Thermodynamics of Vortices in the Plane

P.A. Shah

and

N.S. Manton

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW, U.K.

Abstract

The thermodynamics of vortices in the critically coupled abelian Higgs model, defined on the plane, are investigated by placing $N$ vortices in a region of the plane with periodic boundary conditions: a torus. It is noted that the moduli space for $N$ vortices, which is the same as that of $N$ indistinguishable points on a torus, fibrates into a $\mathbb{F}P_{N-1}$ bundle over the Jacobi manifold of the torus. The volume of the moduli space is a product of the area of the base of this bundle and the volume of the fibre. These two values are determined by considering two 2-surfaces in the bundle corresponding to a rigid motion of a vortex configuration, and a motion around a fixed centre of mass. The partition function for the vortices is proportional to the volume of the moduli space, and the equation of state for the vortices is $P(A - 4\pi N) = NT$ in the thermodynamic limit, where $P$ is the pressure, $A$ the area of the region of the plane occupied by the vortices, and $T$ the temperature. There is no phase transition.

July 1993
1 Introduction

Solitons, as topological defects in field theories and models for elementary particles and magnetic flux tubes in superconductors, have been attracting a lot of interest recently. Defects such as strings and monopoles, if they exist, are recognised to have a significant effect on the dynamics of the early universe [1]. Solitons in Skyrme’s model, a low energy limit of QCD, are a good model for baryons and their low energy interactions [2].

In general, dealing with the interactions of solitons requires a treatment of the full field theory in a non-perturbative regime. However, in certain theories at low energies, the motion of a soliton configuration can be approximated by a continuous sequence of static configurations [3], in much the same way that a moving cinema picture is built from a sequence of still pictures. The energy functional of the fields stays “close” to its minimum, which occurs on the configuration space of stable static solutions of the field equations. This approximation rules out higher energy phenomena, like soliton-antisoliton pair creation, as there are no stable static configurations containing a mixture of solitons and antisolitons. The finite dimensional manifold parametrising static configurations is referred to as the moduli space, and the interaction of the solitons in the field theory is modelled by geodesic motion in this space, with a metric determined from the field theory Lagrangian. This approximation is expected to be good for low soliton impact velocities, becoming exact in the limit of soliton velocities tending to zero.

The moduli space approach to soliton dynamics is possible at a critical value of a coupling constant in the Lagrangian, where the equations describing the minima of the energy functional reduce to first order differential equations, the Bogomolny equations [4], and there are no forces between the solitons. This is why the moduli space has no potential, and the motion on it is geodesic motion. Comparison of the geodesic motion with numerical simulations of the scattering of critically coupled vortices shows a good agreement for velocities up to around \( \frac{1}{3}c \).

For many applications, it would be useful to know the thermodynamic behaviour of a large number of solitons. In a recent paper by one of us [5], it was shown how to derive the thermodynamics of vortices in the abelian Higgs model at critical coupling, with the vortices moving on a 2-sphere. The calculations were performed analytically, but within the framework of the moduli space approximation. This method can be applied to vortices moving on a general orientable compact Riemann surface \( S \). The Lagrangian density of this model is

\[
- \frac{1}{4} f_{\mu \nu} f^{\mu \nu} + \frac{1}{2} D_\mu \phi (D^\mu \phi)^* - \frac{1}{8} (|\phi|^2 - 1)^2
\]

(1)

\( \phi \) is complex scalar Higgs field and \( D_\mu \phi = \partial_\mu \phi - ia_\mu \phi \), where \( a_\mu \) is the \( U(1) \) gauge potential. \( f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \) is the Maxwell field tensor. Although the Bogomolny equations describing the fields of minimal potential energy are first order, explicit solutions are not available. However, a static solution with energy \( N\pi \) and total magnetic flux

\[
\int f_{12} = 2\pi N \quad , \quad N \in \mathbb{Z}
\]

(2)

is uniquely determined (up to a gauge transformation) by specifying the positions of the precisely \( N \) zeros of the Higgs field, which may be identified with the positions of \( N \) vortices [6].
Classically, these vortex positions are an unlabelled set; the configuration is unaffected by interchanging any of the vortices so the moduli space $\mathcal{M}_N$ is $(S)^N / \Sigma_N$ where $\Sigma_N$ is the permutation group of $N$ elements.

There is a naturally defined metric on $\mathcal{M}_N$, derived from the kinetic part of the Lagrangian. Let $g_{ij}(Q)$ denote this metric, where $Q = \{Q^i\}$ is a set of arbitrary coordinates. In the moduli space approximation the energy of moving vortices is $\frac{1}{2} \pi g_{ij}(Q) \dot{Q}^i \dot{Q}^j$. By introducing conjugate momenta $P_i = \pi g_{ij}(Q) \dot{Q}^j$, the energy may be written in the form

$$E(P,Q) = \frac{1}{2\pi} g^{ij}(Q) P_i P_j$$  \hspace{1cm} (3)

The statistical mechanics of $N$ vortices at temperature $T$ can be treated by using a Gibbs distribution, where the partition function is

$$Z = \frac{1}{\hbar^2 N} \int_{\mathcal{M}_N} (dP)(dQ) e^{-\frac{E(P,Q)}{T}}$$  \hspace{1cm} (4)

and $\hbar$ is Planck’s constant. By doing the Gaussian momentum integrals, this reduces to

$$Z = (\int_{\mathcal{M}_N} (dQ) \sqrt{det g_{ij}}) (\frac{2\pi^2 T}{\hbar^2})^N$$  \hspace{1cm} (5)

The first factor is simply the volume of the moduli space. It is known that the metric on the moduli space is Kähler and hence the volume of the space can be determined from the areas of 2-surfaces in it by using homology and symmetry arguments.

