NOTES ON $WGL_n$-ALGEBRAS AND QUANTUM MIURA TRANSFORMATION.

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ABSTRACT

We start from the quantum Miura transformation [7] for the $\hat{W}$-algebra associated with $GL(n)$ group and find an evident formula for quantum $L$-operator as well as for the action of $\hat{W}_l$ currents ($l=1,\ldots,n$) on elements of the completely degenerated $n$-dimensional representation. Quantum formulae are obtained through the deformation of the pseudodifferential symbols. This deformation is independent of $n$ and preserves Adler's trace. Our main instrument of the proof is the notation of pseudodifferential symbol with right action which has no counterpart in classical theory.

1. Introduction

W-algebras orginally introduced by Zamolodchikov [1] as a generalization of the Virasoro algebra in context of two-dimensional Conformal Field Theory [4] turned out to be extremely interesting object. In the past few years considerable progress has been made in an understanding of the deep structures underlying these algebras (see for example refs. [2,5-15]) as well as its classical limits [17-20,24,25].

In this work we want to consider an aspect of the W-algebras (associated with general linear group) which seem to have been overlooked in the literature. In spite of the number of results referring to the $W_n$-algebras with an arbitrary $n$ several principal questions still remain to be answered. It is well-known that classical $\hat{W}$-algebras associated with general linear group are isomorphic to the Poisson algebras of functionals on the manifold of the linear differential operators (with second Gelfand-Dikii bracket, as Poisson bracket) [3]. Other $\hat{W}$-algebras ($\hat{W}_A$, $\hat{W}_B$, $\hat{W}_C$, $\hat{W}_D$) can be considered as certain reductions of $WGL$-algebra [16]. This representation in terms of (pseudo)differential operators is very useful. However, at first glance it would seem that classical theory loses many of its attractive features under quantization. The reason is the lack of any suitable for work formulae for $L$-operator, Lie brackets of the generators etc.

Developing the work by Lukyanov and Fateev [6] we show that there is exist simple

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deformation of (pseudo)differential operators which reproduces formulae for quantum Miura transformation, L-operator and the action of \( W \)-algebra on some invariant operators (from the kernel of L-operator). We hope that in the case of \( \hat{W} \)-algebras associated with other then \( GL \)-group [8] the similar formulae still exist.

The paper is organized as follows. In sect.2 we review some necessary facts from classical theory and derive an explicit formula for \( \hat{W}_n \) algebra action on the Bloch’s solutions of the equation

\[
L\Psi(x) = \left( \sum_{i=1}^{n} W_i(x) \partial^{n-i} \right) \Psi(x) = 0. \tag{1.1}
\]

Our calculation of well-known formula (2.7) in terms of Darboux variables admit simple quantum generalization. In sect.3 we discuss the main difficulties appeared in the quantum case and give the direct calculation of the quantum \( W_i \) \( (l = 2, 3, 4) \) currents action on the highest weight operator \( F_1 = \exp(\phi_1) \) of the \( n \)-dimensional completely degenerated representation of \( WGL_n \). We represent explicit formula for quantum L-operator for \( n = 2, 3, 4 \) cases. In sect.4 we introduce the formalism of right derivations and bi-pseudodifferential operators. Quantum Miura transformation is considered then as the equality of two bi-differential operators. We propose some deformation of the (pseudo)differential operators \( \hat{Y} \) and \( \hat{L} \) preserving Adler’s trace of the product \( \hat{Y} \hat{L} \). In sect.5 we give general proof for an arbitrary \( n \) that such deformations of the differential operators correspond to the quantum \( \hat{W}_n \)-algebras. We work essentially with highest weight operator but all formulae are valid for any of the \( n \) invariant operators obtained by the action of screening operators on \( F_1 \) such as laters represent nilpotent part of quantum group \( U_q(gl(n)) \) which commutes with an current algebra [11,12].

We finally remark that some of the results of this paper were published (without proof) in [23].

2. CLASSICAL CASE.

Let us consider \( WGL(n) \)-invariant classical 2D field theory on the circle. Introduce scalar fields (Darboux variables) \( \phi^b \) with expansion

\[
\phi^b(x) = q^b + p^b \ln(x) + \sum_{m \neq 0} \frac{a_{-m}^b x^m}{im}, \tag{2.1}
\]

\( x = \exp(i\sigma); \sigma \in [\sigma_0, \sigma_0 + 2\pi]; \) \( b = 1, \ldots, n \) and Poisson brackets:

\[
\{ \phi^a(x), \phi^b(y) \} = \pi i n \delta^n(x-y)\delta^{ab},
\]

\[
\{ a^a_m, a^b_n \} = n \delta_{m+n,0} \delta^{bc}; \{ q^a, p^b \} = 2\pi i \delta^{ab}; \{ q^a, a^b_m \} = \{ p^a, a^b_m \} = 0. \tag{2.2}
\]

Then classical \( W \)-currents are expressed via Miura transformation which for the groups \( G = GL(n), (SL(n)) \) has the form [3,16]:

\[
L(x) = \partial^n + \sum_{l=1}^{n} W_l(x) \partial^{n-l} = \prod_{l=1}^{n} \left( \partial - \partial^n L(x) \right) \tag{2.3}
\]
where $\vec{a}_i$ and $\vec{h}_i = \vec{a}_1 - \vec{a}_1 - \ldots - \vec{a}_{i-1}$ are correspondingly positive roots of the Lie algebra $\mathcal{G}$ and fundamental weights of the $n$-dimensional vector representation of $GL(n), SL(n)$. If we consider $n$-order linear differential equation

$$L\psi = 0,$$

then the elements of Bloch’s wave basis $\{\Psi_i(x); i = 1, \ldots, n\}$ of (2.4) is the component of the vector $\vec{\Psi}$ associated with the vector $n$-dimensional representation of $WGL(n)(WSL(n))$ [6].

