\phi_0} \equiv \kappa.
\]

We expect something singular to happen when \(\phi = \phi_0\).

**Conformal invariance**

Since we are dealing with a theory of gravity, we started with a diffeomorphism invariant theory. Then we fixed conformal gauge, leaving as symmetries the subgroup of conformal diffeomorphisms \(\sigma^+ \rightarrow f^+(\sigma^+), \sigma^- \rightarrow f^-(\sigma^-)\). Quantization should preserve these conformal symmetries. In particular, we need to ensure that the resulting theory is a \(\epsilon_{\text{tot}} = 0\) conformal theory. The latter is a necessary condition that relies only on the short distance properties of the quantum theory. They may be inferred even though the full quantum theory might not be known. For \(\kappa > 0\), a non-trivial complication is the presence of the critical value of \(\phi\) where we expect a singularity. Typically \(\phi = \phi_0\) on some line. Although the presence of
this boundary type line complicates the elaboration of a complete quantum theory, it should not affect the short-distance singularities of the propagators away from it. Hence we should be able to check whether or not a theory is conformally invariant away from this line. This is the approach taken here (see refs. [2] and [3]): we will display a class of theories that are conformally invariant, at least when we need not consider the line of singularity, or if it is absent as for \( \kappa < 0 \).

**Transformation to free fields**

The kinetic part of \( S_{cl} + S_{anom} \) can be written as

\[
S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma e^{-2\phi} (-4 \partial_+ \phi \partial_- \phi + 2 \partial_+ \rho \partial_- \phi + 2 \partial_+ \phi \partial_- \rho - \kappa \partial_+ \rho \partial_- \rho) + S_{\text{matter}} . \tag{1.7}
\]

Here, we assume \( \kappa > 0 \), while things work similar for \( \kappa < 0 \). Let now [2]*

\[
\omega = e^{-\phi}/\sqrt{\kappa} , \quad \chi = \rho + \omega^2 . \tag{1.8}
\]

Then

\[
S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ -\kappa \partial_+ \chi \partial_- \chi + 4 \kappa (\omega^2 - 1) \partial_+ \omega \partial_- \omega \right] + S_{\text{matter}} \tag{1.9}
\]

is diagonalized. We can bring the \( \omega \)-kinetic term into a standard form by a further (local) field redefinition:

\[
\Omega = \omega \sqrt{\omega^2 - 1} - \log \left( \omega + \sqrt{\omega^2 - 1} \right) + \frac{1}{2} \left( 1 - \log \frac{\kappa}{4} \right) \Rightarrow \partial \Omega = 2 \sqrt{\omega^2 - 1} \partial \omega , \tag{1.10}
\]

so that finally

\[
S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ -\kappa \partial_+ \chi \partial_- \chi + \kappa \partial_+ \Omega \partial_- \Omega + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] . \tag{1.11}
\]

Note that \( \chi \) and \( \Omega \) have opposite signature, but it now seems that the kinetic term can no longer become singular. What has happened to the singularity (for \( \kappa > 0 \)) at \( \phi = \phi_c \)? Of course, it has been hidden in the transformation from \( \phi \) to \( \Omega \). This transformation is not one to one (for \( \kappa > 0 \)) and we have a singularity when \( d \Omega / d \phi = 0 \) which precisely happens at \( \phi = \phi_c \).

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* Here we rescale \( \chi \) and \( \Omega \) by a factor 2 with respect to ref. [2], and \( \Omega \) also is shifted by a constant.
Not only the kinetic part of the action is very simple when written in terms of the new fields $\chi$ and $\Omega$, but also the stress tensor:

$$T_{\pm\pm} = -\kappa (\partial_{\pm} \chi)^2 + \kappa \partial_{\pm}^2 \chi + \kappa (\partial_{\pm} \Omega)^2 + \frac{1}{2} \sum_{i=1}^{N} (\partial_{\pm} f_i)^2. \quad (1.12)$$

These forms of the kinetic part of the action and of the stress tensor are valid for both signs of $\kappa$, only the precise form of the field transformations are different.

For $\kappa < 0$ quantization is straightforward. For $\kappa > 0$, we disregard the subtleties connected with the presence of a singular line $\phi = \phi_c$ for the moment and proceed with a “naive” quantization. The kinetic part of the action then shows that $\chi$ and $\Omega$ have standard massless free field propagators, and it is straightforward to compute the short distance expansion of the stress tensor with itself. We find that it generates a (continuum) Virasoro algebra with central charge

$$c = 1 + 1 - 12\kappa + N - 26 = N - 24 - 12\kappa \quad (1.13)$$

which vanishes precisely if

$$\kappa = \frac{N - 24}{12}. \quad (1.14)$$

The field $\Omega$ contributes 1 to the central charge. $\chi$ contributes $1 - 12\kappa$ since the $\chi$ part of $T_{\pm\pm}$ has the Feigin-Fuchs form with background charge $\sim \sqrt{\kappa}$. Furthermore, the matter fields just contribute the usual $N$, while, although not written explicitly, we also have ghost from the conformal gauge fixing, and they contribute $-26$, as always.

Thus, with $\kappa = (N - 24)/12$ we expect to have a conformal field theory with vanishing central charge. One might worry however, that when we transformed our fields from $\phi$ and $\rho$ to $\chi$ and $\Omega$, a complicated Jacobian would appear in the functional integral, turning $\chi$ and $\Omega$ into interacting fields. This is not so, as we shall discuss below. Indeed, the initial “measure” for $\phi$ and $\rho$ is precisely such that together with this Jacobian one obtains a standard free field measure for $\chi$ and $\Omega$. 

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1.3. (1, 1)-operators and exact solutions

So far we only discussed the kinetic part of the action. There is also the “interaction” part of the action which contains the cosmological constant $\sim \lambda^2 e^{-2\phi} e^{2\rho}$. This term behaves like a perturbation of our conformal theory. If it is a marginal operator, however, it will preserve the conformal invariance. A necessary condition for a marginal operator is that it has conformal dimension $(1, 1)$. The cosmological constant operator has indeed dimension $(1, 1)$ classically, i.e. if we do a Poisson bracket computation with $T_{\pm\pm}$, but this is no longer true at the quantum level. It is easy to see that the only operators (with no derivatives) of definite conformal dimension $(\Delta, \Delta)$ are $\lambda^{2\Delta} \cdot e^{\alpha \chi + \beta \Omega}$: with

$$\Delta = \frac{\alpha}{2} + \frac{\alpha^2 - \beta^2}{\kappa}. \quad (1.15)$$

Any of these operators with $\Delta = 1$ (if truly marginal) will lead to a conformal theory. However, in the weak coupling limit, $e^{2\phi} \to 0$, we want to recover the classical dilaton-gravity action $S_{cl}$. Hence, we must take $\alpha = 2$ and $\beta = -2$. It is easy to see that this operator is indeed marginal. Then the full action reads:

$$S = \frac{1}{\pi} \int d^2 \sigma \left[ -\kappa \partial_+ \chi \partial_- \chi + \kappa \partial_+ \Omega \partial_- \Omega + \lambda^2 e^{2(\chi - \Omega)} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right]. \quad (1.16)$$

Note that this action differs from the classical dilaton gravity action only by higher order corrections, i.e. terms that are $O(G^2) = O(e^{2\phi})$ with respect to $S_{cl}$.

The equations of motion that follow from the action (1.16) are very simple. First of all, the $N$ matter fields are just free fields, $\partial_+ \partial_- f_i = 0$ with solution $f_i = f_i^+(\sigma^+) + f_i^-(\sigma^-)$. For $\chi$ and $\Omega$ we have:

$$\partial_+ \partial_- (\chi - \Omega) = 0,$$

$$\partial_+ \partial_- (\chi + \Omega) = -\frac{2}{\kappa} \lambda^2 e^{2(\chi - \Omega)}.$$

(1.17)

The general solution reads

$$2(\chi - \Omega) = f^+(\sigma^+) + f^-(\sigma^-) \equiv \log \partial_+ \alpha^+(\sigma^+) + \log \partial_- \alpha^-(\sigma^-),$$

$$\frac{\kappa}{2}(\chi + \Omega) = -\lambda^2 \alpha^+(\sigma^+) \alpha^-(\sigma^-) + \beta^+(\sigma^+) + \beta^-(\sigma^-). \quad (1.18)$$

This solution is just as simple as the one for the classical dilaton-gravity (i.e. the solution of the equations of motion derived from $S_{cl}$ alone) which reads $2(\rho - \phi) = \log \partial_+ \alpha^+(\sigma^+) +$
\[ \log \partial_- \alpha^-(\sigma^-) \text{ and } e^{-2\phi} = -\lambda^2 \alpha^+(\sigma^+) \alpha^-(\sigma^-) + \beta^+(\sigma^+) + \beta^-(\sigma^-). \] Let us also give the stress tensor when evaluated on the solutions (1.18):

\[ T_{\pm \pm} = -\partial_{\pm} f_{\pm} \partial_{\pm} \beta_{\pm} + \partial_{\pm}^2 \beta_{\pm} + \frac{k}{4} \partial_{\pm}^2 f_{\pm}, \tag{1.19} \]

where \( f_{\pm} \equiv \log \partial_{\pm} \alpha_{\pm}. \)

1.4. VARIATIONS OVER THE THEME

There are a few variations of the preceding formalism which we shall briefly discuss.

