EXTENDED DE SITTER THEORY OF
TWO-DIMENSIONAL GRAVITATIONAL FORCES

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ABSTRACT

We present a simple unifying gauge theoretical formulation of gravitational theories in
two-dimensional spacetime. This formulation includes the effects of a novel matter-gravity
coupling which leads to an extended de Sitter symmetry algebra on which the gauge theory
is based. Contractions of this theory encompass previously studied cases.

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I. INTRODUCTION

Recent work has shown that gravitational theories in $1 + 1$ dimensions may be consistently formulated as gauge theories [1]. For string-like models of “dilaton” gravity [2] one uses the Poincaré group ISO$(1, 1)$ [3] or alternatively the extended Poincaré group ISO$(1, 1)$ [4,5], while the constant curvature models involve the de Sitter group SO$(2, 1)$ [6]. Such gauge models are of interest because of the deep connection between diffeomorphism invariance and gauge invariance, and because they provide a picture of two dimensional gravity complementary to the more geometrical string-like approach [7]. Moreover, there is some hope that “black hole” type solutions may yield important new information about the quantum mechanics of realistic four dimensional black holes [8]. It has also been shown that these gauge theories arise as dimensional reductions of $2 + 1$ dimensional gauge theories of gravity [9].

In this paper we seek to unify these gauge theoretical models by considering the effects of a novel type of matter-gravity interaction in the constant curvature de Sitter model. This interaction involves matter being minimally coupled to a gauge field whose field strength two form is everywhere proportional to the area two form. While the physics is very different, at least formally such an interaction is reminiscent of the classic Landau problem of dynamics in the presence of a uniform magnetic field, an interaction which has an interesting generalization to nonzero constant curvature surfaces [10], and even to higher genus surfaces [11]. This type of matter-gravity coupling has already been considered in the zero curvature theory [5]. Here we generalize to nonzero constant curvature and show that this serves as a unifying model from which all other previously studied gauge theories of $1 + 1$ dimensional gravity may be obtained by contraction. In particular, this model clarifies the significance of the extension needed in the Poincaré theory.

This paper is organized as follows. In the remainder of the Introduction we state our conventions for the geometric quantities to be used throughout this paper. In Section II we discuss the matter-gravity coupling in both Lagrangian and Hamiltonian approaches, showing how the extended de Sitter algebra arises as a symmetry algebra in each case. Section III is devoted to a solution of the equations of motion by direct integration, making use of the de Sitter symmetries. In Section IV we show how the theory may be reformulated in an elegant and compact form as a gauge theory based on the extended de Sitter group. The gauge formulation of the matter-gravity coupling is analyzed in Section V, and we conclude with some brief comments in Section VI.

We shall use the convention that lower case Greek and Latin indices refer respectively to space-time and tangent space components. Space-time indices are raised and lowered with the metric tensor $g_{\mu\nu}$, and tangent space indices with the Minkowskian metric $h_{ab} = \text{diag}(1, -1)$. The Zweibein $e^a_{\mu}$ is related to the metric tensor $g_{\mu\nu}$ by

$$ g_{\mu\nu} = e^a_{\mu} e^b_{\nu} h_{ab} \quad (1.1) $$

The spin-connection, when defined in terms of the Zweibein, is given by

$$ \omega_{\mu} = \frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta} e^a_{\mu} \partial_{[\alpha} e^b_{\beta]} \quad (1.2) $$
Here $\varepsilon^\mu{}^\nu$ is the \textit{numerical} anti-symmetric tensor, with sign convention $\varepsilon^{01} = 1$. Thus $\varepsilon^\mu{}^\nu/\sqrt{-g}$ is a contravariant tensor, where

$$g = \det g_{\mu\nu}$$
$$\sqrt{-g} = \det e^a_\mu$$

$$= \frac{1}{2} e^a_\mu e^b_\nu \varepsilon^\mu{}^\nu \varepsilon_{ab}$$

(1.3)

The scalar curvature $R$ is related to the spin connection by

$$\frac{1}{\sqrt{-g}} \varepsilon^\mu{}^\nu \partial_\mu \omega_\nu = \frac{1}{2} R$$

(1.4)

In conformally flat coordinates the space-time metric is

$$g_{\mu\nu} = e^{-2\sigma} h_{\mu\nu}$$

(1.5)

and the corresponding \textit{Zweibein} and spin connection are

$$e^a_\mu = e^{-\sigma} \delta^a_\mu$$
$$\omega_\mu = \epsilon^a_\mu \partial_\nu \sigma$$