$$\Psi_i(x) = \prod_{\lambda=1}^{n-1} \lambda_i (\lambda_i - \lambda_{i-1})^{-1} \exp(\omega_1 \phi_1(x)) \int_{\mathbb{R} \eta_1}^{\eta_{i-1} + (\omega_1 + \omega_2)} d\eta_1 \ldots \int_{\mathbb{R} \eta_{k-2} + (\omega_1 + \omega_2)}^{\eta_{k-1} + (\omega_1 + \omega_2)} d\eta_{k-1} \cdot \exp(-\vec{a}_1 \vec{\phi}(\eta_1)) \ldots \exp(-\vec{a}_{k-1} \vec{\phi}(\eta_{k-1})).$$

Functions $\Psi_i(z)$ satisfy the periodicity condition $\Psi_i(\sigma_0 + 2\pi) = \lambda_i \Psi_i(\sigma_0)$ with $\lambda_i = \exp(i \vec{h}_i)$ and choice of the initial point $x(\sigma_0)$ is unimportant.

In the case of $GL(n)$ group vectors $\vec{h}_i$ constitute the orthogonal basis in the $n$-dimensional vector space: $<\vec{h}_i, \vec{h}_j> = \delta_{ij}$, $\vec{a}(z)\vec{h}_i = \vec{a}_i(z)$.

Let us remind the pseudo differential operator definition:

$$[\partial^{i}, \partial^{j}] = 0, \quad [\partial^{i}, a(x)] = \sum_{p=1}^{\infty} \frac{i(i-1)\ldots(i-p+1)}{p!} a^{(p)}(x)\partial^{i-p}, \quad i, j \in \mathbb{Z}.$$

Given a pseudo differential operator $P = \sum p_i \partial^i$, we define its non-commutative residus and Adler’s trace (14) as:

$$\text{res}(\sum p_i \partial^i) = p_{-1}, \quad \text{Tr}(\sum p_i \partial^i) = \int \frac{dx}{2\pi i} \text{res}(\sum p_i \partial^i).$$

Taking into account eqs. (2.2) and (2.3) it is not difficult to show that the Poisson bracket of the hamiltonian

$$H = \text{Tr}(Y L) = \sum_{i=0}^{n-1} \int \frac{dx}{2\pi i} a_i(x)\Psi_{n-i}(x),$$

$$Y = \sum_{i=0}^{n-1} \partial^{i-1} a_i$$

with $\Psi_i$ (i.e. $W$-algebra action on $\Psi_i$) is given by

$$\delta_H \Psi_j = \{I, \Psi_j\} = -(Y L)_{+} \Psi_j.$$

As usual, $(Y L)_{+}$ stands for the differential part of the pseudo differential operator which is polynomial in $\partial$ (including free term). Really, it is enough to prove (2.7) for $\Psi = \exp(\phi_1)$ and the transformation properties of other fields will be the same. Denoting

$$L(\partial - \phi_1')^{-1} = (\partial - \phi_1')\ldots(\partial - \phi_2')$$

(2.8)

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and using that
\[(\partial - \phi_1')^{-1} = \exp(\phi_1)\partial^{-1} \exp(-\phi_1) \quad (2.9)\]
one can immediately obtain:
\[
\delta_H(\exp \phi_1)(z) = \oint \frac{dk}{2\pi i} \{ \text{res} [Y(z)L(z)], \exp \phi_1(z) \} =
= -\oint \frac{dk}{2\pi i} \text{res} [Y(z)L(z)(\partial - \phi_1(z))^{-1} \delta(z - \zeta)] \exp(\phi_1(z)) =
= -\oint \frac{dk}{2\pi i} \text{res} \{ Y(z)L(z) \exp(\phi_1(z))\partial^{-1} \exp(-\phi_1(z))\delta(z - \zeta) \exp(\phi_1(z)) \} =
= -\text{res} \{ Y(z)L(z) \exp(\phi_1(z))\partial_z^{-1} \} = -(YL)_+ \exp(\phi_1(z)) \quad Q.E.D.
\quad (2.10)

3. QUANTUM CASE

3.1 Basic Definitions. ([6]).

In the quantum case we shall assume that all operators are defined on \( \{ C \setminus \{ 0 \} \}. \) Let \( T \) be the notation of the operator product \( T \):
\[
T(\text{A}(\zeta) \text{B}(z)) = \{ \text{singular terms } \} + \text{A}(z)\text{B}(z) :.
\quad (3.1)
\]

As is well-known, singular terms under \( z \rightarrow \zeta \) determine commutation relation for generators of the fields \([4].\) The Poisson brackets (2.2) for the Darboux's variables \( \phi^a(\eta = 1, \ldots, n) \) (2.1) correspond to the operator product of the free fields:
\[
T(\phi^a(\zeta)\phi^b(z)) = \frac{\delta_{ab}}{\kappa} \log(\zeta - z) + O(\zeta - z),
\quad (3.2)
\]
or, equivalently:
\[
[a^a_m, a^b_i] = \frac{i}{\kappa} \delta_{m+i, 0} \delta^{ab}, \quad [p^a, q^b] = \frac{2\pi i}{\kappa} \delta_{ab}
\quad (3.3)
\]