Strominger’s “decoupled ghost” theory

Strominger proposed [4] to define the measures for the different fields in a functional integral with different metrics. The metric we used so far is \( g_{ij} = e^{2\rho} \delta_{ij}. \) It should be used to define the measure for the matter fields \( f_i. \) To define the measures for the dilaton field \( \phi, \) the conformal factor of the metric \( \rho \) and the reparametrization ghosts, Strominger proposes to take a different, Weyl rescaled metric \( g_{ij} = e^{-2\phi} e^{2\rho} \delta_{ij}. \) Then the anomaly action for the latter fields will be constructed out of \( \rho - \phi \) instead of \( \rho, \) and we have

\[ S_{\text{anom}} = -\frac{1}{\pi} \int d^2 \sigma \left[ -\frac{N}{12} \partial_+ \rho \partial_- \rho + 2\partial_+(\rho - \phi) \partial_- (\rho - \phi) \right]. \tag{1.20} \]

Consider first the case \( N = 0, \) i.e. no matter fields. Then the equations of motion derived from \( S_{\text{cl}} + S_{\text{anom}} \) differ from those derived from \( S_{\text{cl}} \) only by terms \( \sim \partial_+ \partial_- (\rho - \phi), \) which vanishes by these equations of motion: \( S_{\text{anom}} \) has no effect on the solutions of the equations of motion. Thus it was hoped that with this modified anomaly action the ghosts, the dilaton and \( \rho \)-field would not contribute to the Hawking radiation. When \( N \neq 0 \) however, \( \partial_+ \partial_- (\rho - \phi) \) does no longer vanish by the equations of motion, and there seems to be no a priori reason why the modified anomaly action should be more physical than the original one (see the discussion in ref. [2]).

Here we would like to show that the kinetic part of Strominger’s action can again be written in free field form, and that after an appropriate improvement of the cosmological constant term,
one again obtains a conformal theory. Indeed redefine fields as
\[
\omega = e^{-\phi}/\sqrt{\kappa}, \quad \chi = \rho + \omega^2 - 2 \log \omega,
\]
\[
(\partial \Omega)^2 = 4 \left( \omega^2 - \frac{\kappa + 2}{\kappa} + \frac{\kappa + 2}{\kappa^2} \frac{1}{\omega^2} \right) (\partial \omega)^2 ,
\]
(\kappa = (N-24)/12). Then the kinetic part of the action and the stress tensor, when expressed in
terms of the new fields \( \chi \) and \( \Omega \), take exactly the same form (1.11) and (1.12) as before. Going
through the same arguments about how to modify the cosmological constant operator to turn it
into an exactly marginal operator, one finally arrives at the same conformally invariant action
(1.16). The reason that one obtains the same final action is very simple. Since Strominger’s
action, just as the CGHS action, or the actions to be discussed next, differ from the classical
dilaton-gravity action only by higher order corrections, i.e. terms that are \( \mathcal{O}(G^2) = \mathcal{O}(e^{2\phi}) \)
with respect to \( S_{\text{cl}} \), we are bound to obtain the action (1.16) in the end: there is only one
conformally invariant action (with a standard free field kinetic term for two fields of opposite
signature) that reduces in the weak coupling limit to the classical dilaton-gravity action.

The RST variant

So far we started with a given kinetic part of the action and improved the cosmological
constant operator by higher order corrections until it had dimension \((1,1)\). Russo, Susskind
and Thorlacius (RST) [5,6], motivated by the search of simple and exactly solvable equations
of motion, followed a slightly different route. Their procedure amounts to keeping the cosmolo-
gical constant operator fixed and modifying the kinetic part of the action and hence also the
stress tensor until the old cosmological constant operator has dimension \((1,1)\) with respect to
the new stress tensor. It turned out that this was very easy to achieve. All one needs is to add
\[
\delta S_{\text{RST}} = -\frac{\kappa}{4\pi} \int d^2\sigma \sqrt{-g} \phi R = \frac{1}{\pi} \int d^2\sigma \kappa \partial_+ \phi \partial_- \rho ,
\]
where the second expression is valid in conformal gauge only. In this RST-model the field
transformations to the \( \chi, \Omega \)-fields are simplest:
\[
\Omega = \frac{\kappa}{\kappa} e^{-2\phi} + \phi ,
\]
\[
\chi = \frac{\kappa}{\kappa} e^{-2\phi} - \phi + \rho .
\]
Again, when written in terms of these fields, the action and stress tensor take on the form
(1.16) and (1.12). Note also, that for $\kappa > 0$ the modified kinetic action is degenerate at $e^{-2\phi} = e^{-2\phi_c} \equiv \hat{\phi}$. This is precisely the value of $\phi$ where $d\Omega/d\phi = 0$. Although the precise value of $\phi_c$ is shifted, the qualitative feature of a singular line for $\kappa > 0$ is present in all models discussed so far.

The de Alwis models

More generally, one can add higher order corrections to both the kinetic part of the action and to the cosmological constant term. The final action $S[\chi, \Omega]$ and stress tensor $T_{\pm \pm} [\chi, \Omega]$ are always the same, but the transformations between $\phi, \rho$ and $\chi, \Omega$ are different. This program was carried out by de Alwis. The maybe somewhat unexpected result is that there are theories where the transformations are one to one, i.e. have no singularity, even for $\kappa > 0$. We refer the reader to de Alwis’ article [7] for details.

1.5. Singularity and shock-wave scenario

As already repeatedly emphasized, in most models (e.g. [2], [5]) for $\kappa > 0$, the kinetic part of the $\phi, \rho$-action is degenerate at some $\phi = \phi_c$. This translates into a singularity of the transformation function $\Omega(\phi)$: $\frac{d\Omega}{d\phi}(\phi_c) = 0$. Since the scalar curvature $R$ is proportional to $\left(\frac{d\Omega}{d\phi}\right)^{-2}$, in general, the scalar curvature diverges on the line where $\phi = \phi_c$. As shown first by RST [5, 6], when this line of singularity is time-like, one can impose appropriate boundary conditions to avoid the curvature singularity at $\phi = \phi_c$ and match the solution to the LDV configuration.

A typical example is the shock-wave scenario for $\kappa > 0$. Here we only give a very short description. We refer the reader to the RST-paper [5] for any details. One has the LDV for all $\sigma^+ < \sigma_0^+$. Then at $\sigma^+ = \sigma_0^+$ a matter shock-wave, characterised by $T_{++} = m \delta(\sigma^+ - \sigma_0^+)$, passes by, modifying the configuration for $\sigma^+ > \sigma_0^+$. To specify the initial data completely, we have also to give the configuration on $\mathcal{I}_R$. We simply assume LDV asymptotics. Then, the solution is such that a space-like line of singular curvature extends from the shock-wave trajectory into the region $\sigma^+ > \sigma_0^+$, but it is hidden behind an apparent horizon (which starts as a time-like line). This is interpreted as the formation of a black hole. Due to the emission of Hawking

* In the RST-model, the only exception is the LDV where $R = 0$.
† It is defined as the line where $\partial_4 \phi$ or equivalently $\partial_4 \Omega$ vanishes.
radiation (signalled by a non-vanishing $T^M_{\lambda\mu}$), the apparent horizon recedes, and intersects the line of singularity in a finite proper time at $(\sigma^+_s, \sigma^-_s)$. Beyond this point, the singularity turns time-like and naked. One cannot evolve the field equations past this singularity, and one has to impose boundary conditions on the singular line. RST observed that on the future part of the null-line going through $(\sigma^+_s, \sigma^-_s)$ the fields have values precisely such that one can match them continuously to a LDV configuration in the causal future of $(\sigma^+_s, \sigma^-_s)$ (at the cost of a delta-function type singularity in the second derivatives, and hence in the stress tensor: this gives rise to a “thunderpop”, an emission of energy of order $\kappa \lambda$). This avoids the naked time-like curvature singularity, and, maybe more important, it also stops the Hawking radiation at $\sigma^-_s$. This is physically important, since at $\sigma^-_s = \sigma^-_s$ the initially formed black hole has lost almost all of its initial mass $m$, and if Hawking radiation would continue, one would inevitably arrive at configurations of more and more negative total energies. Thus the RST boundary condition serves to stabilize the ground state of the system. Of course, the same applies to the original model of ref. [2] discussed above.