(1.6)

In these coordinates, the relation (1.4) reduces to an equation relating the curvature $R$ and the conformal factor $\sigma$:

$$\partial_+ \partial_- \sigma = \frac{1}{4} Re^{-2\sigma}$$

(1.7)

Here $\partial_+ \equiv \partial/\partial x^+$, $\partial_- \equiv \partial/\partial x^-$, where $x^\pm$ are light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. Later, we shall be interested in the coupling of matter to gravitational fields with \textit{constant} curvature, $R = \Lambda$, in which case equation (1.7) is just the familiar Liouville equation. The general solution is

$$\sigma = \log \frac{1 + \frac{\Lambda}{4} F(x^+) G(x^-)}{F'(x^+) G'(x^-)}$$

(1.8)

for any functions $F$ and $G$. We can, however, always transform it to the solution

$$\sigma = \log(1 + \frac{\Lambda}{4} x^+ x^-).$$

(1.9)

\textbf{II. “MAGNETIC” MATTER-GRAVITY INTERACTION}

\textbf{II.1 Lagrangian Formalism}

The conventional action for a point particle of mass $m$, moving on the world line $x^\mu(\tau)$, is proportional to the arc length

$$I_m = -m \int d\tau \sqrt{\dot{x}^\mu(\tau) g_{\mu\nu}(x(\tau)) \dot{x}^\nu(\tau)}$$

(2.1)
Here, and throughout this paper, the overdot denotes differentiation with respect to the world line parameter $\tau$. It is, however, possible to consider an additional parametrization invariant term in the action [5]. This term is linear in $\dot{x}^\mu$,

$$I_{\text{coupling}} = - \int d\tau \dot{x}^\mu (\tau) \left( A \omega_\mu (x(\tau)) + B a_\mu (x(\tau)) \right)$$  \hspace{1cm} (2.2)

where $A$ and $B$ are constants, $\omega_\mu$ is the spin-connection and $a_\mu$ a one-form defined by

$$\frac{1}{\sqrt{-g}} \varepsilon^{\mu \nu \rho} \partial_\mu a_\nu = 1$$  \hspace{1cm} (2.3)

This addition to the action has the form of a magnetic coupling to an external gauge field with potential $A_\mu = A \omega_\mu + B a_\mu$, but in two dimensional spacetime it is possible to relate this gauge field to the gravitational metric as in (1.4) and (2.3). This has the consequence that the point particle interacts with a gauge field whose curvature two-form is everywhere proportional to the spacetime area two-form.

The effect of this additional matter-gravity coupling is to modify the usual geodesic equation of motion arising from $I_m$ to contain a geometrical force term and to read

$$\frac{d}{d\tau} \left( \frac{1}{\sqrt{\varepsilon^{\alpha \beta}} g_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta} \right) + \dot{x}^\alpha \Gamma^\mu_{\alpha \beta} \dot{x}^\beta + \left( \frac{1}{2} A R + B \right) g^{\mu \alpha} \sqrt{-g} \epsilon_{\alpha \beta} \dot{x}^\beta = 0$$  \hspace{1cm} (2.4)

where $N \equiv \frac{1}{m} \frac{1}{\sqrt{\varepsilon^{\alpha \beta}} g_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta}$.

To understand the symmetries of the total action

$$I \equiv I_m + I_{\text{coupling}},$$  \hspace{1cm} (2.5)

consider the variation of the total Lagrangian induced by a change $\delta x^\mu$ in the spacetime coordinates :

$$\delta L = - \frac{m}{2} \frac{\dot{x}^\mu \dot{x}^\nu}{\varepsilon^{\alpha \beta}} \mathcal{L}_{\delta x} g_{\mu \nu} - \frac{d}{d\tau} \left( (A \omega_\mu + B a_\mu) \delta x^\mu \right) + \left( \frac{1}{2} A R + B \right) \sqrt{-g} \epsilon_{\mu \nu} \delta x^\mu \dot{x}^\nu$$  \hspace{1cm} (2.6)

Here $\mathcal{L}_{\delta x}$ is the Lie derivative along $\delta x^\mu$. If $\delta x^\mu$ is a Killing vector of the metric $g_{\mu \nu}$ (i.e., $\mathcal{L}_{\delta x} g_{\mu \nu} = 0$), one can also verify that $\sqrt{-g} \epsilon_{\mu \nu} \delta x^\mu = \partial_\nu X$ for some function $X$. If moreover the curvature is constant, then the last term in $\delta L$ can be written as a total derivative; hence the action is invariant and the Killing vectors generate corresponding symmetries.