We can consider \( \kappa \) as the deformation parameter. Then \( W \)-currents are defined via quantum Miura transformation \([5]:\)
\[
(\alpha_0 \partial)^n + \sum W_i(z)(\alpha_0 \partial)^{n-i} = : \prod_{l=1}^n (\alpha_0 \partial - \overline{h}_i \overline{\phi}(z)) :,
\quad (3.4)
\]

where \( \alpha_0 \) is some constant. It was proved in ref.[3] that quantum operators in the left hand side of (3.4) form the associative quadratic algebra. Demanding that vertex operators \( \exp(\alpha_1^a \overline{\phi}) \) has conformal dimension \( \Delta = 1 \) in the \( SL(n) \)-theory yields \([4]:\)
\[
\alpha_0 = 1 - \frac{1}{\kappa}.
\quad (3.5)
\]
Now, let us consider the n-dimensional completely degenerated highest weight representation of $\tilde{W}$ algebra determined by highest weight operator

$$F_i(\tilde{\omega}_i|z) =: \exp(\tilde{\omega}_i \tilde{\phi}) :.$$  

The representation space is covered by the following $n$ linearly independent operators:

$$F_i(\tilde{\omega}_i|z) = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_i - \lambda_j} \int_{C_i} \frac{d\eta_1}{2\pi i} \cdots \int_{C_n} \frac{d\eta_n}{2\pi i} \exp(-\tilde{\alpha}_i \tilde{\phi}(\eta_i)) \cdots \exp(-\tilde{\alpha}_i \tilde{\phi}(\eta_n)) =: \exp(\tilde{\omega}_i \tilde{\phi}(\eta_i)) :.$$  

(3.6)

The integration contours in this formula are chosen in accordance with the standard Felder’s prescription (surrounding all singular points [22]).

### 3.2 W-algebra action.

Let us define the action of $W$-currents on local fields of the theory as [10]:

$$\delta_{H_Y} S = k \oint_C \frac{dz}{2\pi i} \sum_{i=1}^{n} a_{n-i}(z) T(W_i(z)) S,$$  

(3.7)

where Hamiltonian $H_Y$ has the form:

$$H_Y = \oint_C \frac{dz}{2\pi i} \sum_{i=1}^{n} a_{n-i}(z) W_i(z) = Tr(YL),$$  

and integration contour $C$ surrounds all singular points of integrand. The action of nilpotent part of quantum group $U_q(gl(n))$ (for $sl(n)$) represented by screening factor has to be commutative with the action of W-algebra (this property can be considered as the definition of $\tilde{W}$ algebra [11,12]). Indeed, it was proved in ref.[5] that

$$T(W_i(z)) \exp(-\tilde{\alpha}_i \tilde{\phi}(\zeta)) = \frac{\partial}{\partial \zeta}X_i(z, \zeta) + o(1),$$  

where $X_i$ is some local operator. Therefore we can afford to ignore screening operators which do not change the transformation properties of the fields. In this section we will consider only the highest weight operator which for $WGL$ has the form $F_1(\tilde{\omega}_1|z) = F_1(z) =: \exp(\tilde{\phi}_1(z)) :$. But all formulae will be valid for any invariant operator $F_i$.

In analogy with classical case we will first evaluate the $\tilde{W}$-algebra action on the operator $F_1$. Using quantum Miura transformation, one can express $W_i$ as normal ordered differential polynomials in terms of the free fields $\tilde{\phi}_a$ and then perform Vick's
pairing with $F_1(z)$ (only one in $GL(n)$ case). Plugging (3.4) into (3.7) after some simple manipulations one finds:

$$
\delta_H F_1(z) = k \oint \frac{d\zeta}{2\pi i} a_{n-i}(\zeta) T(W_i(\zeta)F_1(z))
= \exp(\phi_1 - \frac{\phi_1}{\alpha_0})(\hat{Y} \hat{L})_+ \exp(\frac{\phi_1}{\alpha_0}) : \ ,
$$

(3.8)

where the notation

$$
\hat{Y} = \sum (\alpha_0 \partial)^{-i-1} a_i;
\hat{L} = (\alpha_0 \partial)^n + \sum_i W_i^\phi (\alpha_0 \partial)^{-i} = : \prod_i (\alpha_0 \partial - \phi_i') : \ ,
$$

is introduced. Note that expressions like $\exp(\phi_1(1 - \frac{1}{\alpha_0}))(W_i^\phi (\alpha_0 \partial)^{\phi}) \exp(\frac{\phi_1}{\alpha_0})$ are rather formal. We need to provide all differentiations and then turn to the form $Pol(\phi)F_1 :$, where symbol Pol stands for some differential polynomial in $\phi$. For example, term $\exp(\phi_1(1 - \frac{1}{\alpha_0}))(\alpha_0 \partial)^2 \exp(\frac{\phi_1}{\alpha_0})$ has to be understood as $[(\alpha_0 \partial^2 \phi_1 + (\partial \phi_1)^2)F_1]$ and so on. It is significant that the notation $'$ in (3.8) implies the ordering of $\phi$-fields only. However this is not yet the whole story. To provide the ordering of $W_i$-fields we must change polynomials of $\phi$'s in (3.8) (which formally correspond to $W$-currents) by the first non-vanishing terms of its operator product with $F_1 = \exp(\phi_1) :$. Let us denote coefficient under $W_i^\phi (\alpha_0 \partial)^{\phi}$ in the formula (2.7) as $C_i^\phi$. Clearly, that $C_i^\phi$ after such corrections will change on some value $\delta C_i^\phi$. The difficulty in determination of these quantum corrections is that non-vanishing terms in $T((W_i^\phi (\alpha_0 \partial)^{\phi})F_1^{\phi'})$ include formal differential polynomials in the $\phi$ of the form $(W_i^\phi (\alpha_0 \partial)^{\phi})^{(i-1)} \partial^{\phi'}$, $(W_i^\phi (\alpha_0 \partial)^{\phi})^{(i-1)} \partial^{\phi - 1}$, $(W_i^{\phi_{i-1}} (\alpha_0 \partial)^{\phi})^{(i-1)} \partial^{\phi'}$, ... The later need to be ordered too and so on.