1.6. A LOCAL VERSION OF THE COVARIANT ANOMALY

For various purposes we need to rewrite the covariant anomaly $R \frac{1}{\nabla} R$ in a local form. Consider [8]

$$S_Z = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ -\frac{1}{2}(\nabla Z)^2 + Q R Z \right].$$

(1.24)

If we write

$$Z = \tilde{Z} - Q \frac{1}{\nabla^2} R,$$

(1.25)

we have

$$S_Z = -\frac{1}{4\pi} \int d^2 x \sqrt{-g} (\nabla \tilde{Z})^2 - \frac{Q^2}{4\pi} \int d^2 x \sqrt{-g} R \frac{1}{\nabla^2} R,$$

(1.26)

which is a free-field action for $\tilde{Z}$ plus $S_{\text{anom}}$. However, $\tilde{Z}$ still “remembers” the curvature coupling of the original $Z$ field, since its stress tensor in conformal gauge ($Z = \tilde{Z} + 2Q \rho$) reads

$$T^Z_{\pm \pm} = \frac{1}{2} (\partial_{\pm} \tilde{Z})^2 + Q \partial^2_{\pm} \tilde{Z}.$$

(1.27)

and $T^{Z}_{\pm\pm} = T^Z_{\pm\pm} + T^{\rho, \text{anom}}_{\pm\pm}$, where $T^{\rho, \text{anom}}_{\pm\pm} = -\kappa ((\partial_{\pm} \rho)^2 - \partial^2_{\pm} \rho)$. We see that $T^{Z}_{\pm\pm}$ has a classical central charge equal to $12\kappa = N - 26 + 1 + 1$, hence it really represents the contribution
of the $N$ matter fields, the ghosts and certain quantum fluctuations of $\phi$ and $\rho$. Thus, at the semiclassical level, where we only consider the $\rho$ and $\phi$ (or $\chi$ and $\Omega$) equations of motion and the constraint equations

$$T_{\pm\pm} = T^{\phi, \rho, \text{cl}}_{\pm\pm} + T^Z_{\pm\pm} = T^{\phi, \rho}_{\pm\pm} + T^Z_{\pm\pm},$$

(1.28)

studying $S_{\text{cl}} + S_{\text{anom}} + \delta S_{\text{improvement}}$ is completely equivalent to studying

$$S = S_{\text{cl}} + \delta S_{\text{improvement}} + S^Z.$$  

(1.29)

Here $\delta S_{\text{improvement}}$ stands for whatever higher order corrections we added to the classical dilaton gravity action. For the RST variant e.g. we have

$$S = \frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ \left( e^{-2\phi} - \frac{\kappa}{2} \phi \right) R + e^{-2\phi} \left( 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2}(\nabla Z)^2 + Q R Z \right].$$

(1.30)

Note that by eq. (1.25), the field $Z$ will be real only if $\kappa > 0$.

2. The supersymmetric extension

A supersymmetric extension of the CGHS model was constructed by Park and Strominger [9]. It seems natural to expect that the exactly solvable quantum-improved theories discussed in the previous section also have supersymmetric extensions. In fact there are three different problems one might consider:

1. Find a generally covariant supersymmetric extension of the exactly solvable quantum-corrected actions, e.g. of the RST action (1.30). By supersymmetric extension one means a supersymmetric action that in its bosonic sector (setting all fermions equal to zero, and replacing the auxiliary fields by the solutions of their algebraic field equations) reduces to the exact conformal, exactly solvable quantum-corrected action.

2. Find an exact superconformal theory that reduces in its bosonic sector to the exact conformal, exactly solvable quantum-corrected theory under consideration.

3. Find an exact superconformal theory that reduces in its bosonic sector and in the weak-coupling limit ($e^{2\phi} \to 0$) to the classical dilaton-gravity theory (not conformally invariant).
Obviously, problem 3 is a weaker version of problem 2. It might not be obvious at first sight why problem 1 and 2 should be different. However, problem 1 was solved in ref. [10]. On the other hand, problem 2 has no solution as shown by Nojiri and Oda [11]. Problem 3 was solved by Danielsson [12] who also explained why problems 1 and 2 are different. Here, we will discuss the solution to problem 1 and then show why problem 2 cannot have a solution.

Start from a general supersymmetric dilaton-gravity action in 2D

\[ S^{(1)} = \frac{i}{2\pi} \int d^2x d^2\theta E [J(\Phi)S + iK(\Phi)D_\alpha \Phi D^\alpha \Phi + L(\Phi)] \quad (2.1) \]

(\( \Phi \) is the dilaton superfield, \( S \) the curvature multiplet and \( E \) the super-zweibein, see ref. 9 for all notation and conventions). \( J, K \) and \( L \) are for the moment arbitrary scalar functions of the dilaton superfield. If we expand the superfields, set all fermionic fields to zero and integrate out the auxiliary fields (i.e. replace them with the solutions of their algebraic field equations) we get the bosonic part of the action \( (2.1) \) [9]

\[ S^{(1)}_{\text{bos}} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ JR + 2K(\nabla \phi)^2 + \left( \frac{LL'}{2J'} - \frac{KL^2}{2J'^2} \right) \right], \quad (2.2) \]

where now \( J = J(\phi), K = K(\phi), L = L(\phi) \) are functions of the dilaton field. Choosing e.g. \( J = e^{-2\phi}, K = 2e^{-2\phi}, \frac{LL'}{2J'} - \frac{KL^2}{2J'^2} = 4\lambda^2 e^{-2\phi} \) with solution \( L = \pm 4\lambda e^{-2\phi} \) reproduces \( S_d \).

We now repeat this exercise, including a supersymmetric Z-field:

\[ S^{(2)} = \frac{i}{2\pi} \int d^2x d^2\theta E \left[ -\frac{i}{4}D_\alpha \mathcal{Z} D^\alpha \mathcal{Z} + Q \mathcal{Z} S \right]. \quad (2.3) \]

The bosonic part of this action alone is just \( S_Z \) of eq. (1.24). When combining \( S^{(1)} \) and \( S^{(2)} \), the auxiliary field equations get modified and the resulting bosonic part is not just \( (2.2) \) plus \( S_Z \), but rather

\[
\left[ S^{(1)} + S^{(2)} \right]_{\text{bos}} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ JR + 2K(\nabla \phi)^2 - \frac{1}{2}(\nabla Z)^2 + QRZ + F(\phi) \right],
\]

\[ F(\phi) = \left( 1 - \frac{2\kappa K}{J'^2} \right)^{-1} \left( \frac{LL'}{2J'} - \frac{KL^2}{2J'^2} - \frac{\kappa L^2}{4J'^2} \right). \quad (2.4) \]

All we have to do now is to identify the functions \( J, K \) and \( L \) of \( \phi \) that reproduce e.g. the
RST-action (1.30). For the latter we need

\[ J(\phi) = e^{-2\phi} - \frac{\kappa}{2} \phi, \quad K(\phi) = 2e^{-2\phi}, \quad F(\phi) = 4\lambda^2 e^{-2\phi}. \]  

(2.5)

Substituting these into the equation (2.4) for \( F \) we obtain a non-linear differential equation for \( L(\phi) \):

\[ (L + xL')(L + L') = -\kappa^2 \lambda^2 \frac{(1 - x)^2}{x^2} \]

(2.6)

where \( L' = dL/d\phi \) and \( x = \frac{\kappa}{4} e^{2\phi} \). The solution is very simple:

\[ L(\phi) = \pm 4\lambda \left( e^{-2\phi} + \frac{\kappa}{4} \right). \]

(2.7)

Obviously there are two choices of sign since only \( \lambda^2 \) is relevant. Thus, if \( J, K \) and \( L \) are given by (2.5), (2.7), the action \( S^{(1)} + S^{(2)} \) is a supersymmetric extension of the RST-action.

Similarly, we can construct a supersymmetric extension of the action of ref. 2. In this case \( J(\phi) \) and \( K(\phi) \) are given by the CGHS-functions

\[ J(\phi) = e^{-2\phi}, \quad K(\phi) = 2e^{-2\phi} \]

(2.8)

while the function \( F \) is more complicated [2]. As a consequence, the non-linear differential equation to be solved for \( L(\phi) \) is considerably more involved. After some exercise (see ref. [10] for details), one finds \( L(\phi) \) as a transcendental function of \( \Omega(\phi) \). Here we only give its weak-coupling expansion for small \( \kappa e^{2\phi} \)

\[ L(\phi) \sim \pm 4\lambda \left( e^{-2\phi} + \frac{\kappa}{4} \phi + \hat{c} \right). \]

(2.9)

Note that, as expected, the leading term in the weak-coupling expansion of \( L(\phi) \) for both variants discussed here is \( \pm 4\lambda e^{-2\phi} \) which is the \( L(\phi) \) as appropriate for \( S_{\text{cl}} \). We also note that \( L \) is linear in \( \lambda \) and that the cosmological term \( F \) in the bosonic part obtained after solving the auxiliary field equations is bilinear in \( L \), hence \( \sim \lambda^2 \) as it should.

Now what is the problem with problem 2? As just pointed out, upon integrating out the auxiliary fields, one replaces operators linear in \( L \) by operators bilinear in \( L \). These bilinears have to be regularized, so that the naive procedure of integrating out the auxiliary fields can only be trusted at the semiclassical level [12].

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In other words: at the (semi)classical level, if $L$ has classical conformal dimension $\left( \frac{1}{2}, \frac{1}{2} \right)$ then $L^2$ has dimension $(1,1)$ and vice versa. However, quantum mechanically, due to the necessity to regularize $L^2$, these two statements are in general incompatible. The requirement that the bosonic part be an exact conformal theory means that $L^2$ has to have dimension $(1,1)$ quantum mechanically. Exact superconformal invariance means that $L$ has to have dimension $\left( \frac{1}{2}, \frac{1}{2} \right)$ quantum mechanically. We can arrange for one or the other but not for both at the same time. Problem 2 consists of achieving both which is impossible (or rather has only the trivial solution $L = 0$ [11]). Problem 1 insists on $L^2$ having dimension $(1,1)$, so the susy theory is not an exact superconformal theory. Finally, problem 3 insists on $L$ having dimension $\left( \frac{1}{2}, \frac{1}{2} \right)$, and thus the bosonic part is not an exact conformal theory, but can still be arranged to have the correct weak-coupling classical limit [12].