Let us thus concentrate on the case of a maximally symmetric two dimensional spacetime, which has constant curvature $R = \Lambda$ and three Killing vectors

$$\xi^\mu_{(0)} = (x^1, -x^0)$$

$$\xi^\mu_{(1)} = \left( 1 + \frac{\Lambda}{8} (x^0)^2 + \frac{\Lambda}{4} (x^1)^2, \frac{\Lambda}{4} x^0 x^1 \right)$$

$$\xi^\mu_{(1)} = \left( -\frac{\Lambda}{4} x^0 x^1, 1 - \frac{\Lambda}{8} (x^0)^2 - \frac{\Lambda}{8} (x^1)^2 \right)$$  \hspace{1cm} (2.7)
Each Killing vector generates a symmetry of the action and is associated with a conserved Noether “current”:

\[ Q = \frac{\partial L}{\partial x^\mu} \delta x^\mu + (A \omega_\mu + B a_\mu) \delta x^\mu - (B + \frac{1}{2} A \Lambda) X \]  \hspace{1cm} (2.8)

For the above Killing vectors, the conserved currents are

\[ \delta x^\mu = -\epsilon^{\mu}_{s(J)} : \quad J = \frac{-1}{1 + \frac{A}{\Lambda} x^2} \left( \frac{m}{\sqrt{x^2}} \epsilon_{a b} x^a x^b + \frac{B}{2} x^2 \right) \]

\[ \delta x^\mu = -\epsilon^{\mu}_{s(a)} : \quad P_a = \frac{m}{\sqrt{x^2}} x_a + \epsilon_{a b} x^b (B + \frac{X}{2}) \]  \hspace{1cm} (2.9)

where \( B \equiv B + \frac{1}{2} A \Lambda \) and all indices are lowered and raised with the flat metric \( h_{a b} = \text{diag}(1, -1) \).

Using the canonical symplectic structure \( [\epsilon^{(x)}_{a}, x^b] = \delta^{b}_{a} \), one computes the algebra of these currents to be

\[ [P_a, J] = \epsilon_{a b} P_b \]

\[ [P_a, P_b] = \epsilon_{a b} (\frac{X}{2} J + B) \]  \hspace{1cm} (2.10a)

In the flat case, \( \Lambda = 0 \), one recognizes the extended Poincaré algebra studied in [4], and for \( \Lambda \neq 0 \) with \( B = 0 \) (i.e. without the additional matter-gravity coupling) this is just the de Sitter algebra studied in [6] in relation to two dimensional gravity. We introduce a central element \( I \), which is represented in this symplectic representation by 1

\[ [P_a, P_b] = \epsilon_{a b} (\frac{X}{2} J + B I) \]  \hspace{1cm} (2.10b)

and we shall refer to this symmetry algebra as the “extended de Sitter algebra”. The above discussion shows that this algebra provides a direct bridge between the pure de Sitter and extended Poincaré models. One can, of course, also redefine the generator \( J \) by adding a constant central element

\[ J \equiv J + \frac{2B}{\Lambda} \]  \hspace{1cm} (2.11)

in such a way that the resulting algebra is the conventional de Sitter one. While this is often convenient for computations, the extended form (2.10) is most useful for considering the contraction to the extended Poincaré algebra.

Note also that these generators are related in the following way

\[ P_a h^{a b} P_b - \frac{1}{2} J^2 - 2B I \]  \hspace{1cm} (2.12)

The LHS of this relation is the quadratic Casimir of the extended de Sitter algebra.

We shall see later that it is possible to construct a gauge formulation of this gravitational coupling, with gauge algebra being precisely this extended de Sitter algebra.
II.2 First Order or Hamiltonian Formalism

It is instructive to formulate the above model in a first order formalism. This will prove useful when considering the gauge formulation of the theory, and is also desirable for the eventual quantization of this theory.

