GENERALIZED KDV FLOWS AND NILPOTENT SUBGROUPS OF AFFINE KAC-MOODY GROUPS

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1. Introduction.

In our recent paper [1] we gave a homological construction of the local integrals of motion in Toda field theories.

The system of Toda equations associated to an affine Kac-Moody algebra \( \mathfrak{g} \) reads

\[
\partial_t \partial_t \phi_i(t, \tau) = \sum_{j=0}^{l} (\alpha_i, \alpha_j) e^{-\phi_j(t, \tau)}, \quad i = 0, \ldots, l,
\]

where each \( \phi_i(t, \tau) \) is a family of functions on the circle with the coordinate \( t \), depending on the time variable \( \tau \), \( (\alpha_i, \alpha_j) \) is the scalar product of the \( i \)th and \( j \)th simple roots and \( l + 1 \) is the rank of \( \mathfrak{g} \).

Local integrals of motion of this system are local functionals in \( u_i(t) = \partial_t \phi_i(t), i = 0, \ldots, l \), which are preserved under the time evolution of the \( \phi_i(t) \)'s, defined by the system (1). Recall that local functionals are functionals of the form

\[
F[u(t)] = \int P(u(t), \partial_t u(t), \ldots) dt,
\]

where \( P \) is a differential polynomial in \( u(t) = (u_0(t), \ldots, u_l(t)) \).

The equation (1) can be written in the Hamiltonian form:

\[
\partial_t u(t, \tau) = \{ H, u(t, \tau) \},
\]

where \( \{ \cdot, \cdot \} \) is a certain Poisson bracket on the space of functionals in \( u \), and \( H \) is a Hamiltonian. The space of local integrals of motion is the kernel of the linear operator \( \{ H, \cdot \} \) on the space of local functionals. It turns out that these integrals of motion commute with each other with respect to the Poisson bracket \( \{ \cdot, \cdot \} \). The Hamiltonian equations associated to them coincide with the higher equations of the corresponding generalized KdV hierarchy [2, 3, 4] and so we will call these integrals of motion the KdV Hamiltonians.
The Hamiltonian $H$ of the Toda system is a sum of $l+1$ terms

$$H = \sum_{i=0}^{l} \int e^{-\delta_i(t)} dt.$$  

The operator of Poisson bracket with each of these terms can be lifted to a certain operator $Q_i$ on the space $\pi_0$ of all differential polynomials in $u(t)$. In [1] we proved that these operators satisfy the Serre relations of $\mathfrak{g}$. In other words, these operators define an action of the nilpotent Lie subalgebra $n_+$ of $\mathfrak{g}$ on $\pi_0$. This allowed us to interpret the space of KdV Hamiltonians as the first cohomology of $n_+$ with coefficients in the module $\pi_0$, cf. [1].

In the present work we will investigate further this action and explain the geometric and Lie algebraic meaning of KdV Hamiltonians in this framework.

The space $\pi_0$ is the space of algebraic functions on an infinite-dimensional affine space $U$ with the coordinates $\partial^n u_i, i = 1, \ldots, l, n \geq 0$. The space $U$ can be thought of as the formal space of solutions of the corresponding generalized KdV equation.

We will show that this space is isomorphic to the quotient $N_+/A_+$, where $N_+$ is the Lie group of $n_+$, and $A_+$ is its principal commutative subgroup. The action of the operator $Q_i$ on $\pi_0$ coincides with the operator of the left infinitesimal action of the $i$th generator of $n_+$ on the space of functions on $N_+/A_+$.

One can associate to any local functional $P$ in $u$, a derivation $\xi_P$ of $\pi_0$ and thus a vector field on the space $U$, so that such a vector field uniquely determines this local functional. In particular, we can attach a vector field $\xi_{H_m}$ on $U$ to each KdV Hamiltonian $H_m$. The corresponding hierarchy of generalized KdV equations can then be written as a system of evolution equations on $U$:

$$\frac{\partial u}{\partial \tau_m} = \xi_{H_m} \cdot u, \quad m \in I.$$  

On the other hand, the group $N_+$ can be thought of as a big cell on the flag manifold $B_- \backslash G$ of $\mathfrak{g}$. On this big cell, one has the right infinitesimal action of $\mathfrak{g}$ by vector fields. The Lie algebra $\mathfrak{g}$ contains a principal commutative Lie subalgebra $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where $\mathfrak{a}_+$ is the Lie algebra of $A_+$, and $\mathfrak{a}_-$ is its opposite with respect to a Cartan involution on $\mathfrak{g}$. The right action of $\mathfrak{a}_-$ on $N_+$ commutes with the right action of the group $A_+ \subset N_+$. Therefore each element of $\mathfrak{a}_-$ gives rise to a vector field on $N_+/A_+$.

We will prove that under the isomorphism $N_+/A_+ \cong U$ the vector field of a generator $p_m$ of $\mathfrak{a}_-$ of degree $-m$ coincides with the vector field $\xi_{H_m}$ associated to the KdV Hamiltonian $H_m$. It is well-known that the space of KdV Hamiltonians is isomorphic to the Lie algebra $\mathfrak{a}_-$ as a graded linear space [2, 3, 4]. Our construction unveils the geometric meaning of this isomorphism at the level of Lie algebras.

Thus we see that the KdV Hamiltonians are encoded in the geometry of the group $N_+$. Moreover, the Hamiltonian structure of the KdV hierarchy can be obtained from
a certain structure on the homogeneous space $N_+/A_+$, which can be thought of as a classical limit of the structure of vertex operator algebra of free fields, cf. [1]. It is an interesting question whether this structure is related to the Lie-Poisson structure on the Borel group of $N_+$ [5].

Quantization of Lie-Poisson structures leads to quantum groups [6]. On the other hand, in [1] we proved that all classical KdV hamiltonians can be quantized in the context of vertex operator algebras, and quantum groups appeared in our construction. This makes us to believe that the right quantum counterpart of the classical construction, which we present here, should involve a mixture of quantum groups and vertex operator algebras.

Affine Kac-Moody algebras have also appeared in the tau-function formalism of the KdV hierarchy and its generalizations, cf. [8, 9] and more recent works [10, 11, 12, 13] and references therein. It would be very interesting to understand the connection between this formalism and our construction.

The paper is organized as follows. In Sect. 2 we remind the hamiltonian formalism of affine Toda field theories. We introduce the space $\pi_0$ and show that the Toda hamiltonian gives rise to an action of the nilpotent subalgebra $n_+$ of $g$ on $\pi_0$. In Sect. 3 we prove that the space $\pi_0$ is isomorphic to the space of functions on a coset space $Y$ of the nilpotent group $N_+$. In Sect. 4, using the Bernstein-Gelfand-Gelfand resolution, we explain how the first cohomology classes of $n_+$ with coefficients in $\pi_0$ define the KdV hamiltonians and vector fields on $Y$. Finally, in Sect. 5 we identify these vector fields with vector fields of the right infinitesimal action of $a_-$ on $Y$.

2. The hamiltonian picture.

In this Section we will introduce the main objects involved in the hamiltonian structure of affine Toda field theories. The algebraic formalism that we use was essentially developed in the works of Gelfand and Dickey [14, 15] and of Kupershmidt and Wilson [3, 4]. More details, motivations and references can also be found in [1].

Let $g$ be an affine Kac-Moody algebra, twisted or untwisted, of rank $l+1$, without central element. To $g$ one canonically associates a finite-dimensional Lie algebra $\tilde{g}$, whose Dynkin diagram is obtained from the Dynkin diagram of $g$ by deleting the 0th node. We have the Cartan decomposition: $\tilde{g} = \tilde{n}_- \oplus \tilde{h} \oplus \tilde{n}_+$, where $\tilde{h}$ is the Cartan subalgebra and $\tilde{n}_\pm$ are the nilpotent subalgebras.

We consider $g$ as defined over the formal Laurent power series $\mathbb{C}((t))$, with the topology of inverse limit. If $g$ is untwisted, it is just the “current algebra” $\tilde{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}d$, where $d$ is a $\mathbb{Z}$-grading operator. If $g$ is twisted, it consists of currents, which have special properties with respect to an automorphism of $\tilde{g}$ [7].

Recall that the Lie algebra $g$ has the Cartan decomposition: $g = n_+ \oplus h \oplus n_-$. Here $h$ is the Cartan subalgebra of $g$: $h = h \oplus 1 \oplus \mathbb{C}d$, with the linear basis $h_1, \ldots, h_l, d$. The upper and lower nilpotent Lie subalgebras $n_+$ and $n_-$ are generated by elements
\(e_i, i = 0, \ldots, l\) and \(f_i, i = 0, \ldots, l\), which satisfy the Serre relations [7]:

\[(\text{ad } e_i)^{-a_{ij}+1} \cdot e_j = 0, \quad (\text{ad } f_i)^{-a_{ij}+1} \cdot f_j = 0.\]

The dual space \(\mathfrak{h}^*\) to \(\mathfrak{h}\) is linearly spanned by functionals \(\alpha_i, i = 0, \ldots, l\). The value of \(\alpha_j\) on the \(i\)th generator \(h_i\) of the Cartan subalgebra of \(\mathfrak{g}\) is equal to the element \(a_{ij}\) of the Cartan matrix of \(\mathfrak{g}\). We also have: \(\alpha_0(d) = 1\), and \(\alpha_j(d) = 0, j = 1, \ldots, l\). Denote by \((\cdot, \cdot)\) the invariant scalar product on \(\mathfrak{h}^*\), normalized as in [7]. Introduce the imaginary root \(\delta\) of \(\mathfrak{g}\):

\[\delta = \sum_{0 \leq i \leq l} a_i \alpha_i,\]

where the \(a_i\)'s are the labels of the Dynkin diagram of \(\mathfrak{g}\) [7], which satisfy

\[\sum_{0 \leq j \leq l} a_j a_{ij} = 0\]

for all \(i\). We have: \((\delta, x) = 0\) for any \(x \in \mathfrak{h}^*\).