3. Canonical formalism, ADM energy and functional integral measure

We would like to display a canonical formalism for the dilaton-gravity theories in 2D. The starting point is the covariant, i.e. not gauge-fixed theory, and hence we represent the anomaly term by the local and covariant $Z$-action. We can treat the classical model, the CGHS model and the RST model simultaneously if we consider the following action (cf. (1.30))

$$S = \frac{1}{2\pi} \int d^2 \sigma \sqrt{-g} \left[ \left( e^{-2\phi} - \frac{\kappa}{2} \phi \right) R + 4 e^{-2\phi} \left( (\nabla \phi)^2 + \lambda^2 \right) - \frac{1}{4} (\nabla Z)^2 + Q R Z \right]$$

(3.1)

where $\kappa = Q = 0$ gives back the classical action (with one free matter $Z$-field\footnote{One might add other free (classical) matter fields. These could be included trivially into our subsequent analysis, in particular their contribution to the boundary term $D$ would vanish due to the standard boundary conditions on matter fields.}), $\kappa = 0$, $2Q^2 = \frac{N}{12}$ gives the CGHS model [1], and $\kappa = 2Q^2 = \frac{N-24}{12}$ gives the RST-model [5].

3.1. Canonical formalism

We parametrize the two-dimensional metric in the following way

$$g_{\mu \nu} = e^{2\rho} \begin{pmatrix} A^2 - B^2 & A \\ A & 1 \end{pmatrix}.$$ 

(3.2)

This is inspired by the standard ADM parametrization [13] with $A$ and $e^\rho B$ the analogues of the shift vector and lapse function. Conformal gauge is simply $A = 0$, $B = 1$. Inserting this
parametrization into the above action we obtain \( S[A, B, \phi, \rho, Z, \dot{\phi}, \dot{\rho}, \dot{Z}] \) which does not depend on \( \dot{A} \) or \( \dot{B} \). We refer the reader to ref. [14] for details. It is straightforward to compute the momenta \( \Pi_\phi, \Pi_\rho \) and \( \Pi_Z \). Their general expression can be found in [14]. Here we only give them for \( A = 0, B = 1 \) (conformal gauge)

\[
\Pi_\phi = \frac{1}{\pi} \left( F \dot{\phi} - 2 e^{-2\phi} \dot{\phi} \right), \quad \Pi_\rho = \frac{1}{\pi} \left( F \dot{\phi} - \frac{Q}{2} \dot{Z} \right), \quad \Pi_Z = \frac{1}{4\pi} \left( \dot{Z} - 2Q \dot{\rho} \right),
\]

(3.3)

where \( F = e^{-2\phi} + \frac{1}{4} \). Since no time-derivatives of \( A \) or \( B \) occur in the action there are no momenta conjugate to \( A \) or \( B \). The fields \( A \) and \( B \) are Lagrange multipliers serving to impose constraints.

Writing \( S = \int d\tau L \), the bulk Hamiltonian is given by \( H_1 = \int d\sigma (\dot{\phi} \Pi_\phi + \dot{\rho} \Pi_\rho + \dot{Z} \Pi_Z) - L \) which after integrating by parts reads

\[
H_1 = \int d\sigma \left[ AC_A + BC_B \right]
\]

\[
C_A = \phi' \Pi_\phi + \rho' \Pi_\rho + Z' \Pi_Z - \Pi_\rho'
\]

\[
C_B = \frac{1}{G^2} \left[ \frac{1}{\pi} \left( e^{-2\phi}(\Pi_\rho + 2Q\Pi_Z)^2 + F \Pi_\phi(\Pi_\rho + 2Q\Pi_Z) + \frac{1}{2} Q^2 \Pi_\phi^2 \right) + 2\pi \Pi_Z^2 
\]

\[
+ \frac{1}{\pi} \left[ F \rho' \phi' - (F \phi')' - e^{-2\phi} \phi'^2 \right] - \frac{1}{2} e^{-2\phi+2\rho} + \frac{1}{2} \rho' \phi' + \frac{1}{2} Q \rho' Z' + \frac{1}{2} Q Z'' \right]
\]

(3.4)

where \( G^2 = F^2 - 2Q^2 e^{-2\phi} \).

The Lagrange multipliers \( A \) and \( B \) impose the constraints \( C_A = C_B = 0 \). What is the interpretation of these constraints? We evaluate them in conformal gauge \( (A = 0, B = 1) \) and substitute (3.3) for the momenta. Then (using also \( Z = \dot{Z} + 2Q \dot{\rho} \)) \( C_A \) and \( C_B \) are easily seen to coincide with \( \frac{1}{2\pi}(T_{++} - T_{--}) \) and \( \frac{1}{2\pi}(T_{++} + T_{--}) \), respectively.

Using canonical Poisson brackets,

\[
\{\phi(\sigma), \Pi_\phi(\sigma')\} = \{\rho(\sigma), \Pi_\rho(\sigma')\} = \{Z(\sigma), \Pi_Z(\sigma')\} = \delta(\sigma - \sigma')
\]

(3.5)

we can compute the algebra of the constraints as given by (3.4) (in general gauge). We find that the Poisson bracket of \( C_A + C_B \) with \( C_A - C_B \) vanishes while

\[
\{[C_A(\sigma) \pm C_B(\sigma)], (C_A(\sigma') \pm C_B(\sigma'))\} = 2(\partial_{\sigma} - \partial_{\sigma'})[(C_A(\sigma') \pm C_B(\sigma'))\delta(\sigma - \sigma')]
\]

(3.6)

which is indeed the Poisson bracket algebra of \( T_{\pm \pm} \) with itself. There is no \( \delta'''' \)-term which
means that the total central charge vanishes. This was to be expected: for the classical theory this is obvious, while for the RST-model e.g. $Q^2 (\partial_\pm \rho \partial_\pm \rho - \partial_\pm^2 \rho)$ gives $c = -24Q^2 = -12 \kappa = 24 - N$, and the $Z$-field gives the anomaly for matter, ghosts and the quantum part of $\phi, \rho$ which is $c = N - 26 + 2 = N - 24$. Of course, we just repeated that the Polyakov-anomaly action is designed to cancel the various anomalies present in the theory.

According to the variational principle in Hamiltonian form,

$$
\delta \left( \int d\sigma (\dot{\phi} \Pi_\phi + \dot{\rho} \Pi_\rho + \dot{Z} \Pi_Z) - H_1 \right) = 0
$$

should be equivalent to Hamilton’s equations, $\frac{\delta H}{\delta \phi} + \Pi_i = \frac{\delta H}{\delta \Pi_i} - \dot{\phi}_i = 0$. However, if we carefully carry out the variation, keeping track of boundary terms arising from integrating by parts, we find

$$
\delta \left( \int d\sigma (\dot{\phi} \Pi_\phi + \dot{\rho} \Pi_\rho + \dot{Z} \Pi_Z) - H_1 \right) = -D + X , \quad (3.7)
$$

where $D$ is a boundary term and the vanishing of $X$ is precisely equivalent to Hamilton’s equations. For $A = 0, B = 1$ (conformal gauge), the boundary term $D$ is given by (recall that $F = e^{-2\phi} + 4$

$$
\pi D = \left[ \delta \left( -F \phi' + \frac{1}{2}QZ' \right) + (F \rho' - 2e^{-2\phi} \phi') \delta \phi \right.
$$

$$
+ \left. \left( F \phi' - \frac{1}{2}QZ' \right) \delta \rho + \left( -\frac{1}{2}Q \rho' + \frac{1}{4}Z' \right) \delta Z \right]_{\sigma=+\infty} \left. - \left. \right|_{\sigma=-\infty} . \quad (3.8)
$$

This does not vanish.

### 3.2. The ADM Energy

The same situation occurs in classical general relativity in four dimensions. The ADM Hamiltonian gives the equations of motion only up to boundary terms. As first noticed by Regge and Teitelboim [15] for asymptotically flat space-times this has a very simple resolution. The idea is to take these boundary terms serious and show that under appropriate asymptotic conditions on the fields, the boundary term can be canceled by the variation of an appropriate
boundary Hamiltonian $H_2$ added to the bulk Hamiltonian $H_1$. The sum $H = H_1 + H_2$ then
is interpreted as the true Hamiltonian. Since $H_1$ is only a sum of constraints, its numerical
value on any solution vanishes, and the total energy, i.e. the numerical value of $H$ is given by
that of $H_2$. Thus the total energy is automatically given by the boundary terms.