### 3.3 Examples.

To illustrate the general procedure of ordering, let us now consider examples of $W_2$, $W_3$ and $W_4$-current's action on $F_1$. At first we express $W_1, W_2, W_3$-currents through the free fields using quantum Miura transformation:

$$
W_1 = - \sum_{i=1}^n \phi_i' ,
$$

$$
W_2^\phi = \sum_{i<j} \phi_i' \phi_j' - \sum_{i} (n - i) \alpha_0 \phi_i'' ,
$$

$$
W_3^\phi = - \sum_{i<j<k} \phi_i' \phi_j' \phi_k' + \sum_{i<j} (n - j) \alpha_0 \phi_i'' \phi_j' + (n - i - 1) \phi_i' \phi_i'' - \sum_{i} \frac{(n - i)(n - i - 1)}{2} \alpha_0^2 \phi_i''' ,
$$

(3.9)
From this representation one can immediately find transformation of highest weight operator under the action of \( W_2 \)-current:

\[
T(W_2(\zeta)F_1(z)) = \sum_{i>1} \left[ \frac{1}{k(\zeta - z)} \delta_i(\zeta) - \frac{1}{k(\zeta - z)} a_0(n - i) \right],
\]

(3.10)

\[
\delta_{H_2} F_1 = k \oint \frac{d\zeta}{2\pi i} a_{n-2}(\zeta) T(W_2(\zeta) : \exp(\phi_1(z))) := \nonumber \]

\[= - : [a_0(1-n)a_n + a_{n-2}(W_1 + \partial)] F_1 : .
\]

(3.11)

We have no need for additional ordering procedure because the expression \( W_1 F_1 : \)

coincides with \( W_1^0 F_1 : \). First non-trivial example in which the ordering problem arises

is the \( W_3 \)-current action. As mentioned above, the normal ordering after Vick's pairing

implies the ordering of the free fields only. While \( W_2^0 \)-current which appeared in the

expansion have to be ordered under the scheme:

\[
: W_2^0(z) F_1(z) : = \oint \frac{d\zeta}{2\pi i} \frac{T(W_2(\zeta)F_1(z))}{(\zeta - z)}
\]

It follows from the operator expansion (3.10) that:

\[
: W_2 F_1 := \oint \frac{d\zeta}{2\pi i} \frac{T(W_2(\zeta)F_1(z))}{(\zeta - z)} = : (W_2^0 - \frac{1}{k} W_1^0 - \frac{1}{k} \phi^0) F_1 :
\]

(3.12)

By definition, the action of \( W_3 \) current on the regualrized exponent has the form:

\[
\delta_{H_3} [F_1(z)] = \oint \frac{d\zeta}{2\pi i} a_{n-2}(\zeta) T(W_3(\zeta) : \exp(\phi_1(z))) :
\]

(3.13a)

where \( T \)-product of \( W_3 \) with \( F_1 \) after Vick's pairing and expansion under the powers of

\( (\zeta - z) \) is given by:

\[
T(W_3(\zeta)F_1(z)) = \frac{1}{k} \left[ - \frac{1}{(\zeta - z)^3} (n - 2)(n - 1)a_0^2 + \right.
\]

\[+ \frac{1}{(\zeta - z)^2} (n - 2)a_0 (W_1(z) + \phi_1(z)) + \frac{1}{(\zeta - z)} [- (W_2^0(z) -
\]

\[- \frac{1}{k} W_1^0(z) - \frac{1}{k} \phi_1^0(z)] - W_1(z) \phi_1^0(z) - (\phi_1^0(z) + \phi_1^{2}(z)) +
\]

\[+ ((n - 2)a_0 - \frac{1}{k}) W_1^0(z) F_1(z) : + \{ \text{regular terms} \} .
\]

(3.13b)

Now substituting eqs. (3.12), (3.14) into (3.13b) we obtain after the contour integration:

\[
\delta_{H_3} [F_1(z)] =: \left[ \frac{(n - 2)(n - 1)}{2} a_0^2 a_{n-2}(z) + (n - 2)a_0 a_{n-3}(z)[W_1(z) + \partial_z] +
\]

\[a_{n-3}(z)[- W_2(z) - W_1(z) \partial_z - \partial_z^2 + ((n - 2)a_0 - \frac{1}{k}) W_1^0(z) F_1(z) : .
\]

(3.14)
Analogous but slightly tedious computation for $W_4$-current yields:

$$\delta H_4[F_1(z)] = \oint \frac{dz}{2\pi i} \alpha_{n-4}(\zeta)T(W_4(\zeta) : \exp(\phi_1(z)) : ) =$$

$$= \left( - \frac{(n-3)(n-2)(n-1)}{6} \alpha_0^3 \alpha_{n-4}{(W_1(z) + \partial)} + \frac{1}{2} \right)$$

$$+ (n-3)\alpha_0 \alpha_{n-3}[W_2 + W_1 \partial + \partial^2 - ((n-2)\alpha_0 - \frac{1}{k})W'_1]$$

$$+ \frac{1}{2}((n-3)\alpha_0 - \frac{1}{k})(n-2)\alpha_0 - \frac{1}{k})W'_2 - ((n-3)\alpha_0 - \frac{2}{k})W'_1 \partial +$$

$$\frac{1}{2}((n-3)\alpha_0 - \frac{1}{k})(n-2)\alpha_0 - \frac{1}{k})W'_2 - ((n-3)\alpha_0 - \frac{2}{k})W'_1 \partial +$$