In this paper, we investigate the statistical mechanics of vortices on a torus rather than on a sphere, thereby eliminating undesirable curvature effects. The torus corresponds to a unit cell in the plane $\mathbb{R}^2$ with periodic boundary conditions. One expects that in the thermodynamic limit $N \to \infty$ at fixed number density $n = \frac{N}{A}$, the behaviour of the particles should not depend on the global topology of the manifold they are defined on, although this has not been proven.

## 2 The fibre bundle structure of the moduli space

To model a torus $M = S^1 \times S^1$, we choose a unit rectangular cell in the plane with periodic coordinates $(x_1, x_2)$ with ranges $x_1 \in [0,1), x_2 \in [0,\alpha)$, and give the spacetime the metric

$$ds^2 = dx_0^2 - L^2(dx_1^2 + dx_2^2)$$  \hspace{1cm} (6)

where $L$ is a constant physical scaling factor; the area of the torus is then $A = L^2 \alpha$. We introduce a complex coordinate $z = x_1 + i x_2$. The Lagrangian (1) leads to the Bogomolny equations for critically coupled vortices

$$\begin{align*}
(D_1 + i D_2) \phi &= 0 \\
 f_{12} + \frac{1}{2} L^2 (|\phi|^2 - 1) &= 0
\end{align*}$$  \hspace{1cm} (7)

$$\begin{align*}
(\partial_1 + i \partial_2) \phi &= 0 \\
 f_{12} + \frac{1}{2} L^2 (|\phi|^2 - 1) &= 0
\end{align*}$$  \hspace{1cm} (7)
Here, $\phi$ and $a_\mu$ are not strictly speaking functions on the torus, but rather a section and connection on a $U(1)$ bundle over it. The total flux is quantised as in (2), and $N$ is the first Chern number of the bundle.

The moduli space for $N$ vortices on the torus $M$ is $M_N = M^N/\Sigma_N$. This makes it appropriate to define the configuration in terms of its divisor $d(\phi)$. A divisor of a function is a formal unordered sum of the zeros (counted positively by multiplicity) and poles (counted negatively) of the function. The divisor of the Higgs field of $N$ vortices at positions $\{p_1, \ldots, p_N\}$ is

$$d(\phi) = \sum_{i=1}^N 1 \cdot p_i$$

The magnitude of the Higgs field away from the vortex positions is not important for our purposes. Let us therefore consider the complexification of the $U(1)$ gauge group, namely $\mathfrak{g}^*$. Under transformations of the form $\phi \rightarrow e^{i\alpha(z, \bar{z})}\phi$, where $\alpha(z, \bar{z}) \in \mathfrak{g}$, the absolute magnitude $|\phi|$ is no longer gauge invariant but the positions of the zeros of $\phi$, and hence the vortex positions, still are. Using the enlarged gauge group, we can locally put the fields in the holomorphic gauge where $a_\tau \equiv \frac{1}{2}(a_1 + ia_2) = 0$. Then the first Bogomolny equation, which can be written as $(\partial_\tau - ia_\tau)\phi = 0$, reduces to

$$\frac{\partial \phi}{\partial \bar{z}} = 0$$

so $\phi$ becomes an analytic function of $\bar{z}$. In this gauge we can express the Higgs field as a product of analytic functions with precisely one zero each on the torus. These functions are the theta functions, which are actually defined on the universal covering space of the torus, the complex plane $\mathfrak{g}'$. The torus can be represented as the quotient of $\mathfrak{g}'$ by the lattice generated by 1 and $\tau = i\alpha$. The theta functions are not single valued on this quotient, but the positions of the zeros in the complex plane map to a single point when quotiented by the lattice.

The lattice generated by 1 and $\tau$ defines a normalised choice of what is known as the Jacobi mapping. The Jacobi mapping allows us to define a centre of mass for the vortex configuration (which is not unambiguously possible on the original torus). It is this property that allows a fibration of the moduli space into coordinates related to (but not the same as) the centre of mass and relative positions of the vortices. This uses the theory of generalised theta functions, quasiperiodic functions (or “complex analytic relatively automorphic functions” - caraf for short) which can be defined for any orientable Riemann surface of genus $g > 0$. Our discussion of them will be brief and specific to the torus, but the interested reader is referred to [7] for a fuller and more general discussion.

The Jacobi mapping of an orientable Riemann surface $S$ of genus $g$ maps $S$ to a complex $g$-torus, and is defined in terms of the integrals of normalised abelian differentials (of the first kind). On the torus, there is just one normalised abelian differential, namely $dz$, whose integral is $z$, so the Jacobi mapping is essentially the identity. The fundamental periods of $dz$ are 1 and $\tau$. More precisely, let us choose a base point $p_0$ on $M$, and identify it with the origin of coordinates. We define the Jacobi manifold of the torus $J(M)$ by

$$J(M) = \frac{\mathfrak{g}'}{\mathcal{L}}, \quad \mathcal{L} = \{n_1 + n_2 \tau : n_1, n_2 \in \mathbb{Z}\}$$

\[ (10) \]
and the Jacobi mapping $\Phi^*$ by
\begin{equation}
\Phi^* : M \mapsto J(M) \quad (11)
\end{equation}
\begin{equation}
\Phi^*(x_1, x_2) = x_1 + i x_2, \quad x_1 \in [0, 1), x_2 \in [0, \alpha)
\end{equation}

Given the Jacobi mapping there is a natural map from an $N$-vortex configuration to $\Phi$ defined by $t(\phi) = \sum_{i=1}^{N} \Phi^*(p_i)$. $\frac{t}{N}$ is then a particular choice of a centre of mass for the configuration.