In the present 2D dilaton-gravities the same trick works. Here, we only discuss the case
$A = 0, B = 1$ and refer to [14] for the general case. As in 4D, we assume asymptotically flat
space, and asymptotically Minkowskian coordinates. This is translated into requiring linear
dilaton vacuum asymptotics:

$$\text{as } \sigma \rightarrow \pm \infty : \phi \sim -\lambda \sigma , \rho \sim 0 , \ Z \sim 0 , \ (3.9)$$

(and corresponding conditions on the momenta although in conformal gauge they are not really
needed). For $\sigma \rightarrow +\infty$ we have to add another coordinate condition:

$$\text{as } \sigma \rightarrow \infty : e^{2\lambda \sigma}(\phi + \lambda \sigma - \rho) \rightarrow 0 . \ (3.10)$$

One can indeed convince oneself that this can be satisfied for all models that differ from $S_{\text{cl}}$
only by terms $O(e^{2\phi})$. We now restrict the phase-space to only those configurations that
obey the asymptotic conditions (3.9) and (3.10). This is a perfectly legitimate procedure.
Equation (3.10) implies in particular that $\lim_{\sigma \rightarrow \infty} e^{-2\phi} \delta \phi = \lim_{\sigma \rightarrow \infty} e^{-2\phi} \delta \rho$. Using these
types of relations, one easily sees that

$$D \bigg|_{\text{boundary conditions}} = -\delta H_2 , \ H_2 = \frac{1}{2\pi} \left[ e^{-2\phi} \left( 2\phi' + \lambda \right) + \lambda e^{2\lambda \sigma} \right]_{\sigma=\pm \infty} , \ (3.11)$$

where we adjusted an (infinite) additive field-independent term, not affecting the relation
$\delta H_2 = -D$, so that $H_2$ vanishes for the LDV. Note that as a consequence of the boundary
conditions, $H_2$ receives no contribution from $\sigma = -\infty$. Then

$$\delta \left( \int d\sigma (\dot{\phi} \Pi_{\phi} + \dot{\rho} \Pi_{\rho} + \dot{Z} \Pi_Z) - H_1 - H_2 \right) = 0 \Rightarrow \text{Hamilton’ equations} . \ (3.12)$$

Thus the true Hamiltonian is $H = H_1 + H_2$. Since $H_1$ vanishes on all solutions, the total
energy is given by the value of $H_2$ only (times a conventional normalization factor of $2\pi$). If
we write \( \phi \sim -\lambda \sigma + \delta \phi \) with \( \delta \phi \) at least \( \mathcal{O}(e^{-\lambda \sigma}) \) then we can write the total energy \( M = 2\pi H \)

as

\[
M = \lim_{\sigma \to \infty} 2e^{2\lambda \sigma} (\partial_\sigma + \lambda)(\delta \phi - \delta \phi^2)
\]

(3.13)

Due to our boundary conditions we can rewrite the part linear in \( \delta \phi \) as

\[
M^{(1)} = \lim_{\sigma \to \infty} 2e^{2\lambda \sigma} (\partial_\sigma \delta \phi + \lambda \delta \rho)
\]

(3.14)

which is the formula for the ADM-mass usually given in the literature [16]. However this latter formula is only correct if the \( \delta \phi^2 \) term can be neglected, i.e. if \( \delta \phi = \mathcal{O}(e^{-2\lambda \sigma}) \) as \( \sigma \to \infty \).

Indeed, in general the \( \delta \phi^2 \) term is crucial to make \( M \) time independent. Using (3.10) it is easy to see that the full expression \( M \) as given by (3.13) obeys

\[
\frac{d}{d\tau} M = -2\pi \mathcal{C}_A \bigg|_{\sigma = +\infty},
\]

(3.15)

and since \( \mathcal{C}_A = 0 \) is a constraint, \( \frac{d}{d\tau} M \) vanishes.

Note that the same mass formula applies to all three models, classical, CGHS and RST.

3.3. A NOTE ON THE FUNCTIONAL INTEGRAL MEASURE

In a functional integral approach, any transformation on the fields is accompanied by a Jacobian for the measure. One might wonder whether we should worry about such a Jacobian when we transform from \( \phi, \rho \) to \( \Omega, \chi \). I shall now argue that the final functional integral where one integrates over \( \Omega \) and \( \chi \) contains no Jacobian [17]. The important point is that it is incorrect to start with a functional integral \( \mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\rho \exp\{-S[\phi, \rho]\} \). Of course, we know that the correct starting point for the functional integral is

\[
\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\rho \mathcal{D}\Pi_\phi \mathcal{D}\Pi_\rho \exp\left\{-\int \left(\dot{\phi} \Pi_\phi + \ddot{\rho} \Pi_\rho\right) + \int H[\phi, \rho, \Pi_\phi, \Pi_\rho]\right\}.
\]

(3.16)

It is only when the Hamiltonian \( H \) has a standard kinetic term, \( H = \sum_i a_i \Pi_i^2 + \ldots \) with constant \( a_i \), that one can perform the gaussian integration over the momenta and obtain a functional integral over the fields only. For the dilaton-gravity theories we are considering this is certainly not the case, see e.g. eq. (3.4). The Hamiltonian is still quadratic in the momenta, but the coefficients are complicated \( \phi \)-dependent functions.
Instead of trying to evaluate the functional integral (3.16) we make the canonical transformation

\[ \phi, \rho \rightarrow \Omega, \chi \]
\[ \Pi_\phi, \Pi_\rho \rightarrow \Pi_\Omega, \Pi_\chi . \]  

(3.17)

The only things that matter is that the kinetic part of the action is a free field action in terms of \( \Omega \) and \( \chi \), and that the transformation is canonical, which determines \( \Pi_\Omega \) and \( \Pi_\chi \). For the RST variant we have e.g.

\[ \Pi_\Omega = -\frac{(e^{-2\phi} + \frac{\chi}{\kappa}) \Pi_\rho + \frac{\chi}{\kappa} \Pi_\phi}{e^{-2\phi} - \frac{\chi}{\kappa}} = \frac{\kappa}{2\pi} \dot{\Omega} , \]

\[ \Pi_\chi = \Pi_\rho = -\frac{\kappa}{2\pi} \dot{\chi} . \]  

(3.18)

Now, on the one hand, since the transformation is canonical

\[ \int (\dot{\phi} \Pi_\phi + \dot{\rho} \Pi_\rho) - H[\phi, \rho, \Pi_\phi, \Pi_\rho] \]
\[ = \int (\dot{\Omega} \Pi_\Omega + \dot{\chi} \Pi_\chi) - H[\Omega, \chi, \Pi_\Omega, \Pi_\chi] \]
\[ = \int (\dot{\Omega} \Pi_\Omega + \dot{\chi} \Pi_\chi) - \int \left[ \frac{\pi}{\kappa} \left( \Pi_\Omega^2 - \Pi_\chi^2 \right) + \frac{\kappa}{4\pi} \left( (\Omega')^2 - (\chi')^2 + 2\chi'' \right) - \lambda^2 e^{2(\chi - \Omega)} \right] . \]  

(3.19)

Indeed we see that variation of the latter yields the \( \Omega, \chi \) equations of motion in Hamiltonian form (up to boundary terms, see below). On the other hand, a canonical transformation preserves the phase space measure \( dqdp \). Since our transformation is local and contains no space-time derivatives it is a canonical transformation at any point \((\tau, \sigma)\). Hence it certainly preserves the discretized measure for the functional integral:

\[ \prod_i d\phi(\tau_i, \sigma_i)d\rho(\tau_i, \sigma_i)d\Pi_\phi(\tau_i, \sigma_i)d\Pi_\rho(\tau_i, \sigma_i) \]
\[ = \prod_i d\Omega(\tau_i, \sigma_i)d\chi(\tau_i, \sigma_i)d\Pi_\Omega(\tau_i, \sigma_i)d\Pi_\chi(\tau_i, \sigma_i) . \]  

(3.20)

Thus we expect this equality to hold in the continuum limit,

\[ \mathcal{D}\phi \mathcal{D}\rho \mathcal{D}\Pi_\phi \mathcal{D}\Pi_\rho = \mathcal{D}\Omega \mathcal{D}\chi \mathcal{D}\Pi_\Omega \mathcal{D}\Pi_\chi , \]  

(3.21)

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and hence

\[ Z = \int \mathcal{D}\Omega \mathcal{D}\chi \mathcal{D}\Pi_\Omega \mathcal{D}\Pi_\chi \exp \left\{ -\int \left( \dot{\Pi}_\Omega + \dot{\chi} \Pi_\chi \right) + \int H[\Omega, \chi, \Pi_\Omega, \Pi_\chi] \right\}, \tag{3.22} \]

where the argument of the exponential is given by the negative of (3.19). It is now of the standard quadratic form \( \dot{q} - ap^2 \) with constant \( a \), and we can perform the gaussian integration with the result

\[ Z = \int \mathcal{D}\Omega \mathcal{D}\chi \exp \{ -S[\Omega, \chi] \}, \tag{3.23} \]

where \( S[\Omega, \chi] = \frac{1}{2} \int d^2 \sigma \left[ -\kappa \partial_+ \chi \partial_- \chi + \kappa \partial_+ \Omega \partial_- \Omega + \lambda^2 e^{2(\chi - \Omega)} \right] \), cf. eq. (1.16) (up to a boundary term). We see that this functional integral contains no extra Jacobian. If we however change variables back to \( \phi, \rho \), we get a Jacobian, so that the correct \( \phi, \rho \) functional integral is

\[ Z = \int \mathcal{D}\phi \mathcal{D}\rho \ J e^{-S[\phi, \rho]}, \]

where \( J = \frac{\partial S[\chi]}{\partial (\phi, \rho)} \).