Now let \(\pi_0\) be the free polynomial algebra with generators \(u_i^{(n)}\), \(i = 1, \ldots, l, n \geq 0\). Define a derivation \(\partial\) on \(\pi_0\) by putting \(\partial u_i^{(n)} = u_i^{(n+1)}\). By Leibnitz rule, we extend \(\partial\) to the whole \(\pi_0\). We call \(\pi_0\) the algebra of differential polynomials. Introduce a \(\mathbb{Z}\)-grading on \(\pi_0\) by putting \(\deg u_i^{(n)} = -n - 1\). With respect to this grading, the derivative \(\partial\) is a homogeneous linear operator of degree \(-1\).

We also introduce the variables \(u_0^{(n)}, n \geq 0:\)

\[u_0^{(n)} = -\frac{1}{a_0} \sum_{1 \leq i \leq l} a_i u_i^{(n)} .\]

Let \(\mathcal{F}_0\) be the quotient of \(\pi_0\) by the subspace, generated by the total derivatives and constants. We can introduce a \(\mathbb{Z}\)-grading on \(\mathcal{F}_0\) by adding 1 to the grading induced from \(\pi_0\). It can be interpreted as the space of local functionals of the form (2), since the integral of a total derivative or a constant is equal to 0. Denote by \(f\) the projection \(\pi_0 \rightarrow \mathcal{F}_0\).

**Remark 1.** Conventions in this paper differ slightly from those in [1].

Introduce the operator \(\delta_i\) of \(i\)th variational derivative on \(\pi_0\) by the formula

\[\delta_i P = \sum_{n \geq 0} \sum_{1 \leq j \leq l} (\alpha_i, \alpha_j)(-\partial)^n \frac{\partial P}{\partial u_j^{(n)}}, \quad i = 0, \ldots, l.\]

For \(P \in \pi_0\) let \(\xi_P : \pi_0 \rightarrow \pi_0\) be the derivation given by

\[\xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}} .\]
In particular,

\begin{equation}
(6) \quad \partial = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}} = \xi_P, \quad P = \frac{1}{2} \sum_{1 \leq i \leq l} u_i^{(0)} u_i^{(0)};\end{equation}

where \( u_i^{(0)}; i = 1 \ldots, l, \) are vectors dual to \( u_i^{(0)}; i = 1 \ldots, l, \) with respect to the scalar product defined by \((\cdot, \cdot)\).

Note that if \( P \) is a total derivative or a constant, then \( \xi_P = 0. \) Thus we see that the map \( \xi \) from \( \pi_0 \) to derivations of \( \pi_0 \) factors through \( \mathcal{F}_0. \) We will use the same notation for the corresponding map from \( \mathcal{F}_0 \) to derivations of \( \pi_0. \)

For any element \( P \in \pi_0 \) the derivation \( \xi_P \) commutes with the action of the derivative \( \partial. \) Indeed, from formula (5) we obtain:

\[ [\partial, \partial_i^{(n)}] = -\partial_i^{(n-1)}. \]

Therefore

\[ [\partial, \xi_P] = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+2} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n)}} - \sum_{1 \leq i \leq l, n > 0} (\partial^{n+1} \cdot \delta_i P) \frac{\partial}{\partial u_i^{(n-1)}} = 0. \]

Thus we can define an operator \( \{\cdot, \cdot\} : \mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_0 \) by putting

\begin{equation}
(7) \quad \{ \int P, \int R \} = \int (\xi_P \cdot R).\end{equation}

It is well-known that this map is antisymmetric and satisfies the Jacobi identity, cf., e.g., [15]; therefore it makes \( \mathcal{F}_0 \) into a Lie algebra. One can also check that the Lie bracket (7) preserves the \( \mathbb{Z} \)-grading.

**Remark 2.** If we view elements of \( \mathcal{F}_0 \) as functionals on the space of functions on the circle with values in the Cartan subalgebra of \( \mathfrak{g}, \) \( \mathbf{u}(t) = (u_1(t), \ldots, u_l(t)), \) then we can interpret formula (7) as a Poisson bracket between such functionals, cf. [1]. \( \square \)

Now let us formally introduce variables \( \phi_i; i = 0, \ldots, l, \) such that \( \partial \phi_i = u_i^{(0)}. \) For any element \( \lambda = \sum_{0 \leq i \leq l} \lambda_i \alpha_i; \) of the weight lattice \( \Lambda \) of \( \mathfrak{g} \) define the linear space \( \pi_\lambda = \pi_0 \otimes e^\lambda, \) where \( \lambda = \sum_{0 \leq i \leq l} \lambda_i \phi_i, \) equipped with an action of \( \partial \) by the formula

\begin{equation}
(8) \quad \partial \cdot (P \otimes e^\lambda) = (\partial P) \otimes e^\lambda + \left( \sum_{0 \leq i \leq l} \lambda_i u_i^{(0)} P \right) \otimes e^\lambda.\end{equation}

Let \( \mathcal{F}_\lambda \) be the quotient of \( \pi_\lambda \) by the subspace of total derivatives and \( f \) be the projection \( \pi_\lambda \to \mathcal{F}_\lambda. \)

For any \( P \in \mathcal{F}_0 \) the derivation \( \xi_P : \pi_0 \to \pi_0 \) can be extended to a linear operator on \( \bigoplus_{\lambda \in \Lambda} \pi_\lambda \) by

\[ \xi_P = \sum_{1 \leq i \leq l, n \geq 0} (\partial^{n+1} \cdot \delta_i P) + \sum_{1 \leq i \leq l} \delta_i P \frac{\partial}{\partial \phi_i}. \]
where \( \partial/\partial \phi_i \cdot (S e^\lambda) = \lambda_i S e^\lambda \). This defines the structure of an \( \mathcal{F}_0 \)-module on \( \pi_\lambda \).

For any \( P \in \pi_0 \) the operator \( \xi_P \) commutes with the action of derivative. Hence we obtain the structure of an \( \mathcal{F}_0 \)-module on \( \mathcal{F}_\lambda \), which gives us a map \( \{\cdot, \cdot\} : \mathcal{F}_0 \times \mathcal{F}_\lambda \to \mathcal{F}_\lambda \).

Similarly, any element \( R \in \pi_\lambda \) defines a linear operator \( \xi_R \), acting from \( \pi_0 \) to \( \pi_\lambda \) and commuting with \( \partial \):

\[
\xi_{S \lambda} = \sum_{1 \leq i \leq l, n \geq 0} \partial^n \left( \partial(\delta_i S e^\lambda) - S \frac{\partial e^\lambda}{\partial \phi_i} \right) \frac{\partial}{\partial u_i^{(n)}}.
\]

The operator \( \xi_R \) depends only on the class of \( R \) in \( \mathcal{F}_\lambda \). Therefore it defines a map \( \{\cdot, \cdot\} : \mathcal{F}_\lambda \times \mathcal{F}_0 \to \mathcal{F}_\lambda \). We have for any \( P \in \mathcal{F}_0, R \in \mathcal{F}_\lambda \):

\[
\xi_R \cdot \int P = -\xi_P \cdot \int R.
\]

Therefore our bracket \( \{\cdot, \cdot\} \) is antisymmetric.

One can also check that for any \( P \in \mathcal{F}_0, R \in \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda \), we have

\[
\xi_{[P, Q]} = [\xi_P, \xi_Q].
\]

Denote by \( \tilde{Q}_i, i = 0, \ldots, l \), the linear operators \( \xi_{\int e^{-\phi_i}} : \pi_0 \to \pi_{-1} \), corresponding to \( \int e^{-\phi_i} \in \mathcal{F}_{-1} \). We have:

\[
\tilde{Q}_i = -\sum_{n \geq 0} (\partial^n e^{-\phi_i}) \partial_i^{(n)},
\]

where

\[
\partial_i^{(n)} = \sum_{1 \leq j \leq l} (\alpha_i, \alpha_j) \frac{\partial}{\partial u_j^{(n)}}.
\]

Clearly, \( \partial^n e^{-\phi_i} = B_i^{(n)} e^{-\phi_i} \), where \( B_i^{(n)} \)'s are certain polynomials in \( u_i^{(n)} \)'s. They satisfy a recurrence relation:

\[
B_i^{(n)} = -u_i^{(0)} B_i^{(n-1)} + \partial B_i^{(n-1)},
\]

with the initial condition \( B_i^{(0)} = 1 \).

We can now define derivations \( Q_0, \ldots, Q_l \) of \( \pi_0 \) by the formula

\[
Q_i = -\sum_{n \geq 0} B_i^{(n)} \partial_i^{(n)}.
\]

The following statement was proved in [1].

**Lemma 1.** The operators \( Q_i \) satisfy the Serre relations of the Lie algebra \( \mathfrak{g} \):

\[
(ad Q_i)^{-\alpha_{ij}+1} \cdot Q_j = 0,
\]

where \( \|a_{ij}\| \) is the Cartan matrix of \( \mathfrak{g} \).
According to Lemma 1, we obtain the structure of an \( n_+ \)-module on \( \pi_0 \) by assigning to each generator \( e_i \) of the nilpotent Lie subalgebra \( n_+ \) of \( g \) the operator \( Q_i : \pi_0 \rightarrow \pi_0, i = 0, \ldots, l \).