(3.15)

where formulae for the following normal ordering expressions are used:

$$W_2(z)F_1(z)) := \oint \frac{d\zeta}{2\pi i} T(W_2(\zeta) : \exp(\phi_1(z)) : ) =$$

$$= : W_2^\phi - \frac{1}{2k} (W_{\partial} + \phi_{1''}) F_1 : ,$$

$$W_3(z)F_1(z)) := \oint \frac{d\zeta}{2\pi i} T(W_3(\zeta) : \phi_1(z) \exp(\phi_1(z)) : ) =$$

$$= : W_3^\phi - \frac{1}{2k} (2W_{\partial} + \phi_{1''} + 2\phi_{1'} + W)' F_1 : ,$$

$$W_4(z)F_1(z)) := \oint \frac{d\zeta}{2\pi i} T(W_4(\zeta) : \exp(\phi_1(z)) : ) =$$

$$= : (W_4^\phi - \frac{1}{k} (W_{\partial}^\phi - \frac{1}{2k} (W_{1''} + \phi_{1''})) +$$

$$+ \frac{n-2}{2} \alpha_0 + \frac{1}{2k} W_{1''} + W_{1'} \phi_{1'} + W_1 \phi_{1''} +$$

$$+ 2\phi_{1'} \phi_{1''} (n-3)\alpha_0 - \frac{1}{k}) F_1(z) : ,$$

(3.16)

One can see, for example, that quantum correction $\delta C_{1,3}^{0,3}$ depends not only from the ordering of $W_2^\phi F_1$, $W_3^\phi F_1$ and $W_4^\phi F_1$ terms appeared after Vick’s pairing but also from the ordering of $W_\phi^\phi F_1$-current in the eq.(3.15c). It is clear that for large $n$ direct procedure of finding $\delta C_{1,3}^{0,3}$ becomes more involved.

To find quantum $L$-operator one can determine quantum corrections to the expressions of the form $W_1^{\phi}F_1^{(n-3)}$. Explicit calculations implies that for small $n$ the sum of corrections to the expression $[(\alpha_0 \partial)^n + \sum W_1^{\phi}(z) (\alpha_0 \partial)^{n-1}] F_1$ has the form:

$$: \sum_{p, q, i=0}^{\phi, \phi'} \delta C_{1}^{p, q} \phi_{1'}^{i} W_{1} F^{(p)} :,$$

where $\delta C_{1}^{p, q}$ have precisely the same value as
before. Let us give exact formulæ for small n:

\[ n = 2 : [W_2 + W_1 \partial + \frac{1}{k} W_1'' + \partial^2] F_1 := 0, \]

\[ n = 3 : [W_3 + \frac{1}{k} W_2 + \frac{1}{k^2} W_1'' + (W_2 + \frac{2}{k} W_1') \partial + W_1 \partial^2 + \partial^3] F_1 := 0, \]

\[ n = 4 : [W_4 + \frac{1}{k} W_3 + \frac{1}{k^2} W_2'' + \frac{1}{k^3} W_1'''] + (W_3 + \frac{2}{k} W_2' + \frac{1}{k^2} W_1'''') \partial + (W_2 + \frac{3}{k} W_1''') \partial^2 + \partial^3] F_1 := 0. \]  

(3.17)

Here we replace expressions for : W_3 F_1 : , : W_2 F_1 : , : W_1 F_1 : and : W_2 F_1 : using eqns. (3.12) and (3.16). Unfortunately, formulæ for : W_4 F_1 : , : W_3 F_1 : , : W_2 F_1 : etc. in the terms of free fields are too cumbersome to be represented here.

4. QUANTUM DEFORMATION OF OPERATORS.

4.1 Right derivations.

Let us now introduce the convenient notation of the right derivation \( \bar{\partial} \) and define the deformation of the classical operators \( Y \) and \( L \) preserving Adler’s trace of the product \( (Y L) \),

\[ f(x) \bar{\partial} = f'(x) + \bar{\partial} f(x), \]  

(4.1)

where right derivations commute with ordinary ones \( [\bar{\partial}, \partial] = [\bar{\partial}^{-1}, \partial] = 0 \). Define integral symbol \( \bar{\partial}^{-1} \) as:

\[ \bar{\partial}^{-1} \bar{\partial} = \bar{\partial} \bar{\partial}^{-1} = 1, [\bar{\partial}^{-1}, \partial] = [\bar{\partial}^{-1}, \bar{\partial}^{-1}] = 0, \]

\[ [f(x), \bar{\partial}^{-1}] = \sum_{p=1}^{\infty} \frac{i(i-1) \ldots (i-p+1)}{p!} \bar{\partial}^{-p} f(p)(x), \quad i, j \in \mathbb{Z}. \]

\[ (\partial + \frac{\bar{\partial}}{k})^{-l} = \partial^{-l} \left( 1 + \frac{\bar{\partial}^{-1}}{k} \right)^{-l} = \partial^{-l} - \frac{l}{k} \partial^{-l-1} \bar{\partial} + \frac{l(l-1)}{2k^2} \partial^{-l-2} \bar{\partial}^2 + \ldots \]  

(4.2)

Right Adler’s trace of the pseudodifferential symbol \( P = \sum \bar{\partial}^j f_j(x) \) like an ordinary one is given by

\[ rcs \bar{\partial}^j f_j(x) = f_{j-1}. \]

So we have two copies of derivatives. Such as left and right symbols commute to one another, there are no additional difficulties with definition of Adler’s trace for bi-pseudodifferential operators of the form \( \sum_{i,j} \bar{\partial}^i f_{i,j}(x) \partial^j \):

\[ rcs \left( \sum_{i,j} \bar{\partial}^i f_{i,j}(x) \partial^j \right) = \sum_i \bar{\partial}^i f_{i,-1}(x), \]
and
\[ r\tilde{s}\left[ \sum_{i,j} \partial^i f_{i,j}(x) \partial^j \right] = \sum_j f_{-1,j}(x) \partial^j \] .