As it stands, the functional integral (3.23) does not take into account boundary effects. In particular, we have seen in the preceding subsection that one has to add a boundary Hamiltonian \( H_2 \) so that the solutions to the equations of motion provide a true saddle point to the functional integral. For the RST variant e.g., we have from eq. (3.19) \( D = \frac{\kappa}{2\pi} \left[ \Omega' \delta \Omega' - \chi' \delta \chi + \delta \chi \right]_{\sigma=-\infty}^{\sigma=+\infty} \), and using the same boundary conditions as above one has \( D = -\delta H_2 \) with the boundary Hamiltonian \( H_2 = \left[ \frac{\kappa}{2\pi} (\lambda - \partial_\sigma) \Omega + \frac{\lambda^2 e^{2\lambda \sigma}}{2} \right]_{\sigma=+\infty} \). (Of course, this coincides with eq. (3.11).) It should not be too difficult to impose the appropriate boundary conditions on the fields in the functional integral. Thus, for \( \kappa < 0 \), one should be able to quantize the theory using the functional integral. For \( \kappa > 0 \) however, we have the extra restriction \( \Omega > \Omega_c \) on the range of integration of \( \Omega \). At present, nobody seems to know how to evaluate such functional integrals.
4. A positive energy theorem

4.1. The theorem

We will give a positive energy theorem for the RST variant of the exactly solvable dilaton-gravity theories. Although its formulation involves some spinors, it only relies on the (bosonic) equations of motion, and the spinors are simply a convenient device to express certain dependences on the dilaton and metric fields. Of course, the spinorial formulation is motivated by the existence of a supercharge in the supersymmetric extension, and the deep reason why we can prove a positive energy theorem is probably the existence of this supersymmetric extension. Nevertheless, let us stress again that in this section we are dealing with a purely bosonic theory. We will need the equations of motion for the metric in the covariant form, $T_{\mu \nu} = 0$. Thus we will again make use of the reformulation (1.30) using the $Z$-field. The proof we will give [10] for the RST model is a generalization of the one given by Park and Strominger [9] for the CGHS model. We will assume $\kappa > 0$ throughout this section, so that $Z$ is real.

The equations of motion of the metric are $T^g_{\mu \nu} + T^Z_{\mu \nu} = 0$, where $T^Z_{\mu \nu}$ is obtained from the action (1.24):

$$
T^Z_{\mu \nu} = \nabla^2 Z = \frac{1}{2} \nabla_\mu \nabla_\nu Z - g_{\mu \nu} \nabla^2 Z,
$$

and $T^g_{\mu \nu}$ is the covariant form of $T^\phi_{\mu \nu} - T^\text{anom}_{\mu \nu}$, namely

$$
T^g_{\mu \nu} = -2 \left( e^{-2\phi} g_{\mu \nu} + \frac{\kappa}{4} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu \nu} \nabla^2 \phi \right) - 2 e^{-2\phi} g_{\mu \nu} \left( \nabla^2 \phi \right) - \lambda^2 \right). \tag{4.2}
$$

Note that we have decomposed $T^Z_{\mu \nu}$ into a piece $\hat{T}^Z_{\mu \nu}$ that does obey the dominant energy condition (it is just the standard stress energy tensor for a free matter field) and a piece $\sim Q$ that does not. We then have the following

**THEOREM:** Let

$$
M = \int_{\Sigma} d\sigma^\mu \nabla_\mu \left[ 2 \left( e^{-2\phi} + \frac{\kappa}{4} \right) \nabla_\mu \nabla_\nu \phi - g_{\mu \nu} \nabla^2 \phi \right], \tag{4.3}
$$

where $2Q^2 = \kappa > 0$. $\epsilon$ is a commuting real two-dimensional spinor, and $\nabla = \gamma^\mu \nabla_\mu = e^{\mu a} \Gamma_a \nabla_\mu$. 

\[ e^{\mu a} \] being the zwei-bein. Then

\[ M \geq 0 \quad (4.1) \]

if (i) the line \( \Sigma \) is space-like or null, (ii) \( \phi \) is real on all of \( \Sigma \) (i.e. \( \phi \leq \phi_c \) or \( \phi \) is the LDV), and (iii) \( \epsilon \) is a solution of the ordinary differential equation on \( \Sigma \)

\[
\mathrm{d}\sigma^\mu \left[ \left( 1 + \frac{\kappa}{4} e^{2\phi} \right) \nabla_\mu \epsilon - \frac{1}{2} \gamma_\mu (\nabla \phi - \lambda) \epsilon - \frac{Q}{4} e^{2\phi} (1 + \frac{\kappa}{4} e^{2\phi})^{-1} \gamma_\mu \nabla \epsilon \right] = 0 . \quad (4.5)
\]

Note that the \( \epsilon \)-differential equation determines \( \epsilon \) only up to two functions of integration. Thus \( M \) not only depends on the fields \( \phi, g_{\mu\nu} \) and the line \( \Sigma \), but also on these functions of integration. We will fix the latter in the next subsection.

**Proof** (sketched only, for more details, see [10]): One evaluates \( \nabla_\mu [\ldots] \) in (4.3). Whenever one has \( \nabla_\mu \epsilon \) or \( \nabla_\mu \bar{\epsilon} \) one uses the \( \epsilon \)-differential equation to eliminate it. Using some trivial spinor identities, one can identify various pieces of the stress energy tensor. Using the equations of motion \( T^\phi_{\mu\nu} + T^Z_{\mu\nu} = 0 \) it can be seen that \( M \) equals

\[
M = \int_{\Sigma} \mathrm{d}\sigma^\mu \left( 1 - \frac{\kappa}{4} e^{2\phi} \right) \frac{1}{1 + \frac{\kappa}{4} e^{2\phi}} \epsilon^\rho \epsilon \epsilon^\nu \left( \tilde{T}^Z_{\mu\nu} \epsilon^\gamma \epsilon^\nu \right) . \quad (4.6)
\]

This expression is manifestly non-negative for \( \kappa > 0 \) if \( \phi \) is real everywhere on \( \Sigma \). Indeed, it is easy to see that for any real non-zero \( \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \) (not necessarily a solution of (4.5)), \( \nu^\nu = \bar{\epsilon} \gamma^\nu \epsilon \) is time-like or null and future-directed. Now \( \tilde{T}^Z_{\mu\nu} \) obeys the dominant energy condition, i.e. for time-like or null, future-directed \( \nu^\nu \) the vector \( -\tilde{T}^Z_{\mu\nu} \nu^\nu \) is again time-like or null, future-directed. Note that this is true only if \( Z \) is real, i.e. for \( \kappa > 0 \)! Since \( \epsilon^0_1 = -1 \) it follows that \( M \) as given by (4.6) is non-negative provided \( \Sigma \) is space-like or null and \( \phi \) real on \( \Sigma \).

In ref. [10] it was also shown that the functional \( M \) as given by (4.3) is unique in the following sense: Suppose one replaces the coefficients of \( \nabla \phi, \lambda, \nabla Z \) (which are the only covariant objects of the dimension of a mass) in (4.3) and in (4.5) by arbitrary scalar functions of \( \phi \).

---

* The \( \Gamma_a \) are Minkowski-space Dirac-matrices obeying \( \{ \Gamma_a, \Gamma_b \} = 2\eta_{ab} \). A convenient choice that we adopt here is \( \Gamma_0 = i\sigma_y, \Gamma_1 = \sigma_z \). Let \( \Gamma_5 = \Gamma_0 \Gamma_1 = \sigma_x \) while (following ref. 9) \( \gamma_5 = \gamma^0 \gamma^1 = -\Gamma_5 \). As usual, \( \bar{\epsilon} = \epsilon^* \Gamma_0 \). The anti-symmetric tensor is \( \epsilon_0^1 = \epsilon_1^0 = -1 \).
Then we will be able to use the equations of motion to obtain a non-negative quantity of the type (4.6) involving the \( \hat{T}^Z \) (which obeys the dominant energy condition) only if the functions of \( \phi \) are precisely as given in (4.3) and (4.5).

Although the mass functional (4.3) is uniquely determined, its actual value depends on the boundary or initial conditions imposed on the spinor \( \epsilon \) upon solving its differential equation (4.5). They will be fixed next by imposing physical requirements.

4.2. Physical interpretation and applications

The functional \( M \) is given by a line integral of a derivative along this line and thus reduces to the difference of the values of the expression in the square brackets at “both ends of the world”. Thus \( M \) is given by the asymptotic values of the fields and of the spinor \( \epsilon \). However, through the \( \epsilon \)-differential equation the latter depend on the fields on all of \( \Sigma \). This differs from 4D general relativity.

Now we would like to see whether the non-negative functional \( M \) defines a reasonable mass (energy) and compute it for various physically interesting scenarios. In particular, we will evaluate \( M \) as defined in (4.3) for the case where the field configuration is asymptotic to the LDV at both ends of \( \Sigma \). If \( \Sigma \) is a space-like line one should obtain the ADM-mass while a null-line \( \Sigma \) should lead to the Bondi-mass. This has been discussed in considerable detail in [10]. Here we will just give the main results. All computations in this subsection will be in conformal gauge.