Since each of the operators \( \tilde{Q}_i \) commutes with the action of \( \partial \), it induces a linear operator \( \tilde{Q}_i \) from \( \mathcal{F}_0 \) to \( \mathcal{F}_{-a_i} \). As was explained in [1], the sum of the operators \( \tilde{Q}_i \) coincides with the operator of the bracket \( \{ H_i \cdot \} \) with the Hamiltonian (3) of the Toda field theory associated to \( g \). The elements of the space of local functionals, \( \mathcal{F}_0 \), which lie in the kernel of this operator, are called local integrals of motion of the corresponding Toda theory. It is known that they coincide with the Hamiltonians of the generalized KdV hierarchy associated to \( g \) [2, 3, 4]. This motivates the following definition.

**Definition 1.** The space of KdV Hamiltonians is the intersection of kernels of the operators \( \tilde{Q}_i : \mathcal{F}_0 \rightarrow \mathcal{F}_{-a_i}, i = 0, \ldots, l \).

Denote by \( I \) the set of exponents of \( g \) modulo the Coxeter number. In this paper we will give a new proof of the following result.

**Theorem 1.** The space of KdV Hamiltonians is linearly spanned by elements \( H_m, m \in I, \) where \( \deg H_m = -m \).

3. THE GEOMETRIC PICTURE.

Let \( G \) be the Lie group of \( g \). Accordingly, \( G \) is also defined over formal Laurent series, and it is also equipped with the topology of inverse limit. We consider the induced topology on Lie subgroups of \( G \).

Let \( B_+ \) and \( B_- \) be the Borel subgroups of \( G \). They consist of the \( \mathbb{C}[[t]] \)-points (respectively, \( \mathbb{C}[t^{-1}] \)-points) of \( G \), whose image in the constant Lie subgroup \( \tilde{G} \) of \( G \) belongs to the finite-dimensional Borel subgroup \( \tilde{B}_+ \) (respectively, \( \tilde{B}_- \)), or its invariant part with respect to an automorphism, if \( g \) is twisted.

Let \( N_+ \) be the radical of \( B_+ \); it consists of those elements of \( B_+ \), whose image in \( \tilde{G} \) belongs to the nilpotent subgroup \( \tilde{N}_+ \) of \( \tilde{G} \), or its invariant part with respect to an automorphism, if \( g \) is twisted. The group \( N_+ \) is a pronilpotent proalgebraic group, which is the inverse limit of finite-dimensional algebraic groups \( N_+^{(n)} \backslash N_+, n > 0 \), isomorphic to finite-dimensional affine spaces. Here \( N_+^{(n)} \) denotes the subgroup of \( N_+ \), which consists of elements, which are equal to 1 modulo \( t^n \). By the space of functions on \( N_+ \), we will always mean the space of algebraic functions, which is the inductive limit of the spaces of algebraic functions on \( N_+^{(n)} \backslash N_+, n > 0 \).

In the previous section we introduced the Lie algebra \( n_+ \) of \( N_+ \). For an untwisted affine algebra \( g \) we have \( n_+ = \tilde{n}_+ \oplus \tilde{g} \oplus \mathbb{C}[[t]] \). The exponential map \( \exp : n_+ \rightarrow N_+ \) is an isomorphism, and it introduces the structure of an inverse limit of finite-dimensional linear spaces on the group \( N_+ \).
Let \( p = \sum_{0 \leq i \leq l} a_i e_i \) be the principal element of \( n_+ \), where \( e_i, i = 0, \ldots, l \), are the generators of \( n_+ \), and \( a_i, i = 0, \ldots, l \), are the labels of the Dynkin diagram of \( g \).

Denote by \( a_+ \) the principal commutative Lie subalgebra of \( n_+ \), which is the centralizer of \( p \) in \( n_+ \).

The Lie algebra \( g \) is graded by weights of the Cartan subalgebra \( h \). We can also introduce the principal \( \mathbb{Z} \)-grading on \( g \) by putting \( \deg h_i = \deg d = 0, \deg e_i = -\deg f_i = 1, i = 0, \ldots, l \). It is known that with respect to this grading the Lie algebra \( a_+ \) is linearly spanned by elements of degrees equal to the exponents of \( g \) modulo the Coxeter number. It is also known that the Lie algebra \( n_+ \) splits into the direct sum \( n_+ = \text{Ker} p \oplus \text{Im} p \), where \( \text{Ker} p = a_+ \) and \( \text{Im} p = \bigoplus_{j > 0} n_j^+ \) is the direct sum of homogeneous components \( n_j^+ \), each having the same dimension \( l \). For a proof of these facts, cf. [16], Proposition 3.8 (b).

We can also consider the centralizer \( a \) of \( p \) in \( g \). We have \( a = a_+ \oplus a_- \), where \( a_- \) is a commutative Lie subalgebra of \( n_- \), which can be obtained by applying a Cartan involution of \( g \) to \( a_+ \). In particular, \( a_- \) is linearly spanned by generators of degrees equal to minus the exponents of \( g \) modulo the Coxeter number. Let us choose for each \( m \in I \) a linear generator \( p_m \) of \( a_- \) degree \(-m\). We choose

\[
p_1 = \sum_{0 \leq i \leq l} \frac{(a_i, a_1)}{2} f_i.
\]

The group \( N_+ \) is isomorphic to the the big cell \( X \) of the flag manifold \( F = B_- \setminus G \), which is the orbit of the image of \( 1 \in G \) under the action of \( N_+ \). This flag manifold has been studied, e.g., in [17, 18, 19, 20, 21]. The Lie algebra \( g \) infinitesimally acts from the right by vector fields on \( F \) and hence on \( N_+ \). So does the Lie algebra \( \mathcal{V}ect_- = \mathbb{C}[t^{-1}] t \partial_t \), spanned by the vector fields on the circle \( L_n = t^{-n+1} \partial_t, n \geq 0 \). Indeed, the Lie algebra of vector fields on the circle \( \mathbb{C}((t)) \partial_t \) infinitesimally acts on the group \( G \), cf., e.g., [20]. Its Lie subalgebra \( \mathcal{V}ect_- \) preserves the group \( B_- \) and hence it maps to vector fields on \( F \) and on \( N_+ \). Note that the action of \( L_0 \) coincides with the action of \( d \).

Denote by \( \mathcal{V} \) the Lie algebra of vector fields on \( N_+ \). It contains two commuting Lie subalgebras: \( n_+^R \) and \( n_+^L \), of vector fields of right and left infinitesimal action of \( n_+ \) on its Lie group. The vector field of left infinitesimal action of an element \( \beta \in n_+ \) on \( N_+ \) will be denoted by \( \beta^L \).

The Lie algebra \( n_+^R \) is a part of a larger Lie subalgebra of \( \mathcal{V} \), which is isomorphic to \( \mathfrak{g} = g \times \mathcal{V}ect_- \). The vector field of right infinitesimal action of \( \alpha \in \mathfrak{g} \) on \( N_+ \) will be denoted by \( \alpha^R \).

We will describe a geometric construction of modules contragradient to the Verma modules over \( g \) and homomorphisms between them. This construction is an affine analogue of results proved by Kostant [22] (cf. also cf. also [23, 24, 25]) in the case of simple Lie algebras, although our argument and goal are different. A generalization of this construction to arbitrary symmetrizable Kac-Moody algebras is straightforward.
For $\lambda \in \mathfrak{h}^*$, let $M_\lambda$ be the Verma module over $\mathfrak{g}$ of highest weight $\lambda$:

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ denotes the one-dimensional representation of $\mathfrak{b}_+$, on which $\mathfrak{h} \subset \mathfrak{b}_+$ acts according to its character $\lambda$, and $\mathfrak{n}_+ \subset \mathfrak{b}_+$ acts trivially. Denote by $M_\lambda^\vee$ the module contragradient to $M_\lambda$. As a linear space, it is the restricted dual of $M_\lambda$. Denote by $\langle \cdot, \cdot \rangle$ the pairing $M_\lambda^\vee \times M_\lambda \rightarrow \mathbb{C}$. Let $\omega$ the Cartan anti-involution on $\mathfrak{g}$, which maps generators $e_0, \ldots, e_l$ to $f_0, \ldots, f_l$ and vice versa and preserves $\mathfrak{h}$ [7]. Then the action of $x \in \mathfrak{g}$ on $y \in M_\lambda^\vee$ is defined as follows:

$$\langle x \cdot y, z \rangle = \langle x, \omega(y) \cdot z \rangle, \quad z \in M_\lambda.$$

Suppose first that the highest weight $\lambda$ is integral. Then the module $M_\lambda^\vee$ can be realized as the space of sections of a line bundle over the big cell $X$, corresponding to the character of the group $B_-$, which vanishes on $N_-$ and is defined by $\lambda$ on the Lie group of $\mathfrak{h}$. This line bundle can be trivialized. Therefore the module $M_\lambda^\vee$ is isomorphic to the space of functions on $X$ with respect to the twisted action of $\mathfrak{g}$ by first order differential operators. For an element $\beta$ of $\mathfrak{g}$ this differential operator is equal to $\beta^R + F_\lambda(\beta)$, where $F_\lambda(\beta)$ is a function on $X$. This function is necessarily algebraic. Indeed, if $\beta$ is homogeneous, $F_\lambda(\beta)$ is also homogeneous of the same weight, and all homogeneous functions are algebraic.