4.2 Deformed \( Y \) and \( L \) operators.

In this section we propose some deformation of \( Y \) and \( L \) operators. Our consideration is based on the simple

**Lemma.** Quantum Miura transformation can be rewriting as the equality of two bi-differential operators

\[ \tilde{L} = (\partial + \tilde{\partial})^n + \sum W_i(z)(\partial + \tilde{\partial})^{n-i} = : \prod_1^n (\partial + \frac{\tilde{\partial}}{k} - \phi_i(z)) : \] (4.3)

**Proof.** W-currents expressed from this formula via free bosonic fields coincide with an ordinary ones due to the following property of derivations:

\[ (\partial + \frac{\tilde{\partial}}{k}) f(x) = (1 - \frac{1}{k}) f'(x) + f(x) (\partial + \frac{\tilde{\partial}}{k}) \quad Q.E.D. \]

It would be reasonable to suggest that \( L \) operator can be constructed from \( \tilde{L} \). Let us introduce quantum \( L \)-operator \( \hat{L} \) as

\[ \hat{L} = 1(\tilde{L}) = r\tilde{s}[\tilde{\partial}^{-1} \tilde{L}] \]

\[ = \partial^n + W_1 \partial^{n-1} + [W_2 + \frac{n - 1}{k} W_1'] \partial^{n-2} + \]

\[ + [W_3 + \frac{n - 2}{k} W_2'] + \frac{(n - 2)(n - 1)}{2k^2} W_1'' \partial^{n-3} + \ldots \] (4.4)

For \( n = 2, 3, 4 \) one can recognize in this formula eqs.(3.17). We will show further that eq.(4.4) is the proper expression for the quantum \( L \)-operator in case of an arbitrary natural \( n \).

We define quantum deformation for the pseudodifferential symbol \( \hat{Y} \) as dual to the \( \hat{L} \) with respect to the bilinear form given by Adler’s trace:

\[ \hat{Y} = r\tilde{s}[\tilde{\partial}^{-1}] = r\tilde{s} \sum (\partial + \tilde{\partial})^{-1} q_i \tilde{\partial}^{-1} \]

\[ = [\partial^{-1} q_i + \frac{l+1}{k} \partial^{-2} q_i' + \frac{(l+1)(l+2)}{2k^2} \partial^{-3} q_i'' + \ldots \] (4.5)

It is easy to prove from the properties of bi-pseudodifferential operators the following useful

**Lemma.**

(i)

\[ res(YL) = res(Y\hat{L}) \]
(ii) Hamiltonian $H$ is given by

\[ H = Tr(YL) = Tr((\hat{Y}\hat{L})) = k \oint_C \frac{dz}{2\pi i} \sum_{i=1}^{n} a_{n-i}(z)W_i(z) \quad (4.6) \]

(iii) 

\[ (\hat{Y}\hat{L}) = r\tilde{c}s[\tilde{y}\tilde{\partial}^{-1}\hat{L}] \quad (4.7) \]

5. QUANTUM FORMULAE.

5.1 Quantum $L$ operator.

At first we will prove that quantum analogue of the $L$-operator is given by $\hat{L}$:

**Proposition.**

\[ : [\hat{L}F_s] := 0 \quad s = 1, \ldots, n \quad (5.1) \]

**Proof.** It is convenient for us to extract derivations in $\hat{L}$:

\[ \hat{L} = \sum_i res[\hat{L}\partial^{-i-1}]\partial^i \]

Now using this formula and the definition of the normal ordering one can easily obtain:

\[ : \hat{L}F_s(z) := \sum_i res(\hat{L}\partial^{-i-1})F[i]_s(z) := \]

\[ =: \sum_i \left[ \oint_C \frac{d\zeta}{2\pi i(\zeta - z)} T(r\tilde{c}s[r\tilde{c}s(\tilde{y}\tilde{\partial}^{-1}\hat{L}\partial)^{-i-1}]F[i]_s(z)] := \right. \]

\[ =: \sum_i \left[ \oint_C \frac{d\zeta}{2\pi i(\zeta - z)} r\tilde{c}s[r\tilde{c}s(\tilde{y}^{-1}\hat{L}\partial - \tilde{y}^{-1}\hat{L}\partial + \frac{1}{k}\tilde{\partial} - \phi'(\zeta)^{-1} \phi'(\zeta)^{-1} \tilde{\partial}^{-i-1}]F[i]_s(z) := [\hat{L}\partial F_s(z)] := - \right. \]

\[ - \sum_i \left\{ \oint_C \frac{d\zeta}{2\pi i(\zeta - z)} r\tilde{c}s[(-\tilde{y}^{-1}\hat{L}\partial + \frac{1}{k}\tilde{\partial} - \phi'(\zeta)^{-1} \tilde{\partial}^{-i-1}] \frac{\partial_i C[i]_s}{k(\zeta - z)^{i+1}} F[i]_s(z) := \right. \]