The simplest example of a ‘classical’ theta function is
\begin{equation}
\theta_1(\tau, z) = -i \sum_{n=-\infty}^{n=\infty} (-1)^n e^{(n+\frac{1}{2})^2 i \pi \tau} e^{(2n+1) i \pi z} \quad (12)
\end{equation}

which has zeros at the origin in the complex plane and lattice translates of that. Translation of this function by lattice vectors has the effect of multiplying it by non-vanishing factors, termed factors of automorphy:
\begin{equation}
\begin{aligned}
\theta_1(\tau, z + 1) &= -\theta_1(\tau, z) \\
\theta_1(\tau, z + \tau) &= -\kappa(\tau, z) \theta_1(\tau, z)
\end{aligned} \quad (13)
\end{equation}

where $\kappa(\tau, z) = e^{-2\pi iz} e^{-\pi i \tau}$ (we shall now drop the $\tau$ argument and just write $\theta_1(z)$). There are three other classical theta functions, related to $\theta_1(z)$ by
\begin{equation}
\theta_1(z) = -\theta_2(z + \frac{1}{2}) = -i e^{\frac{i \pi z}{4}} e^{i \pi z} \theta_3(z + \frac{1}{2} + \frac{1}{2} \tau) = -i e^{\frac{i \pi z}{4}} e^{i \pi z} \theta_4(z + \frac{1}{2} \tau) \quad (14)
\end{equation}

Various identities exist between these four functions; the principal ones that we will use are summarised below:

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\theta_1(z)$</th>
<th>$\theta_2(z)$</th>
<th>$\theta_3(z)$</th>
<th>$\theta_4(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$-\kappa$</td>
<td>$\kappa$</td>
<td>$\kappa$</td>
<td>$-\kappa$</td>
<td>$\kappa$</td>
</tr>
</tbody>
</table>

The factors of automorphy of classical theta functions
\begin{equation}
\begin{aligned}
\theta_1(0) &= \theta_2(\frac{1}{2}) = \theta_3(\frac{1}{2} + \frac{\tau}{2}) = \theta_4(\frac{\tau}{2}) = 0 \\
\theta_2^2(z) \theta_4^2 &= \theta_4^2(z) \theta_3^2 - \theta_1^2(z) \theta_3^2 \\
\theta_3^2(z) \theta_4^2 &= \theta_4^2(z) \theta_2^2 - \theta_1^2(z) \theta_2^2 \\
\theta_1(y + z) \theta_1(y - z) \theta_4^2 &= \theta_1^2(y) \theta_4^2(z) - \theta_1^2(y) \theta_2^2(z)
\end{aligned} \quad (16)
\end{equation}

where $\theta_i$ denotes $\theta_i(0)$. The reader is referred to, for example [8], for an extensive list of the properties these functions possess. These functions can be used to represent divisors on the torus $M$. Although they are not single-valued on $M$, the factor of automorphy is non-vanishing, and hence the zeros of $\theta_i(z)$ in the complex plane map to a single point of $M$.

In the holomorphic gauge we can write the Higgs field locally as
\begin{equation}
\phi(z) = \prod_{i=1}^{N} \theta_1(z - z_i), \quad z_i = \Phi^*(p_i) \quad (17)
\end{equation}
which is an example of a caraf. Under translation by a lattice element $\lambda$, this has factor of automorphy

$$
\xi_\phi(\lambda, z) = \begin{cases} 
( -1)^N 
\prod_{i=1}^N \kappa(\tau, z - z_i) & \text{for } \lambda = 1 \\
\prod_{i=1}^N \kappa(\tau, z - z_i) & \text{for } \lambda = \tau 
\end{cases}
$$  
(18)

The characteristic $c(\xi)$ of a factor of automorphy $\xi$ of a scalar caraf is equal to the order of the divisor of the function, i.e. the number of zeros minus the number of poles. Hence $c(\xi_\phi) = N$. It is a special case of a general theorem on factors of automorphy that we can write

$$
\xi_\phi(\lambda, z) = \rho_t(\lambda) \xi_{\theta_0}^N (\lambda, z)
$$  
(19)

where $t = t(\phi) = \sum_{i=1}^N z_i$ as defined above. $\xi_{\theta_0}$ denotes the factor of automorphy of $\theta_1(z)$, and

$$
\rho_t(1) = 1 \quad \rho_t(\tau) = \exp 2\pi it
$$  
(20)

By another theorem, the carafs on the torus with factor of automorphy $\rho_t \xi_{\theta_0}^N$ form at fixed $t$ a vector space with dimension $N$. Any basis functions of this space, $f_i(t, z)$ ($i = 1 \ldots N$) are quasiperiodic in $t$ as well as $z$:

$$
f_i(t, z + \lambda) = \rho_t(\lambda) \xi_{\theta_0}^N (\lambda, z) f_i(t, z)
$$  
(21)

$$
f_i(t + \lambda, z) = \rho_z(\lambda) \chi_{ij}(\lambda, t) f_j(t, z)
$$  
(22)

where $\rho_z(\lambda)$ is defined identically to $\rho_t(\lambda)$. The vector valued function $\theta(t, z) = (f_i(t, z))$ is termed a *generalised theta function* of rank $N$, with associated *theta factor* $\chi_{ij}(\lambda, t)$.

We now give a useful basis for these functions on the torus for general rank $N$, and give the theta factors for first and second rank. For rank 1, a basis having the factors of automorphy (21) and (22) is

$$
f(t, z) = \theta_1(z - t)
$$  
(23)

Using the properties of $\theta_1(z)$ listed in (13) the theta factor can be computed to be

$$
\chi(1, t) = -1
$$  
(24)

$$
\chi(\tau, t) = -e^{-i\pi t} e^{-2\pi i t} = -\kappa(\tau, t)
$$

For rank 2, a basis is provided by the independent functions

$$
f_i(t, z) = \{ \theta_1^2(z - \frac{t}{2}), \theta_2^2(z - \frac{t}{2}) \}
$$  
(25)