We can interpret the functions $F_\lambda(\beta)$ as elements of the group $H^1(\mathfrak{g}, \mathbb{C}[X])$, where $\mathbb{C}[X]$ is the space of algebraic functions on $X$ considered as a $\mathfrak{g}$-module with respect to the right infinitesimal action. Indeed, in the standard complex of Lie algebra cohomology an element of $H^1(\mathfrak{g}, \mathbb{C}[X])$ is realized as a linear functional $f$ on $\mathfrak{g}$ with values in $\mathbb{C}[X]$. Such an element defines a deformation of the $\mathfrak{g}$-module $\mathbb{C}[X]$. The deformed action of $\beta \in \mathfrak{g}$ is obtained by adding to the old action the operator of multiplication by $f(\beta)$. By Shapiro lemma, $H^1(\mathfrak{g}, \mathbb{C}[X]) \approx H^1(\mathfrak{b}_+, \mathbb{C}) \approx (\mathfrak{b}_+/[\mathfrak{b}_+, \mathfrak{b}_+])^* = \mathfrak{h}^*$. We see that all elements of $H^1(\mathfrak{g}, \mathbb{C}[X])$ have weight 0. On the other hand, functions on $X$ can only have non-positive weights and the only functions, which have weight 0 – the constants – are invariant with respect to the action of $\mathfrak{g}$. Therefore the coboundary of any element of the 0th group of the complex, $\mathbb{C}[X]$, has a non-zero weight. Hence any cohomology class from $H^1(\mathfrak{g}, \mathbb{C}[X])$ canonically defines a one-cocycle $f$, that is a map $\mathfrak{g} \rightarrow \mathbb{C}[X]$. Thus, having identified the space of deformations with $\mathfrak{h}^*$, we can assign to each $\lambda \in \mathfrak{h}^*$ and each $\beta \in \mathfrak{g}$ a function on $X$ – this is our $F_\lambda(\beta)$. By linearity, if $\lambda = \sum_{i=0}^l \lambda_i \alpha_i$, then $F_\lambda(\beta) = \sum_{i=0}^l \lambda_i F_i(\beta)$, where we put $F_i(\beta) = F_{\alpha_i}(\beta)$.

Note that, in particular, $F_\lambda(\beta) = 0$ for any $\lambda$, if $\beta \in \mathfrak{n}_+$, and $F_\lambda(\beta) = \lambda(\beta)$, if $\beta \in \mathfrak{h}$.

Thus, for any weight $\lambda \in \mathfrak{h}^*$, we realize the module $M_\lambda^\vee$ as the space of functions on $X$ equipped with the twisted action of $\mathfrak{g}$ by first order differential operators.

Vector $y$ from the Verma module $M_\lambda$ is called singular of weight $\mu \in \mathfrak{h}^*$, if $\mathfrak{n}_+ \cdot y = 0$ and $x \cdot y = \mu(x) y$ for any $x \in \mathfrak{h}$. In particular, the highest weight vector $v_\lambda$ of $M_\lambda$ is a
singular vector of weight \( \lambda \). Since \( M_\lambda \) is isomorphic to \( U(\mathfrak{n}_-) \cdot v_\lambda \) as an \( \mathfrak{n}_- \)-module, any singular vector of \( M_\lambda \) of weight \( \mu \) can be uniquely written as \( P \cdot v_\lambda \) for some element \( P \in U(\mathfrak{n}_-) \) of weight \( \mu - \lambda \). This singular vector canonically defines a homomorphism of \( \mathfrak{g} \)-modules \( i_P : M_\mu \rightarrow M_\lambda \), which sends \( u \cdot v_\mu \) to \( (uP) \cdot v_\lambda \) for any \( u \in U(\mathfrak{n}_-) \).

There is an isomorphism \( U(\mathfrak{n}_-) \rightarrow U(\mathfrak{n}_+) \), which maps the generators \( f_0, \ldots, f_l \) to \( e_0, \ldots, e_l \). Denote by \( \tilde{P} \) the image of \( P \in U(\mathfrak{n}_-) \) under this isomorphism.

The homomorphism \( \mathfrak{n}_+ \rightarrow \mathcal{V} \), mapping \( \alpha \in \mathfrak{n}_+ \) to \( \alpha^L \), can be extended in a unique way to a homomorphism from \( U(\mathfrak{n}_+) \) to the algebra of differential operators on \( X \). Denote the image of \( u \in U(\mathfrak{n}_+) \) under this homomorphism by \( u^L \).

It turns out that using the left infinitesimal action of \( \mathfrak{n}_+ \) on \( X \), one can realize the dual homomorphisms \( i_P^* : M_\lambda^* \rightarrow M_\mu^* \) as differential operators on \( X \).

**Proposition 1.** If \( P \cdot v_\lambda \) is a singular vector in \( M_\lambda \) of weight \( \mu \), then the homomorphism \( i_P^* : M_\lambda^* \rightarrow M_\mu^* \) is given by the differential operator \( \tilde{P}^L \).

**Proof.** As an \( \mathfrak{n}_+ \)-module, the module \( M_\lambda^* \) is isomorphic to the restricted dual module to \( U(\mathfrak{n}_+) \). Hence in addition to the right action of \( \mathfrak{g} \) on \( M_\lambda^* \), we have a left action of \( \mathfrak{n}_+ \). If we realize \( M_\lambda^* \) as the space of functions on \( X \), this right action coincides with the left infinitesimal action of \( \mathfrak{n}_+ \) by vector fields. In other words,

\[
\langle [\beta^L \cdot u, v_\lambda] \rangle = \langle x, (\omega(\beta)) \cdot v_\lambda \rangle, \quad \beta \in \mathfrak{n}_+, u \in U(\mathfrak{n}_-). \]

Accordingly, the left action of \( P \in U(\mathfrak{n}_+) \) on \( M_\lambda^* \) coincides with the action of the differential operator \( \tilde{P}^L \), therefore

\[
\langle \tilde{P}^L \cdot x, u \cdot v_\lambda \rangle = \langle x, (uP) \cdot v_\lambda \rangle.
\]

On the other hand, by definition,

\[
\langle i_P^* \cdot x, u \cdot v_\lambda \rangle = \langle x, i_P \cdot (u \cdot v_\lambda) \rangle = \langle x, (uP) \cdot v_\lambda \rangle,
\]

and Proposition follows. \( \square \)

We can now derive the following crucial statement.

**Lemma 2.**

(a) If \( \alpha \in \mathcal{V} \) is such that for any \( \beta \in \mathfrak{n}_+^L \) (respectively, \( \beta \in \mathfrak{n}_+^R \)) \( [\alpha, \beta] = 0 \), then \( \alpha \in \mathfrak{n}_+^L \) (respectively, \( \alpha \in \mathfrak{n}_+^R \)).

(b) For any \( \beta \in \tilde{\mathfrak{g}} \) we have:

\[
[e_i^L, \beta^R] = -F_i(\beta)e_i^L, \quad i = 0, \ldots, l.
\]

**Proof.** Part (a) is clear, because such a vector field is \( \mathfrak{n}_+^L \)- or \( \mathfrak{n}_+^R \)-invariant, and hence is uniquely defined by its value at the origin of the group \( N_+ \).

To prove part (b), consider the particular case of Proposition 1, when \( \lambda = 0 \) and \( \mu = -\alpha_i \). It is known that vector \( f_i \cdot v_0 \) is a singular vector of \( M_0 \) of weight \( -\alpha_i \). By Proposition 1, \( e_i^L \) is a \( \mathfrak{g} \)-homomorphism from \( M_0^* \) to \( M_{-\alpha_i}^* \). It is therefore a \( \mathfrak{g} \)-homomorphism, since the action of \( \text{Vect}_- \) on any \( M_\lambda \) (and any module from
the category $\mathcal{O}$ of $\mathfrak{g}$-modules) can be expressed in terms of the action of $\mathfrak{g}$ via the Sugawara construction.

Thus the operator $e_i^L$ intertwines the actions of $\tilde{\mathfrak{g}}$ on $M^+_0$ and $M^{-\alpha_i}_0$. But $\beta \in \tilde{\mathfrak{g}}$ acts on $M^+_0$ as $\beta^L$, and on $M^{-\alpha_i}_0$ as $\beta^L - F_i(\beta)$. This gives us formula (14). □

It is clear that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts diagonally on $\mathcal{Y}$, and thus defines a grading of $\mathcal{Y}$ by weights of $\mathfrak{g}$. In particular, vector fields $e_i^L$ and $e_i^R$ both have weight $\alpha_i$. Therefore we have $[e_i^L, h_j^R] = -a_{ji}e_i^L$, and so $F_i(h_j) = a_{ji}$.

Now consider $F_i(f_j)$. Let us apply $e_i^R$ to the left and right hand sides of formula (14) for $\beta = f_j$. We obtain:

\[ [e_i^R, [e_i^L, f_j^R]] = [[e_i^R, e_i^L], f_j^R] + [e_i^R, [e_i^L, f_j^R]] = \delta_{k,j}[e_i^L, h_j^R] = -a_{ji}\delta_{k,j}e_i^L \]

\[ = -(e_i^R \cdot F_i(f_j))e_i^L. \]

Therefore $F_i(f_j)$ satisfies:

\[ (e_i^R \cdot F_i(f_j)) = a_{ji}\delta_{k,j}. \]

There are unique functions $x_i, i = 0, \ldots, l$, on $N_+$, which have the property

\[ e_i^R \cdot x_j = -e_i^L \cdot x_j = \delta_{k,j}. \]

We see that $F_i(f_j) = a_{ji} x_j$.

Formula (13) gives: $F_i(p_1) = \sum_{0 \leq j \leq l} (\alpha_i, \alpha_j)x_j$, because $a_{ji} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$.

Now denote by $A_+$ the image of $\mathfrak{a}_+$ in $N_+$ under the exponential map. Let $Y$ be the homogeneous space $N_+/A_+$. On $Y$ we have the left infinitesimal action of $\mathfrak{n}_+$. Since the right infinitesimal action of the Lie algebra $\mathfrak{a}_-$ commutes with the right action of $A_+$ on $N_+$, we obtain a homomorphism from $\mathfrak{a}_-$ to the Lie algebra of vector fields on the space $Y$.