The action (2.5) may be represented as

\[ I = \int d\tau \left\{ (p_a \epsilon^a - \mathcal{A}_\omega - B a_\mu) \dot{x}^\mu + \frac{1}{2} N(p_a h^a b_p - m^2) \right\} \]  

(2.13)

Upon variation of the fields one obtains \( p_a = -\epsilon^a \dot{x}^\mu / N \) and \( N = \sqrt{x^\mu g_{\mu \nu} \dot{x}^\nu / m} \), in which case the Lagrangian form presented above is regained. Alternatively, one may write

\[ I = \int d\tau \left\{ \pi_\mu \dot{x}^\mu + \frac{1}{2} N \left( (\pi_\mu + \mathcal{A}_\omega + B a_\mu) g^{\mu \nu} (\pi_\nu + \mathcal{A}_\omega + B a_\nu) - m^2 \right) \right\} \]  

(2.14)

where \( \pi_\mu \) is the momentum

\[ \pi_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = p_a \epsilon^a - \mathcal{A}_\omega - B a_\mu \]  

(2.15)

This last form makes explicit the Hamiltonian of the system

\[ H = -\frac{1}{2} N \left( (\pi_\mu + \mathcal{A}_\omega + B a_\mu) g^{\mu \nu} (\pi_\nu + \mathcal{A}_\omega + B a_\nu) - m^2 \right) \]  

(2.16)

In the first order formulation, a change \( \delta x^\mu \) in the spacetime coordinates is induced by the symplectic form \( [\pi_\mu, x^\nu] = \delta^\nu_\mu \) and the generator \( \mathcal{G} \equiv \pi_\mu \delta x^\mu - Y \), where \( Y(x) \) can be an arbitrary function of \( x \):

\[ \delta x^\mu = [\pi_\mu \delta x^\mu - Y, x^\nu] \]  

(2.17)

The resulting change in the Hamiltonian is then

\[ \delta H = [\pi_\mu \delta x^\mu - Y, H] = -N \left( (\frac{\mathcal{A}}{2} + B) \sqrt{-g} \epsilon_{\mu \rho} \delta x^\rho + \partial_\mu \left\{(\mathcal{A}_\omega + B a_\rho) \delta x^\rho + Y \right\} g^{\mu \nu} (\pi_\nu + \mathcal{A}_\omega + B a_\nu) ight) 
- \frac{1}{2} NL \delta x g^{\mu \nu} (\pi_\mu + \mathcal{A}_\omega + B a_\mu) (\pi_\nu + \mathcal{A}_\omega + B a_\nu) \]  

(2.18)

For constant curvature and \( \delta x^\mu \) a Killing vector, the variation of the Hamiltonian will vanish (thereby making \( \mathcal{G} \) a symmetry generator) provided one chooses

\[ Y = (B + \frac{1}{2} \mathcal{A} R) X - (\mathcal{A}_\omega + B a_\rho) \delta x^\rho \]  

(2.19)

where \( X \) is defined as before by \( \sqrt{-g} \epsilon_{\mu \nu} \delta x^\mu = \partial_\nu X \).
If one applies this to the constant curvature case and the three Killing vectors described earlier, one finds the generators

\[
J = -\pi_a \epsilon^{ab} b_a b_b \\
P_a = -(1 + \frac{\Lambda}{8} x^2) \pi_a + \frac{1}{2} (B + \frac{\Lambda}{2} J) \epsilon_a b^b
\]  

(2.20)

It is straightforward to verify that these generators satisfy the same algebra (2.10) as that obtained above by Noether’s theorem. Moreover, the Hamiltonian (2.16) can be expressed as a quadratic form in these generators

\[
H = -\frac{1}{2} N \left( P_a h^{ab} P_b - \frac{\Lambda}{2} J^2 - 2B J - m^2 \right)
\]  

(2.21)

The expression in parentheses is simply the quadratic Casimir of the extended de Sitter algebra, confirming indeed that the Hamiltonian is invariant under transformations generated by this algebra.