Let us introduce the following shorthand notation

\[ -r\tilde{c}s[\tilde{y}^{-1}\hat{L}\partial - \frac{1}{k}\tilde{\partial} - \phi'(\zeta)^{-1}] = - \sum_{j=0}^{n-1} r\tilde{c}s[\tilde{Q}_j\partial^j], \quad (5.2) \]

where $\tilde{Q}_j$ stands for right pseudodifferential operator $\tilde{Q}_j = \sum_i \tilde{\partial}^i q_{ij}$ and $q_{ij}$ are some differential polynomials in terms of the $\phi$'s.
After the differentiation procedure we obtain taking the residus:

\[ [L F_s(z)] := \]

\[ =: \bar{L}^{\theta} F_s(z) := \sum_{i,j} [\frac{d^{\dagger}}{2\pi i} \bar{Q}_j \frac{1}{(z - \bar{\zeta})^{(\theta - 1)}} C_s^{\theta} \partial_s^{\theta - i - 1} F_s(z) := \]

It is rather simple calculation to show that sum in the second term reduces to

\[ \left[ - \sum_j (\bar{Q}_j \partial_s^{\theta}/k) \exp(\phi_1) \right] . \]

In the other hand, from the structure of the first term we find

\[ : r = \bar{s} \bar{\partial}^{-1} \bar{L}^{\theta} F_s = r \bar{s} \bar{\partial}^{-1} (\partial + \frac{1}{k} \bar{\partial} - \phi_s) \cdots (\partial + \frac{1}{k} \bar{\partial} - \phi_s) \]

\[ (\partial + \frac{1}{k} \bar{\partial} - \phi_s) F_s := : \bar{r} \bar{s} \bar{\partial}^{-1} \bar{L}^{\theta} (\partial + \frac{1}{k} \bar{\partial} - \phi_s)^{-1} \frac{1}{k} \bar{\partial} F_s := \]

\[ = \sum_j (\bar{Q}_j \partial_s^{\theta}/k) F_s : . \]

Finally we get

\[ [L F_s] := 0 . \]

Q.E.D.

5.2 W-algebra action. Proof.

Now turn our attention to the finding of the W algebra action. Following the scheme above we will prove that there exist simple quantum deformation of formulae (2.7).

Proposition.

\[ \delta_H F = : - \left[ \left( \bar{Y}\bar{L} \right)_s F_s \right] ; \quad s = 1, \ldots, n , \]

(5.4)

where \( \bar{Y} \) and \( \bar{L} \) are given by eqs. (4.4) and (4.5).

Proof. Explicit calculation shows that eqs.(3.11),(3.14),(3.15) can be recast in such a form. To prove this in general case let us represent both expressions in (5.4) through the free fields and show that they are identical.

Unique Vick’s pairing (due to eq.(3.2)) of \( \bar{Y} \) and \( F_s \) can be found as for the classical case:

\[ \delta_H (F_s(z)) = kT(Tr [Y L] F_s(z)) = \]

\[ = kT(Tr [r \bar{\partial} (\bar{Y} \bar{\partial}^{-1} \bar{L}) F_s(z)]) = \]

\[ = - : \frac{d^{\dagger}}{2\pi i} r \bar{\partial} [r \bar{\partial} (\bar{Y} \bar{\partial}^{-1} \bar{L}) F_s(z)] := \]

\[ = - : r \bar{\partial} [r \bar{s} (\bar{Y}(z) \bar{\partial}^{-1} \bar{L} \bar{\partial} + \frac{1}{k} \bar{\partial} - \phi_s(z))^{-1} \frac{1}{\zeta - z} ] F_s(z) := \]

(5.5)
This is exactly the same equation as (3.8). Let us denote for the sake of simplicity

\[ [\hat{Y} \tilde{\partial}^{-1} \tilde{L}(\partial + \frac{1}{k} \tilde{\partial} - \phi'_1)^{-1}] = \sum_j \tilde{P}_j \tilde{\partial}^j, \]  
(5.6)

where symbol \( \tilde{P}_j \) stands for some right differential operator \( \tilde{P}_j = \sum f_{ij} \tilde{\partial}^i \) with differential polynomials of \( \phi \) as coefficients. Now one can rewrite the formula (5.5) as:

\[ \delta_H(F_s(z)) = - \sum_j r\tilde{s}[\text{res}(\tilde{P}_j \tilde{\partial}^j)]F_s(z) := - r\tilde{s}[\tilde{P}_{-1}]F_s(z) : . \]  
(5.7)

Consider the expression \( - (\hat{Y} \check{L})_+ F_s \) : At first let us note that

\[ (\hat{Y} \check{L})_+ = \sum_{i \geq 0} (\hat{Y} \check{L})_+ \tilde{\partial}^i = \sum \text{res}(\hat{Y} \check{L} \tilde{\partial}^{-i-1}) \tilde{\partial}^i. \]

Try to find now the operator product \( T[(\hat{Y} \check{L})_+ \zeta F_s^{(i)}(z)] \). Of course, after Vick’s pairing and expansion under the powers of \( (\zeta - z) \) in the point \( z \) we don’t obtain quantum corrections. However the special structure of \( (\hat{Y} \check{L})_+ = r\tilde{s}(\hat{Y} \tilde{\partial}^{-1} \check{L})_+ \) make it possible to apply some trick and find the proper expression.

\[ r\tilde{s}[:(\hat{Y} \check{L})_+ \tilde{\partial}^i F_s(z):] \sim \sum_i r\tilde{s}[T[(\hat{Y} \check{L})_+ \tilde{\partial}^i F_s^{(i)}(z)]] = 
\]

\[ = \sum_i T[r\tilde{s}[\text{res}(\hat{Y} \check{L} \tilde{\partial}^{-i-1}) F_s^{(i)}(z)]] = 
\]

\[ = (\hat{Y} \check{L})_+ F_s(z) : - \sum_i r\tilde{s}[\text{res}(\hat{Y} \check{L} \tilde{\partial}^{-i-1} \tilde{\partial}^i (\partial \zeta + \frac{1}{k} \tilde{\partial} - \phi'_1)^{-1}(\zeta)] C^i \hat{L} F_s^{(i)}(z^{i-i}) : . \]  
(5.8)