There exists $\Delta \in \mathfrak{h}$, such that $[\Delta, e_i] = 1$ for $i = 0, \ldots, l$. The action of $\Delta$ on $\mathfrak{g}$ coincides with the action of the principal $\mathbb{Z}$-grading. Therefore it preserves the Lie algebra $\mathfrak{a}_+$, and hence $\Delta^R$ defines a vector field on $Y$, for which we will use the same notation. The commutator with $\Delta^R$ defines a $\mathbb{Z}$-grading on the space of functions on $Y$.

The functions $F_i(p_1)$ are $\mathfrak{a}_+^R$-invariant algebraic functions on $N_+$ of degree $-1$. Indeed, let us apply $x^R \in \mathfrak{a}_+^R$ to the left hand side of formula (14) with $\beta = p_1$. We obtain

\[ [x^R, [e_i^L, p_1^R]] = [[x^R, e_i^L], p_1^R] + [e_i^L, [x^R, p_1^R]] = 0, \]

because $\mathfrak{a}_+^R \subset \mathfrak{n}_+^R$. The right hand side of formula (14) then gives $(x^R \cdot F_i(p_1))e_i^L = 0$ for any $i$. Therefore $F_i(p_1)$ is $\mathfrak{a}_+^R$-invariant. In the same way we can show that the functions $F_i(x)$ are $\mathfrak{a}_+^R$-invariant for any $x \in \mathfrak{a}_-$.

Denote by $u_i^{(0)}$ the algebraic function on $Y$, corresponding to $F_i(p_1), i = 1, \ldots, l$, and let $u_i^{(m)}, i = 1, \ldots, l, m \geq 0$, be the algebraic function $p_1^m \cdot u_i^{(0)}$. Denote by $U^{(m)}$
the linear space with coordinates $u_i^{(j)}, j = 1, \ldots, l, j = 0, \ldots, m$, and by $U$ the inverse limit of the spaces $U^{(m)}$.

**Proposition 2.** The homogeneous space $Y = N_+ / A_+$ is isomorphic to the space $U$.

**Proof.** Consider the values of the differentials $du_{i,1}^{(n)}$ of the functions $u_i^{(n)}$ at the image $\bar{1} \in Y$ of $1 \in N_+$. They are vectors in the cotangent space to $Y$ at $\bar{1}$, which is naturally isomorphic to $(n_+/a_+)^*$. The covectors $du_{i,1}^{(0)}, i = 1, \ldots, l$, form a linear basis in $(n_+/a_+)^*$. The element $p_1$ of $a_-$, which sends $du_{i,1}^{(m)}$ to $du_{i,1}^{(m+1)}$, maps $(n_+/a_+)^*$ isomorphically to $(n_+^{m+1})^*$. Hence the vectors $du_{i,1}^{(m)}$ are linearly independent. Therefore the functions $u_i^{(n)}, i = 1, \ldots, l, n \geq 0$, are algebraically (and functionally) independent. With respect to the $\mathbb{Z}$-grading defined by the vector field $\Delta^R$, the function $u_i^{(n)}$ has degree $-n$.

On the other hand, the image of the adjoint action of $a_+$ coincides with $\text{Im} \ p \simeq n_+/a_+$. From Campbell-Baker-Hausdorff formula we derive that any element of $N_+$ can be uniquely presented as the product of an element of $A_+$ and an element $\exp x$ of $N_+$, where $x \in \text{Im} \ p \subset n_+$. Therefore $Y$ is isomorphic to $n_+/a_+$. Hence with respect to the $\mathbb{Z}$-grading defined by the vector field $\Delta$, the space $\mathbb{C}[Y]$ of algebraic functions on $Y$ is a free polynomial algebra with $l$ generators of each negative degree. Thus $\mathbb{C}[Y]$ is isomorphic to $\mathbb{C}[U]$ and the Proposition follows. \qed

We can identify the algebra $\mathbb{C}[U]$ with the algebra of differential polynomials $\pi_0$ from the previous section; the operator $p_1$ gets identified with $\partial$.

**Proposition 3.** $\mathbb{C}[Y]$ is isomorphic to $\pi_0$ as $n_+$-modules.

**Proof.** By definition of $u_i^{(0)}$, we have

$$[\epsilon_i, p_1] = -u_i^{(0)} \epsilon_i.$$  \hspace{1cm} (16)

We have

$$\epsilon_i = \sum_{1 \leq i \leq l, n \geq 0} C_{i,j}^{(m)} \frac{\partial}{\partial u_i^{(m)}},$$

where $C_{i,j}^{(m)}$ are certain algebraic functions on $U$. Since by definition,

$$p_1 = \sum_{1 \leq i \leq l, n \geq 0} u_i^{(m+1)} \frac{\partial}{\partial u_i^{(m)}},$$

from formula (16) we find the recurrence relations for the coefficient of $\partial_j^{(m)}$ in the vector field $\epsilon_i$:

$$C_{i,j}^{(m)} = -u_i^{(m-1)} C_{i,j}^{(m-1)} + p_1 \cdot C_{i,j}^{(m-1)}.$$  

These recurrence relations coincide with the recurrence relations for the coefficients of the vector fields $Q_i$ on $U$. We also have, according to (15), $\epsilon_i \cdot u_j = -\alpha_i \cdot \alpha_j$, where
therefore \( C_{ij}^{(0)} = -\langle \alpha_i, \alpha_j \rangle \). Formulas (12) and (11) then show that the vector field \( e_i \) coincides with \( Q_i \) and Proposition follows. \( \square \)

**Remark 3.** This result was proved in [1], Propositions 3.1.10 and 3.2.5, by other methods. \( \square \)

In the next section we will explain how to construct KdV hamiltonians by the first cohomology classes of \( \mathfrak{n}_+ \) with coefficients in \( \pi_0 \).

### 4. The resolution.

Let \( B^*(\mathfrak{g}) \) be the dual of the Bernstein-Gelfand-Gelfand (BGG) resolution [26, 27]. Recall that \( B^*(\mathfrak{g}) = \bigoplus_{j \geq 0} B^j(\mathfrak{g}), \) where \( B^j(\mathfrak{g}) = \bigoplus_{l(w)=j} M^*_{w(\rho)-\rho} \). Here \( M^*_\lambda \) is the module contragradient to the Verma module of highest weight \( \lambda \), and the differentials of the resolution commute with the action of \( \mathfrak{g} \). The 0th cohomology of \( B^*(\mathfrak{g}) \) is one-dimensional and all higher cohomologies of \( B^*(\mathfrak{g}) \) vanish, so that \( B^*(\mathfrak{g}) \) is an injective resolution of the trivial representation of \( \mathfrak{n}_+ \).

Using Proposition 1, one can explicitly construct the differenitals of the dual BGG resolution.

It is known that for each pair of elements of the Weyl group, such that \( w \prec w' \), there is a singular vector \( P_{w,w'} \cdot v_{w(\rho)-\rho} \) in \( M_{w(\rho)-\rho} \) of weight \( w'(\rho) - \rho \). By Proposition 1, this vector defines the homomorphism \( \overline{P}_{w,w'} : M^*_{w(\rho)-\rho} \to M^*_{w'(\rho)-\rho}. \)

Let us normalize all \( P_{w,w'} \)'s in such a way that \( P_{w,w_1} P_{w_1,w_2} = P_{w,w_2} P_{w_2,w_3} \) for any quadruple of elements of the Weyl group, satisfying \( w \prec w_1, w_1 \prec w_2 \).

The differential \( \delta^j : B^j(\mathfrak{g}) \to B^{j+1}(\mathfrak{g}) \) of the BGG complex can be written as follows

\[
\delta^j = \sum_{l(w)=j} \epsilon_{w,w'} \overline{P}_{w,w'},
\]

where \( \epsilon_{w,w'} = \pm 1 \) are chosen as in [26, 27].

The right action of the Lie algebra \( \mathfrak{g} \) on this complex commutes with the differentials. Also the right action of the group \( N_+ \) commutes with the differentials. Therefore we can take the quotient of \( B^*(\mathfrak{g}) \) by the action of the Lie subgroup \( A_+ \) of \( N_+ \). Denote the corresponding complex by \( F^*(\mathfrak{g}) \).

We have: \( F^*(\mathfrak{g}) = \bigoplus_{j \geq 0} F^j(\mathfrak{g}) \), where

\[
F^j(\mathfrak{g}) = \bigoplus_{l(w)=j} \pi^{(w(\rho)-\rho)},
\]

where \( \pi^{(w(\rho)-\rho)} \) denotes the space of \( \mathfrak{a}_+ \)-invariants of the module \( M^*_{w(\rho)-\rho} \). As \( \mathfrak{n}_+ \)-modules all of \( \pi^{(w(\rho)-\rho)} \) are isomorphic to \( \pi_0 \).

The action of the Lie algebra \( \mathfrak{a}_+^0 \) on \( B^*(\mathfrak{g}) \) gives rise to an \( \mathfrak{a}_+ \)-action on the complex \( F^*(\mathfrak{g}) \). Indeed, if \( x \) is an element of \( \mathfrak{a}_+ \), then each of the functions \( F_i(x) \) is invariant under the \( A_+ \)-action. Denote by \( \overline{F}_i(x) \) the corresponding function on \( N_+/A_+ \), \( \overline{F}_i(x) \in \)
The action of $x$ on $\pi^{(\lambda)}$, where $\lambda = \sum_{0 \leq i \leq l} \lambda_i \alpha_i$, is given by the first order differential operator

$$x^R + \sum_{0 \leq i \leq l} \lambda_i \tilde{F}_i(x).$$

(17)

By construction, this action commutes with the differentials of the complex $F^*(\mathfrak{g})$. Hence this action defines an action of $a_-$ on the cohomologies of the complex $F^*(\mathfrak{g})$. Since $B^*(\mathfrak{g})$ is an injective resolution of the trivial representation of $\mathfrak{n}_+$, the cohomology of the complex $F^*(\mathfrak{g})$ coincides with $H^*(\mathfrak{n}_+, \pi_0)$. According to Proposition 3, as an $\mathfrak{n}_+$-module, $\pi_0$ is isomorphic to the module coinduced from the trivial representation of $\mathfrak{a}_+$. Therefore, by Shapiro lemma, $H^*(\mathfrak{n}_+, \pi_0)$ is isomorphic to $H^*(\mathfrak{a}_+, \mathbb{C}) = \Lambda^\ast(\mathfrak{a}_+^\ast)$, cf. [1].