The computation of the theta factor utilises identities on the squares of theta functions, and leads to

$$
\chi(1, t) = \frac{1}{\theta_4^4} \begin{pmatrix} -\theta_3^2 & \theta_2^2 \\ -\theta_2^2 & \theta_3^2 \end{pmatrix}
$$  
(26)

$$
\chi(\tau, t) = \begin{pmatrix} 0 & -\kappa(\tau, \frac{t}{2}) \\ -\kappa(\tau, \frac{t}{2}) & 0 \end{pmatrix}
$$
For rank $N$ even, we will find the following basis useful

$$f_i(t, z) = \{\theta_i^N, \theta_i^{N-2}\theta_4^2, \ldots, \theta_i^4, \theta_i^{N-3}\theta_4\theta_3, \theta_i^{N-5}\theta_4^2\theta_3, \ldots, \theta_i^4\theta_3\theta_4^2\theta_3\} \tag{27}$$

where all these theta functions are evaluated at $z = \frac{i}{\sqrt{N}}$. For $N$ odd we shall use

$$f_i(t, z) = \{\theta_i^N, \theta_i^{N-2}\theta_4^2, \ldots, \theta_i\theta_4^{N-1}, \theta_i^{N-3}\theta_4\theta_3, \theta_i^{N-5}\theta_4^2\theta_3, \ldots, \theta_i^2\theta_4^{N-4}\theta_3, \theta_i^{N-2}\theta_4\theta_3\} \tag{28}$$

These functions all have the same factors of automorphy, and are clearly independent, as they have zeros of orders $N, N - 2, N - 3, \ldots, 2, 1, 0$ at $z = \frac{i}{\sqrt{N}}$, and lattice translates of it. It is interesting to note the absence of a basis element with a zero of order $N - 1$. If such an independent function existed, division by $\theta_1^N(z - \frac{i}{\sqrt{N}})$ would lead to an elliptic function with one simple pole in a unit cell, but this contradicts a theorem on elliptic functions. Computation of the theta factors of these bases is in principle possible, but in practice difficult; we shall anyway not need to use the theta factors in our computation of the volume of the moduli space.

The relevance of these generalised theta functions and their bases to the moduli space of $N$ vortices is clear from the following theorem [7] :

**Theorem 1** For an orientable Riemann surface $S$ of genus $g$, and any integer $N > 2g - 1$, the space $\frac{(S)^N}{\Sigma_N} = \{d_\phi : d_\phi = \sum_{i=1}^{N-1} p_i, p_i \in S\}$ of positive divisors of order $N$ on the surface can be given the structure of a fibre bundle with fibre $\mathfrak{P}_{N-g}$ and base space $J(S)$, the Jacobi manifold of $S$. This bundle has a natural projection

$$\Phi^*_N : \frac{(S)^N}{\Sigma_N} \mapsto J(S) \tag{29}$$

$$\Phi^*_N(p_1 \ldots p_N) = \Phi^*(p_1) + \cdots + \Phi^*(p_N) \mod \mathcal{L}$$

where $\mathcal{L}$ is the lattice of periods of normalised abelian differentials.

We can understand this for the torus $M$, with $g = 1$, as follows. $\Phi^*_N$ may be identified with $t \mod \mathcal{L}$, with $t$ defined as above. As noted above, this coordinate is related to a choice of centre of mass of a configuration; its most important property is that it is invariant under the permutation group acting on the vortices. For fixed $t$ we have the $N$-dimensional vector space with a basis given by (27) or (28) above, and a divisor $d_\phi$ determines an element of this vector space via the intermediary of the function

$$f_\phi(u, t, z) = \sum_{i=1}^{N} u_i f_i(t, z) \tag{30}$$

such that $d(f_\phi(u, t, z)) = d_\phi$ for a choice of $u_i$. The $u_i$ give homogeneous coordinates in the $\mathfrak{P}_{N-1}$ fibre, and are only defined up to a constant (non-zero) complex multiple, as $cu_i$ represents the same divisor as $u_i$. If $t$ is restricted to lie in one unit cell of the lattice, the
\( u_i \) are unique modulo such a constant multiple. The \( t \) coordinate is unique modulo lattice translations, and translation of \( t \) by a lattice spacing changes the coordinates \( u_i \) by the appropriate theta factor (dependent on the choice of caraf basis functions \( f_i(t, z) \)). More explicitly, \((t, u_i) \rightarrow (t + \lambda, u_j \chi_{ji}^{-1}(t, \lambda)) \) (where an irrelevant homogeneous factor of \( \rho_z^{-1}(\lambda) \) has been omitted) leads to the same divisor, as writing \( t' = t + \lambda \) and \( u_i' = u_j \chi_{ji}^{-1} \)

\[
\begin{align*}
    f_\phi(u', t', z) &= \sum u_i' f_i(t + \lambda, z) \\
    &= \sum u_i' \rho_i(\lambda) \chi_{ij}(\lambda, t) f_j(t, z) \\
    &= \rho_i(\lambda) \sum u_j f_j(t, z) \\
    &= \rho_i(\lambda) f_\phi(u, t, z)
\end{align*}
\]

Hence \( d(f_\phi(u', t', z)) = d(f_\phi(u, t, z)) \). The theta factor therefore describes the non-trivial nature of the bundle over \( J(M) \).

With the understanding that \( t \) will now be restricted to lie in the unit cell containing the origin in the complex plane, we now seek to investigate the form of the metric on the bundle, using the symmetry of the torus.

## 3 The metric on the moduli space \( \mathcal{M}_N \)

We begin with the most general Hermitian metric on the moduli space, which has the structure of a fibre bundle as discussed in section 2:

\[
ds^2 = a(t, v) dt d\bar{t} + b_\alpha(t, v) dt d\bar{v}_\alpha + \overline{b_\alpha(t, v)} dt d\bar{v}_\alpha + c_{\alpha\beta}(t, v) dv_\alpha dv_\beta
\]

\( v_\alpha \) are the inhomogeneous coordinates on the fibre \( \mathbb{C}P_{N-1} \), \( \alpha, \beta = 1 \ldots N - 1 \), and \( t \) is the base coordinate which covers \( J(M) \) once. \( a \) is real and \( c_{\alpha\beta} \) is Hermitian. We note first that from translation symmetry of the torus, \( a, b_\alpha, c_{\alpha\beta} \) must be functions of \( v \) only. Furthermore, we can use the 180° rotational symmetry of the torus to eliminate the functions \( b_\alpha \). The proper distance associated with a small displacement \((\delta t, \delta v)\) satisfies

\[
\delta s^2(t, v, \delta t, \delta v) = \delta s^2(-t, v, -\delta t, \delta v)
\]

and by choosing \( \delta t \) real so that \( \delta t = \delta \bar{t} \), this implies that

\[
b_\alpha \delta \bar{v}_\alpha + \overline{b_\alpha} \delta v_\alpha = 0
\]

Choosing \( \delta v_\alpha = i \delta_{\alpha\beta} \) implies that \( \text{Im}(b_\beta) = 0 \), and choosing \( \delta v_\alpha = \delta_{\alpha\beta} \) implies that \( \text{Re}(b_\beta) = 0 \), hence \( b_\beta = 0 \) for each \( \beta \).