\( \Sigma \) a space-like line of constant \( \tau \): ADM-mass

If we denote the expression in square brackets in (4.3) by \( \mathcal{M} \) we have

\[
M(\tau) = \mathcal{M}(\tau, \sigma = \infty) - \mathcal{M}(\tau, \sigma = -\infty) .
\]  

(4.7)

We assume LDV asymptotics, i.e. as \( \sigma \to \pm \infty \) : \( \phi \sim -\lambda \sigma + \delta \phi \), \( \rho \sim \delta \rho \) where \( \delta \phi \) and \( \delta \rho \) vanish as \( \sigma \to \pm \infty \). Let furthermore \( \ddot{Z} \to 0 \) as \( \sigma \to \pm \infty \) so that \( Z \sim 2Q \rho \). More precisely, the LDV asymptotics as \( \sigma \to +\infty \) together with the equations of motion and constraints imply

\[
\rho \sim a_1(\tau)e^{-\lambda \sigma} + a_2(\tau)e^{-2\lambda \sigma} + \ldots , \quad \phi = -\lambda \sigma + \rho
\]

\[
\dot{a}_2 = 2a_1 \dot{a}_1 , \quad \ddot{a}_1 = \lambda^2 a_1 ,
\]

(4.8)

where we use a suitable set of coordinates so that \( \phi = -\lambda \sigma + \rho \). It is related to the “Kruskal”
coordinates $x^\pm$ where $\phi = \rho$ by the usual transformation $\lambda x^\pm = e^{\pm \lambda \sigma^\pm}$. Then the asymptotics of the differential equation for $\epsilon$ implies

$$\epsilon = \frac{c_0}{\sqrt{2}} \left( \frac{1}{1} \right) + \frac{c_1}{\sqrt{2}} \left( \frac{1}{1} \right) e^{-\lambda \sigma} + O(e^{-2\lambda \sigma})$$

(4.9)

where $c_0$ and $c_1$ may depend on $\tau$ and are the two functions of integration. Note that this implies $(\nabla \phi - \lambda) \epsilon |_{\sigma = +\infty} = 0$ which ensures that $\mathcal{M}(\tau, \sigma = +\infty)$ and hence $M$ do not diverge for configurations asymptotic to the LDV. Given $c_0$ and $c_1$, the differential equation completely determines $\epsilon$, and in particular its asymptotics as $\sigma \to -\infty$.

For the LDV, it is easy to solve the $\epsilon$-differential equation exactly and show that

$$M_{\text{LDV}} = 0$$

(4.10)

independent of the choice of the functions of integration $c_0$ and $c_1$. It is worthwhile noting however, that unless $c_1 = 0$ the total energy for the LDV receives contributions from both ends of $\Sigma$ (which cancel each other). One sees that one has to take carefully into account $\mathcal{M}(\tau, \sigma = -\infty)$ as well as the subleading term $(\sim O(e^{-\lambda \sigma}))$ in $\epsilon$ when evaluating $\mathcal{M}(\tau, \sigma = +\infty)$.

For the general LDV-asymptotic configuration (4.8) we have to fix the functions of integration $c_0$ and $c_1$. We first impose

$$\bar{\epsilon} \gamma_5 \epsilon |_{\sigma = \infty} = 1.$$  

(4.11)

which fixes $c_0 = 1$. The other function of integration $c_1$ is determined by requiring

$$\lim_{\sigma \to -\infty} e^{-\phi} (\nabla \phi - \lambda) \epsilon = 0.$$  

(4.12)

Indeed, this fixes the subleading term from the expansion (4.9) of $\epsilon$, since the leading term vanishes automatically by the differential equation. Equation (4.12) determines $c_1$ as $c_1 = \frac{i}{2\pi} c_0 = \frac{i}{2\pi}$. Remark, that an alternative choice would be to replace (4.12) by the following condition at $\sigma = -\infty$: $(\nabla \phi - \lambda) \epsilon |_{\sigma = -\infty} = 0$. Then $\mathcal{M}(\tau, \sigma = -\infty) = 0$ and $M = \mathcal{M}(\tau, \sigma = +\infty)$. On the other hand, $c_1$ then has to be obtained by solving the $\epsilon$-differential equation for all $\sigma$. This type of approach will be used when we compute the Bondi-mass, but it could also be carried out for the present discussion of the ADM-mass.
With our choice (4.11) and (4.12) we obtain

\[ \mathcal{M}(\tau, \sigma = +\infty) = 2\lambda(a_1^2 - a_2) + \frac{1}{\lambda}(a_1^2 - \lambda^2 a_1^2), \]
\[ -\mathcal{M}(\tau, \sigma = -\infty) = \frac{\kappa}{2} \lambda \gamma(1 + \Gamma_1)\epsilon|_{\sigma = -\infty} \geq 0. \]  

(4.13)

We note that, by equation (4.9) the terms \( O(e^{-2\lambda \sigma}) \) in the asymptotic expansion of \( \epsilon \) do not contribute to \( \mathcal{M}(\tau, \sigma = +\infty) \). Using (4.8) we get

\[ \frac{d}{d\tau} \mathcal{M}(\tau, \sigma = +\infty) = 0. \]

(4.14)

Thus we find that at least the contribution from \( \sigma = +\infty \) does not depend on time.

Let us compare (4.13) with the expression for the “true” total energy we derived in the previous section: \( M_{\text{true}} = \lim_{\sigma \to -\infty} 2e^{2\lambda \sigma}(\delta \sigma + \lambda)(\delta \phi - \delta \phi^2) \). From (4.8) we have \( M_{\text{true}} = 2\lambda(a_1^2 - a_2) \). This differs from \( \mathcal{M}(\tau, \sigma = +\infty) \) only by \( \Delta = \frac{1}{\lambda}(a_1^2 - \lambda^2 a_1^2) \). Now by eq. (4.8),

\[ a_1 = a_e^{\lambda \tau} + a_\omega e^{-\lambda \tau}, \]

so that \( \Delta = -4\lambda a_e a_\omega \) is a constant. In many situations, due to the initial conditions, either \( a_+ \) or \( a_- \) vanishes and so does \( \Delta \). If this is the case, \( \mathcal{M}(\tau, \sigma = +\infty) = M_{\text{true}} \).

More generally we have proven that

\[ 0 \leq M = M_{\text{true}} + \Delta + \frac{\kappa \lambda}{2} \epsilon(-\infty) \gamma(1 + \Gamma_1)\epsilon(-\infty). \]

(4.15)

If \( \Delta \neq 0 \) this is not of much use to prove positivity properties of \( M_{\text{true}} \), but if one can show on general grounds that \( \Delta \) has to vanish, one has \( M_{\text{true}} \geq M_{\text{min}} \) where \( M_{\text{min}} = -\frac{\kappa \lambda}{2} \epsilon(-\infty) \gamma(1 + \Gamma_1)\epsilon(-\infty) \) is some “slightly” (i.e. naively of order \( \kappa \lambda \)) negative amount of energy. This suggests the interpretation that the true total energy is non-negative except for some amount \( O(\kappa \lambda) \), due to quantum effects, which is bounded by \( M_{\text{min}} \).

\( \Sigma \) a null line of constant \( \sigma^- = \tau - \sigma \): Bondi-mass

On can solve the \( \epsilon \)-differential equation exactly in terms of integrals of functions of \( \phi \) and \( \rho \). We have in mind to study the shock-wave scenario or any other scenario with \( T_{++}^M \neq 0 \) over a finite interval in \( \sigma^+ \) only, where we have the exact LDV for all \( \sigma^+ < \sigma_0^+ \) with some finite \( \sigma_0^+ \).