**Lemma 3.** The action of $a_-$ on $H^*(\mathfrak{n}_+, \pi_0)$ is trivial.

**Proof.** Let $\Omega^\ast(\mathfrak{g})$ be the de Rham complex $\Omega^\ast(\mathfrak{g})$ of the big cell $X$ of the flag manifold. This complex is an injective resolution of the trivial representation of $\mathfrak{n}_+$. It is isomorphic to the tensor product of the space of functions on $X$ and the exterior algebra $\Lambda^\ast(\mathfrak{n}_+^\ast)$. The Lie algebra $\mathfrak{g}$ infinitesimally acts on it by right vector fields, and also the group $N_+$ acts from the right, and both actions commute with the differentials of the complex. The complex $B^*(\mathfrak{g})$ is a subcomplex of $\Omega^\ast(\mathfrak{g})$ [26, 27]; the corresponding embedding commutes with the actions of $\mathfrak{g}$ and $N_+$ and induces an isomorphism on cohomologies.

We can now take the quotient of $\Omega^\ast(\mathfrak{g})$ by the right action of $A_+ \subset N_+$. The quotient complex is isomorphic to the standard complex $C^*(\mathfrak{g}) = \pi_0 \otimes \Lambda^\ast(\mathfrak{n}_+^\ast)$, which computes the cohomology $H^*(\mathfrak{n}_+, \pi_0)$. But now we have an action of $a_-$ on $C^*(\mathfrak{g})$. This action is a sum of its actions on $\pi_0$ and on $\Lambda^\ast(\mathfrak{n}_+^\ast)$ (note that $\mathfrak{n}_+^\ast$ is isomorphic to $\mathfrak{n}_-$ by the invariant scalar product on $\mathfrak{g}$). By construction, we have an embedding $F^*(\mathfrak{g}) \rightarrow C^*(\mathfrak{g})$, which induces an isomorphism on cohomologies. This embedding is also a homomorphism of $a_-$-modules.

The complex $\Lambda^\ast(a_+) \otimes \Lambda^\ast((n_+/a_+)^\ast)$ with the trivial differentials is a quotient of the complex $C^*(\mathfrak{g})$ by the subcomplex $\pi_0 \otimes \Lambda^\ast((\mathfrak{n}_+/\mathfrak{a}_+)^\ast) \oplus \pi_0 / \mathbb{C} \otimes \Lambda^\ast(\mathfrak{a}_+^\ast)$. Since $a_-$ commutes with $\mathfrak{a}_+$, the map $C^*(\mathfrak{g}) \rightarrow \Lambda^\ast(\mathfrak{a}_+)$ is an $a_-$-homomorphism. On the other hand, by Shapiro lemma, the complex $C^*(\mathfrak{g})$ has the same cohomology as the complex $\Lambda^\ast(\mathfrak{a}_+^\ast)$ and this map induces an isomorphism on cohomologies. But the action of $a_-$ on $\Lambda^\ast(\mathfrak{a}_+)$ is trivial. Therefore it is trivial on the cohomologies of $C^*(\mathfrak{g})$. □

In particular, the operator $p_1$ acts trivially on the cohomologies. We already know that its action on $\pi_0$ coincides with the action of $\partial$. Consider now the action of $p_1$ on $\pi^{(\lambda)}$. According to formula (17), it is given by $\partial + \sum_{0 \leq i \leq l} \lambda_i u_i^{(0)}$. But this coincides with the action of $\partial$ on $\pi^{(\lambda)}$, given by (8). Therefore $\pi^{(\lambda)} \simeq \pi^{(\lambda)}$ with respect to the action of $\mathfrak{n}_+$ and $p_1 = \partial$. 

Proof of Theorem 1. Since $\partial = p_1$ commutes with the differentials of the complex $F^*(\mathfrak{g})$, we can consider the double complex

$\begin{equation}
\mathbb{C} \longrightarrow F^*(\mathfrak{g}) \xrightarrow{\partial \circ p_1} F^*(\mathfrak{g}) \longrightarrow \mathbb{C}.
\end{equation}$

Here $\mathbb{C} \rightarrow \pi_0 \subset F^*(\mathfrak{g})$ and $F^*(\mathfrak{g}) \rightarrow \pi_0 \rightarrow \mathbb{C}$ are the embedding of constants and the projection to constants, respectively.

In the spectral sequence, in which $\pm p_1$ is the 0th differential, the first term is the complex $F^*(\mathfrak{g})[-1]$, where

$$F^j(\mathfrak{g}) \simeq \bigoplus_{\{\mu\}} F_{\mu} \rho, \rho \rightarrow -\rho.$$ 

Indeed, if $\lambda \neq 0$, then in the complex

$$\pi_\lambda \xrightarrow{p_1} \pi_\lambda$$

the 0th cohomology is 0, and the first cohomology is, by definition, the space $F_\lambda$. If $\lambda = 0$, then in the complex

$$\mathbb{C} \longrightarrow \pi_0 \xrightarrow{p_1} \pi_0 \longrightarrow \mathbb{C}$$

the 0th cohomology is 0 and the first cohomology is, by definition, the space $F_0$.

In particular, $F^0(\mathfrak{g}) = F_0$, $F^1(\mathfrak{g}) = \bigoplus_{0 \leq i \leq j} F_{\alpha_i}$, and the corresponding differential is given by $\delta^1 = \sum_{0 \leq i \leq j} Q_i$. By definition, the 1st cohomology of the complex $F^*(\mathfrak{g})$ and hence the 1st cohomology of the double complex (18) is isomorphic to the space of KdV hamiltonians.

We can also compute this cohomology, using the other spectral sequence associated to our double complex. Since $H^*(F^*(\mathfrak{g})) \simeq \Lambda^*(a_+^*)$, we obtain in the first term the following complex

$$\mathbb{C} \longrightarrow \Lambda^*(a_+^*) \xrightarrow{\partial \circ p_1} \Lambda^*(a_+^*) \longrightarrow \mathbb{C}.$$ 

By Lemma 3, the action of $p_1$ on $\Lambda^*(a_+^*)$ is trivial and hence the cohomology of the double complex (18) is isomorphic to $\Lambda^*(a_+^*)/\mathbb{C} \oplus \Lambda^*(a_+^*)/\mathbb{C}[-1]$. In particular, we see that the space of KdV hamiltonians is isomorphic to $H^1(n_+, \pi_0) \simeq a_+^*$.

Therefore the space of KdV hamiltonians is linearly spanned by elements $H_m$, of degrees $-m, m \in I$. \[\Box\]

Remark 4. In [1], Theorems 3.1.11 and 3.2.6, we proved this result in the case when all exponents of $\mathfrak{g}$ are odd and the Coxeter number is even (this excludes $D_{2n}^{(1)}$, $E_6^{(1)}$, and $E_8^{(1)}$). In this case the degrees of all elements $p_m$ are odd. The statement of Lemma 3 then follows from simple degree counting in this case, since the image of a cohomology class of odd degree under the action of an operator of odd degree should be of even degree and hence should vanish. In particular, it follows that $\partial = p_1$ acts trivially on cohomologies, and we can apply the proof of Theorem 1. \[\Box\]
Let us explain how to construct a KdV hamiltonian starting from a class in the first cohomology of the complex $F^*(g)$.

Consider such a class $\mathcal{H} \in \oplus_{0 \leq i \leq l} \pi_{-\phi_i}$. If we apply $\partial$ to $\mathcal{H}$, we obtain a trivial cycle. Therefore there exists such $h \in \pi_0$ that $\delta^1 \cdot h = \partial \mathcal{H}$.

By construction, the element $h$ has the property that $\tilde{Q}_i \cdot h \in \pi_{-\phi_i}$ is a total derivative for $i = 0, \ldots, l$. But it itself is not a total derivative, because otherwise $\mathcal{H}$ would also be a trivial cocycle. Therefore, $\int h \neq 0$. But then $\int h$ is a KdV hamiltonian, because $\tilde{Q}_i \cdot h = 0$ for any $i = 0, \ldots, l$.

For $m \in I$ denote by $H_m \in \mathcal{F}_0$ the KdV hamiltonian of degree $-m$, and let $\eta_m$ be the derivation $\xi_{H_m}$. In particular, simple calculation shows that we can choose as $\mathcal{H}_1$ vector $\sum_{0 \leq i \leq l} e^{-\phi_i}$. Then $\partial \mathcal{H}_i = -\sum_{0 \leq i \leq l} u_i^{(0)} e^{-\phi_i}$ and $h_1 = \frac{1}{l} \sum_{0 \leq i \leq l} u_i^{(0)} w_i^{(0)}$. Hence $\eta_1 = \partial$, by (6).

Now $\eta_m$ is a vector field on $Y \cong U$. On the other hand the right infinitesimal action of the generator $p_m$ of the Lie algebra $\mathfrak{a}_- \subset \mathfrak{g}$ on $U$ also defines vector field $\mu_m$ on $U$.

**Theorem 2.** The vector field $\mu_m$ coincides with the vector field $\eta_m$ up to a non-zero constant multiple for any $m \in I$.

Note that we have already established this for $m = 1$. Indeed, we have just shown that $\eta_1 = \partial$, and we know that the action of $\partial$ coincides with the action of $p_1$.