For later use, we record the form of the conserved generators (2.20) in light cone form:

\[
P_+ = \frac{1}{\sqrt{2}} (P_0 + P_1) = -\frac{\Lambda}{2} (x^-)^2 \pi_- - \pi_+ + \frac{B}{2} x^-
\]

\[
P_- = \frac{1}{\sqrt{2}} (P_0 - P_1) = -\frac{\Lambda}{2} (x^+)^2 \pi_+ - \pi_- - \frac{B}{2} x^+
\]

\[
J = x^+ \pi_+ - x^- \pi_-
\]  

(2.22)

These generators satisfy the extended de Sitter algebra

\[
[P_+, P_-] = \frac{\Lambda}{2} J + B I \\
[P_\pm, J] = \pm P_\pm
\]  

(2.23)

III. MATTER-GRAVITY COUPLING

III.1 Equations of Motion

The equations of motion can be computed equivalently in both the Lagrangian and Hamiltonian formalisms. Let us consider the first one. The geodesic equation obtained by varying the action is

\[
\frac{d}{d\tau} \left[ \frac{m \dot{x}^\mu}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} \right] + \frac{m}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho + \left( \frac{1}{2} \mathcal{A} R + B \right) g^{\mu\nu} \sqrt{-g} \epsilon_{\nu\rho} \dot{x}^\rho = 0
\]  

(3.1)

In the maximally symmetric case, with \( R = \Lambda \) a constant, the symmetry generators (2.9) are first integrals of motion. Due to the algebraic relation they satisfy, it is sufficient to consider two of them, namely

\[
P_a = \frac{m}{\sqrt{2}} \dot{x}_a + (B + \frac{\Lambda}{4} J) \epsilon_a b^b = (B + \frac{\Lambda}{4} J) \epsilon_a b^b = \text{constant}
\]  

(3.2)
where $\mathcal{F}'$ are constants of integration. This leads to classical trajectories of hyperbolic form:

$$(\mathcal{F} + \frac{\Lambda}{4} J)^2 (x - \mathcal{F})^2 + m^2 = 0 \quad (3.3)$$

Note that $J$ may be expressed in terms of the $\mathcal{F}'$ using the above Casimir relation (2.12). We also remark that in the limit of flat spacetime, $\Lambda \to 0$, the trajectory remains hyperbolic, as found in the extended Poincaré case [4].

### III.2 Gravity Sector

We propose an action for gravity which enforces the condition of a maximally symmetric spacetime and provides the needed one-form $a_\mu$ whose curvature two-form is everywhere proportional to the spacetime area two-form. This was first considered in Ref. 12 and may be achieved using two Lagrange multipliers $\eta, \lambda$:

$$I_{\text{grav}} = \frac{1}{4\pi G} \int d^2x \sqrt{-g} \left[ \eta (\mathcal{F} - \Lambda) + \lambda \left( \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \partial_\mu a_\nu - 1 \right) \right] \quad (3.4)$$

The equations of motion obtained by varying with respect to the fields $\eta, \lambda, g_{\mu\nu}, a_\mu$ are then

$$\begin{align*}
R &= \Lambda \\
\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \partial_\mu a_\nu &= 1 \\
\left( \nabla_\mu \partial_\nu - g_{\mu\nu} \nabla_\rho \partial^\rho \right) \eta &= \frac{1}{2} \Lambda g_{\mu\nu} \eta + \frac{1}{2} \lambda g_{\mu\nu} \\
\partial_\mu \lambda &= 0
\end{align*} \quad (3.5)$$

where $\nabla_\mu$ is the metric covariant derivative.

From the first two equations, we recover the geometric quantities

$$\begin{align*}
g_{\mu\nu} &= \frac{h_{\mu\nu}}{(1 + \frac{\Lambda}{8} x^2)^2} \\
e^a_\mu &= \frac{\delta^a_\mu}{1 + \frac{\Lambda}{8} x^2} \\
\omega_\mu &= \frac{\Lambda}{4} \sigma_\mu = \frac{\Lambda}{4} \frac{e_{\mu\nu} x^\nu}{1 + \frac{\Lambda}{8} x^2}
\end{align*} \quad (3.6)$$

These agree (as they must) with the constant curvature solutions (1.5) and (1.6) found before from the Liouville solution (1.9). Notice that according to the last equation the field $a_\mu$ appears to be redundant. This is true unless one wants to perform a $\Lambda \to 0$ limit; besides providing an equation for $\lambda$, it is in that case necessary for the gauge formulation.

The two last equations determine the Lagrange multiplier fields

$$\begin{align*}
\lambda &= \lambda_0 = \text{constant} \\
\eta &= \frac{1}{2} e_{a b} x^a \varphi^b + (\varphi_2 + \frac{2\Lambda_0}{\Lambda})(1 - \frac{\Lambda}{8} x^2) - \frac{2\Lambda_0}{\Lambda}
\end{align*} \quad (3.7)$$

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where $\lambda_0, \varphi^a, \varphi_2$ are integration constants. Once again, in the flat space limit, $\Lambda \to 0$, the solutions of the extended Poincaré theory are recovered.