Hence it is convenient to fix the initial conditions for \( \epsilon \) in the LDV region or on its boundary
at \( \sigma^+ = \sigma_0^+ \) (where \( \rho = 0 \)):

\[
\epsilon(\sigma^+\sigma^-) = \begin{pmatrix} d_1(\sigma^-) \\ d_2(\sigma^-) \end{pmatrix}
\]

(4.16)

i.e. \( d_1(\sigma^-) \) and \( d_2(\sigma^-) \) are our functions of integration. Then the asymptotics as \( \sigma^+ \to -\infty \) are relatively easy to obtain, while those for \( \sigma^+ \to \infty \) are more involved. We will not give any details here but refer the interested reader to [10]. The outcome of the computation is that the Bondi-mass equals

\[
M_B(\sigma^-) = M(\sigma^-,\sigma^+ = +\infty) - M(\sigma^-,\sigma^+ = -\infty)
\]

\[
= \frac{2d^2(\sigma^-)}{1 - \frac{\lambda}{4} e^{\lambda \sigma^-}} \left[ m + \frac{\kappa}{4} \lambda \log \left( 1 - \frac{p}{\lambda} e^{\lambda \sigma^-} \right) + \frac{\kappa}{4} p e^{\lambda \sigma^-} \right]
\]

(4.17)

\[- \lambda e^{-\lambda \sigma^-} L^2(\sigma^-) + 2(d_1 + d_2)^2 \lambda \left( e^{\lambda (\sigma_0^+ - \sigma^-)} + \frac{\kappa}{4} \right), \]

where \( m \) and \( p \) are the total energy and momentum carried by the infalling matter. (The shock-wave corresponds to \( p = a \) and \( m = ae^{\lambda \sigma_0^+} = a\lambda x_0^+ \) in the usual notation.) The last term in (4.17) is the contribution from \( \sigma^+ = -\infty \). The function \( L(\sigma^-) \) is determined by the asymptotics of \( \epsilon \) as \( \sigma^+ \to \infty \).

\[
\epsilon = -e^{a_0} d_2 \left( e^{\frac{1}{2} a_0} \right) \left( -e^{-\frac{1}{2} a_0} \right) + \frac{1}{\sqrt{2}} e^{\frac{3}{2} a_0} L \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\frac{3}{2} \sigma^+} + O \left( e^{-\lambda \sigma^+} \right).
\]

(4.18)

where \( a_0 = \lim_{\sigma^+ \to \infty} \rho \). The function \( L(\sigma^-) \) is given explicitly by some integral of functions of \( \phi \) and \( \rho \) [10]. In general we do not have an explicit expression for \( \phi, \rho \) at our disposal since they are given by solving the transcendental relation between \( \phi, \rho \) and \( \Omega, \chi \). Thus in the generic case one has to evaluate the integrals numerically. There are however certain cases where one can still evaluate \( L \) analytically (e.g., as an expansion in \( e^{\lambda \sigma^-} \) as \( \sigma^- \to -\infty \)).

Again, we have to fix the functions of integration \( d_1(\sigma^-) \) and \( d_2(\sigma^-) \) before we can try to interpret \( M_B \) as Bondi-mass. Rather than fixing those functions “by hand” and then deducing the properties of \( M_B \), we will impose a few physical requirements that will actually fix the two functions almost completely. We require

1. for \( \kappa = 0 \) : \( M_B = m \). Indeed, in the limit \( \kappa \to 0 \), the model under consideration has no Hawking radiation and the Bondi-mass should remain constant and equal to the total energy carried by the in-flux of matter.
2. as \( \sigma^- \to -\infty : M_B \to m \). Again, at \( \sigma^- = -\infty \) no Hawking radiation yet had a chance to occur, and the Bondi-mass must equal \( m \).

3. “causality” : The solution \( \epsilon \) in the LDV region shall not depend on the position and strength of the infalling matter that is in its causal future.

The last requirement implies \( d_1 + d_2 = 0 \) as can be seen from the explicit solution for \( \epsilon \) for \( \sigma^+ < \sigma_0^+ \). The second requirement gives \( d_2(\infty) = 1/2 \), while the first gives \( d_2^2(\sigma^-) = 1/2 + \mathcal{O}(\kappa) \). This suggests to take \( d_2 = 1/2 \) although it does not exclude a choice like \( d_2^2 = 1/2 + \mathcal{O}(\kappa e^{\lambda \sigma}) \). The latter complication however seems physically unmotivated and we discard it. Thus we are lead to take

\[
d_1 + d_2 = 0 , \quad d_1^2 = d_2^2 = \frac{1}{2} , \quad \forall \sigma^- ,
\]

so that in the LDV region \( \epsilon = \pm \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \). It is remarkable that we can satisfy the above requirements with such a simple choice of \( d_1, d_2 \). The conditions (4.19) can be written more elegantly as

\[
e^{-2\rho \varepsilon \gamma_5 \epsilon}_{\sigma^+ = +\infty} = -1 \quad \text{(4.20)}
\]

which fixes \( d_2 = \frac{1}{2} \), and (note that the l.h.s. is taken at \( \sigma^+ = -\infty \), and not at \( \sigma^+ = +\infty \))

\[
(\nabla \phi - \lambda) \epsilon |_{\sigma^+ = -\infty} = 0 \quad \text{(4.21)}
\]

which fixes \( d_1 + d_2 = 0 \). With this choice, \( \mathcal{M}(\sigma^- , \sigma^+ = -\infty) = 0 \), and

\[
M_B(\sigma^-) = \frac{1}{1 - \rho e^{\lambda \sigma^-}} \left[ m + \frac{\kappa}{4} \lambda \log \left( 1 - \frac{\rho}{\lambda} e^{\lambda \sigma^-} \right) + \frac{\kappa}{4} \rho e^{\lambda \sigma^-} \right] - \lambda e^{-\lambda \sigma^-} L^2(\sigma^-) .
\]

We can evaluate \( L(\sigma^-) \) and hence \( M_B(\sigma^-) \) for large negative but finite \( \sigma^- \). For the shockwave scenario \( (\rho e^{\lambda \sigma^-} = m) \), we arrive at

\[
M_B(\sigma^-) \sim m - \kappa \left( m I_{1/2} + \frac{\lambda \kappa}{4} I_{1/2}^2 \right) e^{-\lambda \sigma^-} e^{\lambda \sigma^-} + \mathcal{O}(e^{2\lambda \sigma^-}) ,
\]

where \( I_{1/2} \) is a logarithm integral given by \( I_{1/2} = \int_0^{\infty} dx \frac{e^{-x}}{x + \frac{\rho}{\lambda} m} = -e^{\frac{\rho}{\lambda} m} \text{li} \left( e^{-\frac{\rho}{\lambda} m} \right) > 0 \). Let us comment on this equation. First, as already observed, \( M_B \) is constant for \( \kappa = 0 \): semiclassically
there is no Hawking radiation for $\kappa = 0$. Second, $M_B$ is decreasing as $\sigma^{-}$ (i.e. time) increases (at least to the first order in $e^{\lambda \sigma^{-}}$ we computed): Hawking radiation carries energy away from the black hole.* Note that for $\frac{\lambda \kappa}{m} << 1$ the leading term in (4.23) reads $M_B \sim m - \kappa m \log \left( \frac{2m}{\lambda \kappa} \right) e^{-\lambda \sigma^{-}} e^{\lambda \sigma^{-}}$. This differs from the CGHS prediction for the very early Hawking radiation by the extra factor of $\log \left( \frac{2m}{\lambda \kappa} \right)$. However, there is nothing wrong with this difference, since the RST and CGHS models represent different $O(\kappa)$ corrections to the same classical dilaton-gravity.

Finally we give $M_B(\sigma^{-})$ for the shock-wave scenario at $\sigma^{-} = \sigma_{s}^{-}$, the point where the singularity and apparent horizon intersect, and the solution is matched to the LDV. As shown in ref. [10], $M_B(\sigma_{s}^{-})$ can be computed as a series in $\frac{m}{\lambda \kappa}$. It was found that to first order in $\frac{m}{\lambda \kappa}$, we simply have

$$M_B(\sigma_{s}^{-}) \sim m + O\left( \frac{m}{\lambda \kappa} \right)^2. \quad (4.24)$$

Thus if we start with a very small black hole (small $m$) or a very large number of matter fields (large $\kappa$), the black hole is matched to the shifted LDV before any substantial Hawking radiation has occurred: its mass is still the initial mass $m$ up to $O\left( \left( \frac{m}{\lambda \kappa} \right)^2 \right)$ corrections. This positive amount of energy must then be sent off by the thunderpop. In ref. 5, RST find (up to their sign ambiguity) that the thunderpop carries energy $\frac{\lambda \kappa}{4} \left( 1 - e^{-\frac{\lambda \kappa}{m}} \right) = m + O\left( \left( \frac{m}{\lambda \kappa} \right)^2 \right)$ in agreement with (4.24).

In conclusion, we have found that our functional $M$ as given by (4.3) with $\epsilon$ subject to the differential equation (4.5) and the boundary conditions (4.20) and (4.21) defines a satisfactory Bondi-mass: it is non-negative, equals the ADM-mass $m$ at $\sigma^{-} = -\infty$, decreases for $\kappa > 0$ and is constant for $\kappa = 0$. It also gives correctly the energy of the thunderpop (at least to the order we computed), and for $\sigma^{-} \to -\infty$ has an expansion in $\kappa e^{\lambda \sigma^{-}}$ as expected.

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* Recall that the we assume $\kappa \geq 0$. 
5. Conclusions

There is a whole class of exact conformal 2D dilaton gravity theories (differing by $\mathcal{O}(\epsilon^2\phi)$ terms) that all have an action $S = \frac{1}{2} \int \left[ \kappa \partial_+ H \partial_- \Omega - \kappa \partial_+ H \partial_- \chi + \lambda^2 e^{2(\chi - \Omega)} \right]$. The corresponding equations of motion are exactly solvable. There is no Jacobian in a $\Omega, \chi$ functional integral formulation.

These theories have (semiclassical) supersymmetric extensions. If one insists on exact superconformal invariance, the bosonic part is not exactly conformal, and vice versa.

The Regge-Teitelboim method gives a well-defined (constant) total ADM-energy (true energy).

We can prove a positive energy theorem. The positive energy however differs from the previous true energy by an $\mathcal{O}(\kappa)$-term. The Bondi energy can be defined to satisfy all reasonable physical requirements.

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17. These remarks are based on discussions with I. Kogan.