**Corollary 1.** The KdV hamiltonians commute with each other:

$$\{H_n, H_m\} = 0$$

in $\mathcal{F}_0$ for any $n, m \in I$.

**Proof.** Since $p_m, m \in I$, lie in a commutative Lie algebra, they commute with each other. So do the corresponding vector fields: $[\mu_n, \mu_m] = 0$. By Theorem 2, the same holds for the vector fields $\eta_m, m \in I$: $[\eta_n, \eta_m] = 0$. By formula (10), injectivity of the map $\xi$, on $\mathcal{F}_0$, and the definition of the vector fields $\eta_m$, the corresponding KdV hamiltonians also commute with each other. $\square$

The proof of Theorem 2 will be based on formula (19) below.

From the definition of the KdV hamiltonians and (10) we obtain:

$$[\tilde{Q}_j, \eta_m] = [\xi \int e^{-\phi_j}, \xi_{H_m}] = \xi_{\int e^{-\phi_j}, H_m} = 0,$$

since $\{\int e^{-\phi_j}, H_m\} = \int (\tilde{Q}_j \cdot H_m) = 0$. Therefore

$$[Q_j, \eta_m] = - (\delta_j H_m) Q_j.$$

(19)

Indeed, in contrast to the operator $\tilde{Q}_j$, which acts from $\pi_0$ to $\pi_{-\phi_j}$, the operator $Q_j$ acts from $\pi_0$ to itself. Hence this commutator should be equal to $\Delta^j_m \cdot Q_j$, where
\( \Delta_{F_m} \) is the difference between the actions of the operator \( \xi_m \) on \( \pi_{-\alpha_j} \) and \( \pi_0 \). This difference is equal to

\[
\sum_{1 \leq i \leq l} \delta_i H_m \frac{\partial e^{-\phi_j}}{\partial \phi_i} = -\delta_j H_m.
\]

Our strategy will be the following. We will first find all vector fields on \( N_+ \), which have commutation relations as in (19) (we call this property (P)). We will then show that vector fields, which in addition commute with \( a^R_+ \), are exhausted by the vector fields \( p^R_m, m \in I \). On the other hand, we will prove that vector fields \( \eta_m \) can be lifted to \( a^R_+ \)-invariant vector fields on \( N_+ \), which satisfy property (P).

5. VECTOR FIELDS.

We will say that a vector field satisfies property (P), if it satisfies (14). Clearly, if \( \alpha \) and \( \beta \) satisfy property (P), so does their commutator.

**Definition 2.** The Lie algebra \( \mathcal{L} \) is the Lie algebra of all vector fields on \( N_+ \), which satisfy property (P).

According to Lemma 2, \( \mathfrak{g} \) is a Lie subalgebra of \( \mathcal{L} \).

**Proposition 4.** \( \mathcal{L} \) is isomorphic to \( \mathfrak{g} \).

**Proof.** The Lie algebra \( \mathfrak{g} \) acts on \( \mathcal{V} \) by commutation. We have the exact sequence

\[
0 \longrightarrow \mathfrak{g} \longrightarrow \mathcal{L} \longrightarrow M \longrightarrow 0
\]

of \( \mathfrak{g} \)-modules. We will show that the module \( M \) belongs to the category \( \mathcal{O} \) of modules over \( \mathfrak{g} \) [26, 27]. Modules from the category \( \mathcal{O} \) are those, which satisfy two properties: the Lie algebra \( \mathfrak{n}_+ \subset \mathfrak{g} \) acts on them locally nilpotently, and the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) acts on them diagonally. Let us show that these properties are satisfied for the \( \mathfrak{g} \)-module \( M \).

But we have already seen that the latter property is satisfied on the whole Lie algebra \( \mathcal{V} \). Recall that the Cartan subalgebra \( \mathfrak{h} \) maps into \( \mathcal{V} \). One easily checks that the adjoint action of the image of \( \mathfrak{h} \) defines a grading of \( \mathcal{V} \) by the weights of \( \mathfrak{h} \), and that \( \mathcal{L} \), and hence \( M \), are graded Lie subalgebras of \( \mathcal{V} \). Note that in \( M \) only non-positive weights can occur and in \( \mathfrak{n}^L_+ \subset \mathfrak{n}_+ \) only positive weights occur. Indeed, by definition, the commutation relations in \( \mathcal{V} \) preserve the grading. Formula (14) then implies that the weight of a vector field \( x \in \mathcal{L} \) is equal to the weight of each of the functions \( F_i(x) \). But functions on \( X \) can only have non-positive weights, and if all \( F_i(x) \)'s vanish, then \( x \in \mathfrak{n}^L_+ \subset \mathfrak{g} \) by Lemma 2, (a).

To show that the first property is satisfied, consider an element \( x \) of \( M \) of weight \( \gamma \). If \( y \) is an element of \( \mathfrak{n}_+ \), then, since \( y^R \) commutes with \( \mathfrak{n}^L_+ \), we obtain:

\[
[[y^R, x], e^L_\gamma] = -(y^R \cdot F_i(x)) e^L_\gamma.
\]
As an $n_+^R$-module, the space of functions on $X$ is isomorphic to the dual of the free module with one generator. The weight of each of $F_i(x)$ is equal to $\gamma$, therefore any element of $U(n_+^R)$ of weight greater than $-\gamma$ maps $x$ to a vector field, which commutes with all the $e_i$'s and hence lies in $n_+^R \subset \mathfrak{g}$. Therefore the action of $n_+$ on $M$ is locally nilpotent.

Suppose that $M \neq 0$. Then it should contain a highest weight vector, i.e. a vector field $\nu$, which satisfies $[n_+^R, \nu] \in \mathfrak{g}$. But then $\mathbb{C} \nu \oplus \mathfrak{g}$ should be an extension of a trivial one-dimensional $n_+^R$-module by the $n_+^R$-module $\mathfrak{g}$. This extension must be non-trivial, because otherwise we would be able to find $\nu' \in \mathbb{C} \nu \oplus \mathfrak{g}$, such that $[n_+^R, \nu'] = 0$. But then by Lemma 2, (a), $\nu' \in n_+^R$, and then $\nu'$ can not satisfy property (P), because $n_+^R$ is a nilpotent Lie algebra.

Thus, this extension should define a non-zero element in the group $H^1(n_+, \mathfrak{g})$. The cohomology $H^1(n_+, \mathfrak{g})$ was computed in [28]: $H^1(n_+, \mathfrak{g}) \cong H^{l-1}(n_+, \mathbb{C}) \otimes H^1(n_+, \mathfrak{g})$.

The space $H^1(n_+, \mathfrak{g})$ is naturally identified with the Lie algebra of vector fields on the circle: such a vector field $\delta$ defines a 1-cocycle $f_\delta$ on $n_+$ with coefficients in $\mathfrak{g}$ by the formula $f_\delta(x) = [\delta, x]$.

From the long exact sequence associated with the short exact sequence

$$0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow \text{Vect}_{-} \rightarrow 0$$

of $n_+$-modules we obtain: $H^1(n_+, \mathfrak{g}) \cong H^1(n_+, \mathbb{C}) \otimes \text{Vect}_{+}$, where $\text{Vect}_{+} = \mathbb{C}[t] \otimes \mathbb{C}[\partial]$. In particular, $H^1(n_+, \mathfrak{g}) \cong \text{Vect}_{+}$. But then the vector field $\nu$, defining the extension, should have a positive weight. Therefore the functions $F_i(\nu), i = 0, \ldots, l$, in formula (14) should also have positive weights and hence they must vanish. But then $\nu \in n_+^R \subset \mathfrak{g}$, by Lemma 2, (a), and so $M = 0$. \(\square\)

Remark 5. The Lie algebra $\mathcal{L}$ preserves a certain geometric structure on the flag manifold $F$. Denote by $P_i, i = 0, \ldots, l$, the parabolic subgroup of $G$, obtained by adjoining to the Borel subgroup $B_+$ the one-parameter subgroup of the negative simple root generator $f_i$ of $\mathfrak{g}$. We have natural bundles: $F \rightarrow G/P_i, i = 0, \ldots, l$. The fiber of such a bundle is a projective line. The tangent spaces to the fibers of these bundles defines $l+1$ tangent directions at each point of $F$. The Lie algebra $\mathcal{L}$ consists of vector fields, which are infinitesimal symmetries of this structure restricted to the big cell $X$ of $F$.

An analogue of the Lie algebra $\mathcal{L}$ can be defined for an arbitrary Kac-Moody group as the Lie algebra of vector fields on the big cell of the flag manifold, satisfying the relations (14). Clearly, $\mathfrak{g}$ itself is a Lie subalgebra of $\mathcal{L}$. The knowledge of $\mathcal{L}$ is important for understanding representation theory of $\mathfrak{g}$. Let $\beta$ be an element of $\mathcal{L}$ and $\lambda = \sum_{i=1}^{n} \lambda_i \alpha_i$ be a weight of the Cartan subalgebra of $\mathfrak{g}$, where $\alpha_i, i = 1, \ldots, l$, are the simple roots of $\mathfrak{g}$ and the $\lambda_i$'s are complex numbers. Define a map from $\mathcal{L}$ to the Lie algebra of differential operators of the first order on the big cell of $F$, which sends $\beta \in \mathcal{L}$ to the sum of the vector field, corresponding to $\beta$, and the function $\sum_{i=0}^{n} \lambda_i F_i(\beta)$, where the $F_i(\beta)$'s are defined by the relations (14). This is
clearly a homomorphism of Lie algebras. As we already mentioned above, the space of functions on the big cell of $\mathcal{F}$ with respect to this action of the Lie algebra $\mathfrak{g} \subset \mathcal{L}$ coincides with the contragradient module to the Verma module with highest weight $\lambda$, $M_\lambda^*$. Thus we obtain a structure of $\mathcal{L}$-module on $M_\lambda^*$. Since any irreducible module from the category $\mathcal{O}$ of $\mathfrak{g}$ can be realized as a submodule of some $M_\lambda^*$, we obtain a structure of $\mathcal{L}$-module on an arbitrary $\mathfrak{g}$-module.