It would seem physically reasonable that for a small rigid motion of the vortex configuration, which would correspond to shifting the \( t \) coordinate but staying stationary in the fibre, the distance moved in the moduli space should not depend on the relative positions of the vortices.
This is confirmed when we apply the result that the moduli space is Kähler. The Kähler form is
\[ \omega = \frac{i}{2} (a(v) dt \wedge d\bar{t} + c_{\alpha\beta}(v) dv_\alpha \wedge d\bar{v}_\beta) \]  

(35)

On a Kähler manifold \( d\omega = 0 \), which implies
\[
\frac{\partial a}{\partial v_\alpha} dv_\alpha \wedge dt \wedge d\bar{t} + \frac{\partial a}{\partial \bar{v}_\alpha} d\bar{v}_\alpha \wedge dt \wedge d\bar{t} + \frac{\partial c_{\alpha\beta}}{\partial v_\gamma} dv_\alpha \wedge dv_\beta \wedge d\bar{v}_\gamma + \frac{\partial c_{\alpha\beta}}{\partial \bar{v}_\gamma} d\bar{v}_\alpha \wedge d\bar{v}_\beta \wedge dv_\gamma = 0
\]

(36)

so
\[
\frac{\partial a}{\partial v_\alpha} = \frac{\partial a}{\partial \bar{v}_\alpha} = 0
\]

(37)

and therefore \( a \) is a constant on the fibre bundle. The resultant form of the metric is then
\[
ds^2 = adt d\bar{t} + c_{\alpha\beta}(v) dv_\alpha d\bar{v}_\beta
\]

(38)

so the volume of the moduli space is a product of the area of the base \( J(M) \) and the volume of the fibre \( \mathbb{F}P_{N-1} \).

Samols [9] has given a formula for the metric at a point on the moduli space in terms of the corresponding solution of the field equations. Writing \( f = \log |\phi|^2 \) (in the original formulation with gauge group \( U(1) \)), it follows from the Bogololny equations (7) that \( f \) satisfies
\[
\partial^2 f + L^2 (1 - e^f) = 4\pi \sum_{i=1}^{N} \delta^{(2)}(z - y_i)
\]

(39)

where \( y_i \) are the vortex positions in \( J(M) \). \( f \) can be expanded as a series about each vortex position
\[
f(z, \bar{z}) = \log |z - y_i|^2 + a_i(y) + \frac{1}{2} b_i(y)(z - y_i) + \frac{1}{2} \bar{b}_i(y) (\bar{z} - \bar{y}_i) + \frac{1}{2} c_i(y)(z - y_i)^2 + \frac{1}{2} \bar{c}_i(y) (\bar{z} - \bar{y}_i)^2 + \frac{1}{2} d_i(y)(z - y_i)(\bar{z} - \bar{y}_i) + \ldots
\]

(40)

where \( y \) denotes the set of all vortex positions. \( b_i \) measures how the centres of the contours of \( |\phi|^2 \) drift away from the vortex position \( y_i \) as the magnitude of \( |\phi|^2 \) increases from zero, and this depends on the positions of all the other vortices. The metric on the moduli space is then
\[
ds^2 = \sum_{i,j=1}^{N} (L^2 \delta_{ij} + 2 \frac{\partial \bar{b}_j}{\partial y_i}) dy_i d\bar{y}_j
\]

(41)

and we shall show in the next section how to relate this to (38).

4 Collective motion of N coincident vortices

We now turn to the calculation of the first factor in the volume of \( \mathcal{M}_N \), the area of the base of the fibre bundle. From (38), this is \( a \alpha \), since the range of \( t \) is \( \{0 \leq Re(t) < 1, 0 \leq Im(t) < \alpha \} \). To calculate \( a \) we look for a 2-surface of vortex configurations that is locally orthogonal to the
fibre. Pursuing our observation that the base coordinate defines a choice for the centre of mass of the vortices, such a surface should correspond to a rigid motion of the vortices. A simple choice of such a surface is obtained by placing all the $N$ vortices at one point, and allowing that point to vary around the torus. We define

$$
\mathcal{M}_c = \{ d(\phi) : d(\phi) = N \cdot p, \; p \in M \} \subset \frac{M^N}{\Sigma_N}
$$

(42)

We can express $\mathcal{M}_c$ algebraically in terms of the fibre bundle coordinates as follows. Write the Higgs field in the holomorphic gauge as

$$
\phi(z) = \theta_1^N(z - y)
$$

(43)

where $y = \Phi^*(p) \in J(M)$. Recall the basis of carafes of rank $N$ given in (27) and (28), $f_i = \{ \theta_1^N(z - \frac{t_i}{N}), \ldots \}$, where now $t = Ny \bmod \mathcal{L}$. Keeping $y$ restricted to the small patch of the unit cell delimited by $0 \leq Re(y) < \frac{1}{N}, 0 \leq Im(y) < \frac{\alpha}{N}$, which we shall denote $\mathcal{U}_{11}$ (we shall use $\mathcal{U}_s$, $r, s = 1 \ldots N$ to mean the small patch $\frac{r+1}{N} < Re(y) < \frac{r}{N}, \frac{(s-1)\alpha}{N} < Im(y) < \frac{s\alpha}{N}$), then

$$
\phi(z) = f_1(Ny, z)
$$

(44)