Let us consider the first term in this formula:

\[ : (\hat{Y} \check{L})_+ F_s := (r\tilde{s}[(\hat{Y} \tilde{\partial}^{-1} \check{L})_+ F_s] := 
\]

\[ = (r\tilde{s}[(\hat{Y} \tilde{\partial}^{-1} \check{L})(\partial + \frac{1}{k} \tilde{\partial} - \phi'_1)^{-1}(\partial + \frac{1}{k} \tilde{\partial} - \phi'_1)])_+ F_s := 
\]

\[ = \sum_{j \geq 0} r\tilde{s}[\tilde{P}_j \tilde{\partial}^j + \sum_{j \geq 0} \tilde{P}_j \tilde{\partial}^j + \tilde{P}_{-1}] F_s := 
\]

\[ = r\tilde{s} \sum_{j \geq 0} \tilde{P}_j \tilde{\partial}^j + \sum_{j \geq 0} \tilde{P}_j \tilde{\partial}^j + \tilde{P}_{-1} F_s := : \]  
(5.9)

Direct calculation shows that quantum corrections in the formulae (5.8) cancel sum in the expression (5.9) and the remainder term \( r\tilde{s} \tilde{P}_{-1} F_s \) gives formula (5.7). Indeed
let us turn to the second term in (5.8). After the changing (5.6) and differentiation one can immediately obtain that singular terms in (5.8) can be represented as:

$$- \sum_{i \neq l} \frac{1}{k} r \tilde{e}_s \left[ \text{res} \left( \tilde{P}_j (\zeta) \frac{\partial^i }{\partial \zeta^i} (z) \right) \right] \frac{1}{(\zeta - z)^{i-1}} C_i \tilde{F}_j^{(i-s)} (z) =$$

$$= - \frac{1}{k} \sum_{i \neq l} r \tilde{e}_s \left[ C_i \tilde{P}_j (\zeta) \frac{(l+q)(-1)^j}{(\zeta - z)(i+q+1)^{j-l}} \tilde{F}_j^{(j-l-s)} (z) \right]. \quad (5.10)$$

We have need for one more

**Lemma.** Expression (5.10) gives proper formulae for quantum corrections if the expansion of $r \tilde{e}_s [\tilde{P}_j (\zeta)]$ in terms of $(\zeta - z)$ has the form:

$$r \tilde{e}_s [\tilde{P}_j (\zeta)] \to \sum_q r \tilde{e}_s [\tilde{P}_j \tilde{\partial}^q \frac{(\zeta - z)^q}{q!}]$$

**Proof.** From (5.6) one can recognize that if right derivations act on the coefficients $a_k (\zeta)$ in $\tilde{Y}$ then right residues becomes zero such as under the definition there are no right integral symbols in $\tilde{Y}$ and $\tilde{L} (\partial + \frac{1}{k} \tilde{\partial} - \phi (\zeta))^{-1}$. Therefore, functions $a_k (\zeta)$ do not take part in the ordering in accordance with the procedure represented in the section 3.

Q.E.D.

Now we are capable to complete computation of the quantum corrections:

$$: r \tilde{e}_s (\tilde{Y} \tilde{L}^{-1} \tilde{L}_+ F_s) := - : \sum \frac{1}{k} r \tilde{e}_s [C_i \tilde{P}_j (\zeta) \tilde{\partial}^{i+q+1} \tilde{F}_j^{(j-l-s)} (z) :] \quad :$$

Now we are capable to complete computation of the quantum corrections:

$$: \tilde{Y} \tilde{L}_s + F_s := - : \sum \frac{1}{k} r \tilde{e}_s [C_i \tilde{P}_j (\zeta) \tilde{\partial}^{i+q+1} \tilde{F}_j^{(j-l-s)} (z) :] \quad :$$

Taking into account eq.(5.9) one can see that the following equality is obeyed

$$- \tilde{r} \tilde{e}_s [\tilde{Y} \tilde{L}^{-1} (\tilde{Y}_L) F_s ] := - \tilde{r} \tilde{e}_s [\tilde{P}_s (\zeta)] F_s (z) :=$$

These together with (5.7) prove (5.4) i.e.

$$\delta_H F_s = - : r \tilde{e}_s [\tilde{Y} \tilde{L}^{-1} (\tilde{Y}_L) F_s ] := - : (\tilde{Y} \tilde{L}) F_s : \quad Q.E.D. \quad .$$

**5. CONCLUSION.**

Thus we have used right derivation technik to prove that the vertex operators $F_s$ $(s = 1, ..., n)$ (3.6) are transformed under the $\tilde{W}$ algebra action as

$$\delta_H F_s = - : [Y \tilde{L}, F_s ] : \quad .$$
Another rigorous result is that these operators must satisfy the "null-vector equation":

$$\hat{L} F_a := 0 .$$

We have seen above that provided choose suitable deformation of the (pseudo)differential operators these formulae and quantum Miura transformation (3.16) for $W$-algebras have precisely the same form as classical ones. It is worth noting that our deformation is independent of the $n$.

It will be very important to develop these results for the $W$-algebras associated to $SL(n)$ group. Another vital problem is the finding of the quantum analogue of the Gelfand-Dikii bracket in terms of pseudodifferential operators. Unfortunately, direct application of the monodromic deformation method proposed in [23] gives improper result because it is necessary to have into account some subtle problems with normal ordering [13].

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