We do not know a description of $\mathcal{L}$ for Kac-Moody algebras other than finite-dimensional or affine.

Proposition 4 gives a description of $\mathcal{L}$ in the case when $\mathfrak{g}$ is an affine algebra. We can also describe $\mathcal{L}$ in the case when $\mathfrak{g}$ is a finite-dimensional simple Lie algebra.

Proposition 5. If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra other than $\mathfrak{sl}_2$, then the Lie algebra $\mathcal{L}$ coincides with $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_2$ the Lie algebra $\mathcal{L}$ coincides with the Lie algebra of vector fields on the line.

Proof. In the same way as in the proof of Proposition 4, we reduce the problem to the calculation of the cohomology $H^1(\mathfrak{n}_+, \mathfrak{g})$. Since $\mathfrak{g}$ is finite-dimensional, we can use the Borel-Weil-Bott-Kostant theorem [29, 30], which gives:

$$H^1(\mathfrak{n}_+, \mathfrak{g}) = \bigoplus_{i=1, \ldots, l} \mathbb{C} \xi_i (\Lambda_{\text{adj}} + \rho) - \rho.$$ 

Here $l$ is the rank of $\mathfrak{g}$, $\Lambda_{\text{adj}}$ is the highest weight of the adjoint representation, $\rho$ is the half-sum of the positive roots of $\mathfrak{g}$, and $s_i, i = 1, \ldots, l$, are the reflections from the Weyl group of $\mathfrak{g}$. We denote by $\mathbb{C}_\lambda$ a one-dimensional representation of the Cartan subalgebra of $\mathfrak{g}$, on which it acts by the weight $\lambda$.

If $\mathcal{L} \neq \mathfrak{g}$, then there should exist a vector field $\beta \in \mathcal{L}$, whose weight and hence the weights of the corresponding functions $F_j(\beta)$ should be equal to $s_i(\Lambda_{\text{adj}} + \rho) - \rho$ for some $i = 1, \ldots, l$ (cf. the proof of Proposition 4). If $\mathfrak{g}$ is not $\mathfrak{sl}_2$, all of the weights $s_i(\Lambda_{\text{adj}} + \rho) - \rho$ are non-negative. Therefore $\beta$ can not satisfy property (P), because functions on the big cell can only have negative weights, and hence $\mathcal{L} = \mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{sl}_2$, one can check by hand that all vector fields on the big cell satisfy property (P). \qed

Let us now consider the vector field $\eta_m$ for some $m \in I$, corresponding to a KdV Hamiltonian. By formula (19), it satisfies property (P) on the homogeneous space $N_+/A_+$.

Proposition 6. For any $m \in I$ there exists an $\mathfrak{a}_+^R$-invariant vector field $\tilde{\eta}_m$ on $N_+$ satisfying property (P), such that its projection to $N_+/A_+$ coincides with $\eta_m$.

Proof. We can define a trivial one-cocycle $f_m$ on the Lie algebra $\mathfrak{n}_+$ with coefficients in vector fields on $N_+/A_+$ by putting $f_m(x) = [x, \eta_m]$. Any one-cocycle $f$ satisfies the relation

$$f([x, y]) = [x, f(y)] - [y, f(x)].$$

(20)
Since \( n_+ \) is generated by \( e_i, i = 0, \ldots, l \), \( f \) is uniquely defined by its values on \( f_m(e_i) \). Vice versa, if we assign to each \( e_i \) an element of the module – a vector field in our case – we can consider these elements as the values of a one-cocycle \( f \) on the \( e_i \)'s and try to extend \( f \) to a one-cocycle, using the formula (20). It is possible if and only if the value of \( f \) on any of the Serre relations, constructed this way, vanishes.

For example, if we have the relation \([e_i, e_j] = 0\) in \( n_+ \), then we can extend \( f \) to a one-cocycle, if the relation \([e_i, f(e_j)] - [e_j, f(e_i)] = 0\) holds.

Our cocycle \( f_m \) has a specific form due to the formula (19): the value of \( f_m \) on each \( e_i \) is proportional to \( e_i \), i.e. equal to \( e_i \) multiplied by a function. By induction, one can show that in this case the value of \( f_m \) on the Serre relation

\[
(ad e_i)^{-n_{ij} + 1} \cdot e_j = 0
\]

is a linear combination

\[
h_i e_i + h_j e_j + h_{ij} [e_i, e_j] + \ldots + h_i \ldots ij (ad e_i)^{-n_{ij}} \cdot e_j,
\]

where \( h_\bullet \) are certain functions on \( N_+/A_+ \).

For example, if we have \( f_m(e_i) = g_i e_i, f_m(e_j) = g_j e_j \), and the relation \([e_i, [e_i, e_j]] = 0\), then

\[
f_m([e_n, [e_i, e_j]]) = (e_j e_i g_i - 2e_i e_j g_i) e_i + (e_i^2 g_j) e_j + (e_i g_i + 2e_i g_j) [e_i, e_j].
\]

The linear term of the vector field \((ad e_i)^k \cdot e_j\) is non-zero and has degree \( k + 1 \) (cf. [1], proofs of Propositions 3.1.10 and 3.2.5). Therefore the linear terms of these vector fields with \( k > 0 \) are linearly independent from each other and from the linear terms of the vector fields \( e_i \) and \( e_j \), which have degree 1. The linear terms of the latter are given by \(-\partial_i^{(0)}\) and \(-\partial_j^{(0)}\) and hence are also linearly independent, if \( g \) is not \( sl_2 \). If \( g = sl_2 \), then the linear term of the vector field \( e_0 + e_1 \) vanishes, but this vector field itself is non-zero at a generic point of \( Y \). Thus we see that the vector fields \( e_i, e_j, [e_i, e_j], \ldots, (ad e_i)^{-n_{ij}} \cdot e_j \), are linearly independent at a generic point of \( Y \) (and even at each point of \( Y \), if \( g \neq sl_2 \)).

Vanishing of the linear combination (21) of these vector fields multiplied by certain functions then implies that each of the functions \( h_\bullet \) vanishes identically on \( N_+/A_+ \).

We now want to lift the one-cocycle \( f_m \) of \( n_+ \) with coefficients in vector fields on \( N_+/A_+ \) to a one-cocycle \( f_m \) of \( n_+ \) with coefficients in vector fields on \( N_+ \) with respect to the left action. For the one-cocycle \( f_m \) we have:

\[
f_m(e_i) = -(\delta_i H_m)e_i^L,
\]

by (19).

So we want to put

\[
\tilde{f}_m(e_i) = e^*(\delta_i H_m)e_i^L, \quad i = 0, \ldots, l,
\]
where ε is the projection $N_+ \to N_+/A_+$. For $\bar{f}_m$ to be a one-cocycle, the value of $\bar{f}_m$ on any of the Serre relations must vanish. This value is given by formula (21), where we should replace each of the functions $h_\bullet$ by its image under the map $\epsilon^*$. Therefore this value is equal to 0. Thus we obtain a one-cocycle $\bar{f}_m$ of $n_+$ with coefficients in vector fields on $N_+$.

The cohomology $H^i(n_+^L, \mathcal{V})$ vanishes for $i > 0$, because as a module over $n_+^L$, $\mathcal{V}$ is dual to a free module, with $n_+^R$ as the space of invariants. In particular, $H^i(n_+^L, \mathcal{V}) = 0$, and therefore there exists a vector field $\bar{\eta}_m$ on $N_+$, such that

$$[e_i^L, \bar{\eta}_m] = -\epsilon^*(\delta_i H_m) e_i^L, \quad i = 0, \ldots, l.$$  

(22)

By construction, the vector field $\bar{\eta}_m$ satisfies property (P). By Proposition 4, it lies in $\mathcal{L}$. Let $p^R$ be the vector field of the right infinitesimal action of an element $p \in a_+ \subset n_+$ on $N_+$. We have

$$[e_i^L, [p^R, \bar{\eta}_m]] = [[e_i^L, p^R], \bar{\eta}_m] + [p^R, [e_i^L, \bar{\eta}_m]] = 0,$$

for any $i = 0, \ldots, l$, because $[e_i^L, p^R] = 0$ and

$$[p^R, [e_i^L, \bar{\eta}_m]] = -\left( p^R \cdot \epsilon^*(\delta_i H_m) e_i^L \right) - \epsilon^*(\delta_i H_m)[e_i^L, p^R] = 0.$$

Therefore, by Lemma 2, (a), $[p^R, \bar{\eta}_m] \in n_+^R$. But the degree of $\bar{\eta}_m$ is equal to $-m < 0$, so that the degree of the commutator with $p$ is equal to $-m + 1 \leq 0$, whereas the degree of any element of $n_+^R$ should be positive. We conclude that $[p^R, \bar{\eta}_m] = 0$. But then, since $\bar{\eta}_m \in \mathcal{L}$, $\bar{\eta}_m$ commutes with the whole $a_+^R$. Proposition is proved. □

**Proof of Theorem 2.** The only elements of $\mathcal{L}$, which commute with $a_+^R$, are elements of $a_+^L$. Therefore the vector field $\bar{\eta}_m$ coincides with the vector field of the right infinitesimal action of an element of $a_+^L$. Comparing degrees we see that this element is equal to $p_m$ up to a normalization factor. By Proposition 6, the vector field $\mu_m$ of the infinitesimal action of the element $p_m$ on $N_+/A_+$ coincides with the vector field $\bar{\eta}_m$ of the $m$th KdV hamiltonian up to a normalization factor. □

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