Hence the coordinates of $\mathcal{M}_c$ for $y \in \mathcal{U}_{11}$ are

$$
\mathcal{M}_c = \{(t, u) : (Ny, 1, 0 \ldots 0), \; t \in J(M), \; u \in \mathcal{F}P_{N-1}\}
$$

(45)

This surface is locally orthogonal to the fibre since $u$ is constant, and as $y$ varies over $\mathcal{U}_{11}$, $t$ varies over the whole of $J(M)$. To find the coordinates of the surface for $y \in \mathcal{U}_s$, one must apply the quasiperiodicity of the caraf basis in the $t$ coordinate, corresponding to the non-trivial structure of the fibre bundle. For $y$ varying over the whole of $J(M)$, the surface intersects each fibre $N^2$ times and always orthogonally. For example, for $y \in \mathcal{U}_{11}$, equation (22) implies

$$
u_i(y + \frac{1}{N}) = \chi_{ij}(1, Ny)u_j(y)
$$

(46)

where the LHS are the coordinates in $\mathcal{U}_{11}$, and a homogenous factor has been omitted. Using $N = 2$ as an example, we have determined the $\chi(\lambda, t)$ matrices and hence the surface $\mathcal{M}_c$ has coordinates

$$
(t, u) = \begin{cases} 
(2y, 1, 0) & y \in \mathcal{U}_{11} \\
(2y - 1, \theta_3^2, \theta_2^2) & y \in \mathcal{U}_{21} \\
(2y - \tau, 0, 1) & y \in \mathcal{U}_{12} \\
(2y - 1 - \tau, \theta_2^2, \theta_3^2) & y \in \mathcal{U}_{22} 
\end{cases}
$$

Returning to $N$ vortices, the algebraic expression for the coordinates of $\mathcal{M}_c$ can be inserted in the metric on the fibre bundle, restricted to $\mathcal{M}_c$. For $y \in \mathcal{U}_{11}$, $dt = Ny$ and $dv = \frac{\partial v}{\partial y}dy = 0$ so recalling equation (38)

$$
\left. ds^2 \right|_{\mathcal{M}_c} = aN^2dyd\bar{y}
$$

(47)
We can now find $a$ by determining the metric on $\mathcal{M}_c$ from Samols’ formula (41) which for $N$ coincident vortices reduces to

$$ds^2 |_{\mathcal{M}_c} = N(L^2 + 2 \frac{\partial b}{\partial \bar{y}})dyd\bar{y}$$

(48)

where $\frac{1}{2} b(y)$ is the linear coefficient in the expansion of $f = \log |\phi|^2$ around the vortex position $y$. The reflection symmetry $z - y \to -(z - y)$ can be used in this local expansion, and implies $b(y) = 0$; the contours of the Higgs field are locally centred on the $N$-vortex position. Hence (48) simplifies to

$$ds^2 |_{\mathcal{M}_c} = NL^2 dyd\bar{y}$$

(49)

which is the same result as would be obtained for $N$ coincident vortices in a plane, using the more powerful circular symmetry. Comparison with (47) yields

$$a = \frac{L^2}{N}$$

(50)

Hence

$$\text{Area}(J(M)) = \frac{L^2 \alpha}{N} = \frac{A}{N}$$

(51)

5 Simultaneous motion of two diametrically opposite vortices

In the previous section, we used a rigid collective motion of the vortex configuration to determine the area of the base space of the fibre bundle. We now turn to determining the volume of the fibre, $\mathcal{CP}_{N-1}$, by looking for a motion that lies entirely in the fibre, keeping the centre of mass fixed. The simplest example seems to be the motion where $N - 2$ of the vortices are fixed at the point $p_0$, and the other two are at positions $p_1$ and $p_2$ symmetrically placed on opposite sides of $p_0$ and moving over some suitable region of the torus. This motion of vortices corresponds to a motion in a submanifold $\mathcal{M}_d$ of the moduli space,

$$\mathcal{M}_d = \{d(\phi) : d(\phi) = (N - 2) \cdot p_0 + 1 \cdot p_1 + 1 \cdot p_2, \ p_1, p_2 \in M\}$$

(52)

where $\Phi^*(p_1) = x$, say, and $\Phi^*(p_2) = -x \text{ mod } \mathcal{L}$, so the coordinate $t(\phi) = 0$. $\mathcal{M}_d$ is covered once if $x$ ranges over the half cell $\mathcal{U}_x = \{x : 0 \leq Re(x) < \frac{1}{2}, 0 \leq Im(x) < \alpha\}$. As before, we seek to find an expression for the fibre coordinates on $\mathcal{M}_d$. Again in the holomorphic gauge

$$\phi(z) = \theta_1(z - x)\theta_1(z + x)\theta_1^{N-2}(z)$$

(53)

We can use the addition identities of theta functions (16) to write this as

$$\phi(z) = \frac{1}{\theta_4^2}(\theta_1^N(z)\theta_4^2(x) - \theta_4^2(z)\theta_1^{N-2}(z)\theta_1^2(x))$$

(54)
Recall our basis, with \( t = 0 \) here,

\[
f_i(0, z) = \{ \theta_1^N(z), \theta_1^{N-2}(z) \theta_4^2(z), \ldots \}
\]  

(55)

Hence, omitting the irrelevant \( \theta_4^{-2} \) factor, the base and homogenous fibre coordinates of \( \mathcal{M}_d \) are

\[
\mathcal{M}_d = \{(t, u_i) : (0, \theta_4^2(x), -\theta_4^2(x), 0 \ldots 0)\}
\]  

(56)

Therefore \( \mathcal{M}_d \) lies in the \( \Phi P_1 \) line \( u_3 = u_4 = \ldots = u_N = 0 \). The inhomogenous fibre coordinates are

\[
v_\alpha = \begin{cases} 
-\frac{\theta_2'(x)}{\theta_2(x)} & \alpha = 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(57)

which are even elliptic functions of \( x \), as expected from the symmetry of the motion. Now for \( x \) varying over a unit cell, the number of solutions of \( v_1 = c \in \mathbb{C} \) is independent of \( c \), and is equal to the number of zeros or poles of \( v_1 \), counted by multiplicity. In this case, there are two solutions, but the symmetry \( v_1(x) = v_1(-x) \) implies that there is exactly one solution in the half cell \( \Gamma_4 \). Hence, for \( x \) varying over \( \Gamma_4 \), \( \mathcal{M}_d \) covers a \( \Phi P_1 \) line of \( \Phi P_{N-1} \) exactly once.

We now need to apply this result to the metric on the fibre bundle, whose restriction to \( \mathcal{M}_d \) is given by

\[
ds^2|_{\mathcal{M}_d} = 2(L^2 + \frac{\partial b_x}{\partial \bar{x}}) dx d\bar{x}
\]  

(58)

(remembering that as we vary \( x \) we are varying the positions of two vortices simultaneously). \( \frac{1}{2} b_x \) is the linear coefficient of the expansion of \( f = \log |\phi^2| \) around the vortex at \( x \) and \( \frac{1}{2} b_{-x} \) the linear coefficient around the vortex at \( -x \). Set \( b_x = b \), and by 180° rotational symmetry \( b_{-x} = -b \). However, it is now not possible to solve for \( b \) using the symmetry of the torus. Fortunately, we do not need to know \( b \) everywhere; we can instead consider the integral of \( \frac{\partial b(x)}{\partial \bar{x}} \) over \( x \in \Gamma_4 \), and use Stokes’ theorem. The area of \( \mathcal{M}_d \) is

\[
I = \int_{\Gamma_4} (2L^2 + 2 \frac{\partial b_x}{\partial \bar{x}} - 2 \frac{\partial b_{-x}}{\partial \bar{x}}) \left( \frac{1}{2i} d\bar{x} \wedge dx \right)
\]  

(59)

\[
= A - 2i \int_{\partial \Gamma_4} b \ dx
\]

\[
= A + 2i \int_{\partial \Gamma_4} b \ dx
\]

where the orientation of the contour \( \partial \Gamma_4 \) is shown in figure 1. \( b \) has poles at the points where the vortices at \( x \) and \( -x \) coincide, and these are marked. Consider the integral on the two line segments \( l_1 = \{ x : x = ce, c \in (e, \frac{1}{2} - e) \cup (\frac{1}{2} + e, 1 - e) \} \). For \( x \in l_1 \), \( |\phi(z)| = |\phi(-z)| = |\phi(z)\bar{z}| = |\phi(-z)\bar{z}| \), and

\[
\int_{l_1} b \ dx = 0
\]  

(60)
as the parts from \( c < \frac{1}{2} \) and \( c > \frac{1}{2} \) cancel. In a similar way, the integral on \( l_3 \) (see figure) is zero, and \( l_2 \) and \( l_4 \) are lattice translations of each other with opposite direction, and so their contributions to \( I \) cancel. The only contribution to \( I \) comes from the poles indicated. Consider the pole at \( x = 0 \). To investigate it, we expand \( f \) about \( z = 0 \), assuming \( x \) is near 0.

\[
f = (N - 2) \log |z|^2 + \log |z + x|^2 + \log |z - x|^2 + A(x) + \frac{1}{2}B(x)z + \frac{1}{2}B(x)\bar{z} + C(x)z^2 + \bar{C}(x)\bar{z}^2 + D(x)z\bar{z} + \ldots
\]

where \( A, C, D \) are all regular around \( x = 0 \). The symmetry \( z \to -z \) implies that \( B(x) \), and all other coefficients of terms odd in \( z \), vanish. Likewise, the symmetry \( x \to -x \) implies \( A(x), C(x), D(x) \) are all even functions of \( x \). To find the singularity in \( b \), compare this with an expansion about \( z = x \)

\[
f = \log |z - x|^2 + a(x) + \frac{1}{2}b(x)(z - x) + \frac{1}{2}b(x)(\bar{z} - \bar{x}) + c(x)(z - x)^2 + \bar{c}(x)(\bar{z} - \bar{x})^2 + d(x)(z - x)(\bar{z} - \bar{x}) + \ldots
\]

Using \( \log |z + x|^2 = \log |2x|^2 + \frac{1}{2}\bar{x}(z - x) + \frac{1}{2}(z - x) - \frac{1}{8x^2}(z - x)^2 - \frac{1}{8x^2}(\bar{z} - \bar{x})^2 + \ldots \) (the expansion is valid for \( |z - x| < |2x| \), which includes \( z = 0 \), and similarly for \( \log |z|^2 \) we obtain

\[
a(x) = A(x) + \log |2x|^2 + (N - 2) \log |x|^2 + C(x)x^2 + \bar{C}(x)\bar{x}^2 + D(x)x\bar{x} + O(x^4)
\]

\[
\frac{1}{2}b(x) = \frac{1}{2x} + \frac{N - 2}{x} + C(x)2x + D(x)\bar{x} + O(x^3)
\]

\[
c(x) = -\frac{1}{8x^2} - \frac{N - 2}{2x^2} + C(x) + O(x^2)
\]

\[
d(x) = D(x) + O(x^2)
\]

Hence, \( a(x), b(x), c(x) \) have logarithmic, first order and second order pole at \( x = 0 \), with the singularities completely determined by the log terms in the expansion around \( z = 0 \). The integral of \( 2ib \) on the contour near \( x = 0 \), \( l_c = \{x: x = ce^{i\theta}, \theta \in (0, \frac{\pi}{2})\} \) just has a contribution from the pole (the linear and all higher order terms are odd in \( x \) and vanish under integration):

\[
2i \int_{l_c} b \, dx = 2i \int_{0}^{\frac{\pi}{2}} (2N - 3)i d\theta = -\pi(2N - 3)
\]

A calculation for the pole of \( b \) at \( x = \frac{\pi}{2} \), where the two vortices again coincide (but this time the expansions are unaffected by the other vortices at \( z = 0 \)), yields a contribution to the integral of \(-2\pi \) (this time the integral is around a half circle), and likewise for the pole at \( x = \frac{1}{2} + \frac{\pi}{2} \). The poles at \( x = \frac{1}{2}, \frac{1}{2} + \tau \) contribute \( -\pi \) as the integral there is around a quarter circle. Finally, the pole at \( x = \tau \) makes the same contribution as \( x = 0 \). The residues of all the poles of \( b \) on \( \partial U_\frac{1}{4} \) are shown in figure 1. Their contributions to the integral \( I \) sum to give

\[
I = A - 2\pi(2N - 3 + 2 + 1) = A - 4\pi N
\]
The result is always positive, because of the observation in [6] that the Bogomolny equations for vortices on a compact Riemann surface only have non-trivial solutions if the physical area $A$ of the surface is larger than $4\pi N$.

The value of $I$ obtained for $x \in \mathcal{U}_1$ represents the area of a complex line, $\mathcal{C}P_1$, in the fibre $\mathcal{C}P_{N-1}$ (had we integrated instead over the whole unit cell $\mathcal{U}$, we would have obtained double the result from the sum of pole residues, but would then divide by two as this surface would have covered $\mathcal{C}P_1$ twice). Now in a complex manifold with a Kähler metric, the area of any complex curve is the integral of the Kähler form over it. So the integral of the Kähler form (35) over a complex line in the fibre is

\[ [\omega] = \int_{\mathcal{C}P_1} \omega = A - 4\pi N \] (66)

\[ \tilde{\omega} = \omega / [\omega] \] has integral of unity over a complex line, and so is a basis element for $H_2(\mathcal{C}P_{N-1}, \mathbb{Z})$, the second cohomology group of the fibre over the integers. Then, from the topology of complex projective space (see e.g. [10]), $\tilde{\omega}^{N-1}$ is a basis element for $H_2^{N-2}(\mathcal{C}P_{N-1}, \mathbb{Z})$. Now the volume of a Kähler manifold of dimension $2N - 2$ is the integral of $\omega^{N-1}/(N-1)!$ over it. Therefore the fibre volume is

\[ \text{Vol}(\mathcal{C}P_{N-1}) = \frac{(A - 4\pi N)^{N-1}}{(N-1)!} \] (67)

### 6 The partition function of $N$ vortices

We can now easily combine our results for the area of the base space of the fibre bundle (51) and the volume of the $\mathcal{C}P_{N-1}$ fibre (67); their product gives the volume of the moduli space

\[ \text{Vol}(\mathcal{M}_N) = \frac{A}{N!}(A - 4\pi N)^{N-1} = \frac{A^N}{N!}(1 - 4\pi n)^{N-1} \] (68)

where $n = \frac{N}{A}$ is the vortex number density. The partition function is then

\[ Z_{\text{torus}} = \frac{A^N}{N!}(1 - 4\pi n)^{N-1}(\frac{2\pi^2 T}{\hbar^2})^N \] (69)

This should be compared with the result obtained in [5]

\[ Z_{\text{sphere}} = \frac{A^N}{N!}(1 - 4\pi n)^N(\frac{2\pi^2 T}{\hbar^2})^N \] (70)

The additional single factor of $(1 - 4\pi n)$ is thermodynamically insignificant. We can calculate the free energy $F = -T\ln Z_{\text{torus}}$, using $N! \simeq N\ln N - N$ for large $N$,

\[ F \simeq -NT(-\ln N + \ln(A - 4\pi N) + \ln \frac{2e\pi^2 T}{\hbar^2}) \] (71)

The pressure $P = -\frac{\partial F}{\partial A}$ is

\[ P = \frac{NT}{A - 4\pi N} \] (72)
and the entropy $S = -\frac{\partial F}{\partial T}$ is proportional to $N$:

$$S = N(- \ln n + \ln (1 - 4\pi n) + \ln \frac{2e^2 \pi^2 T}{\hbar^2})$$

(73)

So, in the thermodynamic limit $N \to \infty$ at fixed $n$, we obtain the same physical results on the torus as on the sphere, a remarkable result given the disparity of the mathematical methods used to arrive at the result for the two manifolds. This result supports the general principle that in the thermodynamic limit the behaviour of the gas is independent of the global topology of the physical manifold it occupies, even though topological methods have been used to calculate the partition function. Since $A \to \infty$ as $N \to \infty$ and since the torus is intrinsically flat, we can interpret the results (72) and (73) as describing the thermodynamics of vortices in the plane.

In the low density limit we find

$$PA = NT(1 + 4\pi n + O(n^2))$$

(74)

as would be obtained for a gas of hard discs of area $2\pi$, and in the high density limit $n = \frac{1 - \epsilon}{4\pi}$, with $\epsilon$ small

$$P = T\epsilon^{-1} + O(1)$$

(75)

This describes what would happen if, keeping the number of vortices $N$ fixed, we were to decrease the physical area available to the vortices towards the limit $A = 4\pi N$; the work needed to compress the surface $-\int PdA$ would diverge as $A \to 4\pi N$.

We also note that the pressure and free energy are differentiable, provided $A > 4\pi N$, and hence there is no phase transition in a vortex gas at critical coupling. This is perhaps not unexpected due to the smooth interactions of the vortices and the absence of forces between them.

It would be interesting to investigate the behaviour of the vortices some small distance away from the critical coupling either towards a Type I superconductor (so that the vortices would be weakly attracting) or a Type II superconductor (with vortices repelling). In these cases, it is possible that the vortex gas could undergo a phase transition between a gaseous and a condensed (type I) or crystalline (type II) phase.
References