For the sake of later comparison with the gauge formulation of this gravitational theory, it is helpful to rewrite this gravity action (3.4) in terms of the Zweibein and spin connection fields. To do so, we recall the relation (1.4) between the curvature and the spin connection, and we include a Lagrange multiplier to enforce the relation (1.2) between the Zweibein and spin connection. Then the gravity action becomes:

$$I_{\text{grav}} = \frac{1}{4\pi G} \int d^2x \, \epsilon^\mu \nu \left\{ 2\eta (\partial_\mu \omega_\nu + \frac{\Lambda}{4} \epsilon^a_\mu \epsilon^b_\nu \epsilon_{ab}) + \lambda (\partial_\mu a_\nu + \frac{1}{2} \epsilon^a_\mu \epsilon^b_\nu \epsilon_{ab}) \right\} + \eta_a (\partial_\mu e^a_\nu + \epsilon^a_\nu \omega_\mu e^b_\nu) \right\} \tag{3.8}$$

The equations of motion easily determine $e^a_\mu, \omega_\mu$ and $a_\mu$ as above, and the equations of motion for the Lagrange multiplier fields may be expressed most succinctly in light-cone notation:

$$\partial_+ \lambda = 0$$

$$\partial_+ \eta = \pm \frac{1}{2} \frac{\eta_+}{1 + \frac{\Lambda}{8} x^2}$$

$$\partial_+ \eta_\pm = \frac{-\frac{1}{8} x^\mp \eta_+}{1 + \frac{\Lambda}{8} x^2}$$

$$\partial_+ \eta_\mp = \pm \left( \Lambda \eta + \lambda \right) + \frac{\Lambda}{8} x^\mp \eta_\mp \tag{3.9}$$

Integrating these equations yields the solutions

$$\lambda = \lambda_0$$

$$\eta = \frac{1}{2} \left( \frac{x^+ \varphi^- - x^- \varphi^+ - \frac{2}{\Lambda} (1 - \frac{\Lambda}{8} x^2)}{1 + \frac{\Lambda}{8} x^2} - \frac{2\lambda_0}{\Lambda} \right)$$

$$\eta_\pm = \frac{\varphi_\mp + \frac{\Lambda}{4} (x_\mp)^2 \varphi_\pm \pm \alpha x_\mp}{1 + \frac{\Lambda}{8} x^2} \tag{3.10}$$

where $\lambda_0, \alpha$ and $\varphi_\pm$ are constants of integration. Note that in order to compare with (3.7) and to obtain a finite solution in the $\Lambda \to 0$ limit (viz. the extended Poincaré theory [4]) we can write the arbitrary constant $\alpha$ as

$$\alpha = -\lambda_0 - \Lambda \varphi_2 \tag{3.11}$$

where $\varphi_2$ is the integration constant in (3.7).
IV. GAUGE FORMULATION OF GRAVITATIONAL THEORY

IV.1 Invariant Inner Product for Extended de Sitter Algebra

As a precursor to the introduction of a gauge formulation for this gravitational theory, we consider the extended de Sitter algebra in more detail. Representing the four generators $P_4, J, I$ as $Q_A$, we can express the commutation relations (2.10) as

$$[Q_A, Q_B] = f_{AB}^C Q_C$$

and define an invariant inner product

$$\langle Q_A | Q_B \rangle = h_{AB}$$

The term “invariant” means invariant under conjugation,

$$\langle U^{-1} Q_A U | U^{-1} Q_B U \rangle = \langle Q_A | Q_B \rangle$$

This leads to the following form for $h_{AB}$:

$$h_{AB} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & c & -\frac{1}{B}(1 + \frac{cA}{2}) \\
0 & 0 & -\frac{1}{B}(1 + \frac{cA}{2}) & \frac{A}{2B}(1 + \frac{cA}{2})
\end{pmatrix}$$

(4.4)

The inverse matrix $h^{AB}$ ($c \neq -\frac{2}{A}$)

$$h^{AB} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -\frac{B}{A} & -\frac{B}{1 + \frac{cA}{2}} \\
0 & 0 & -\frac{B}{1 + \frac{cA}{2}} & -\frac{cB}{1 + \frac{cA}{2}}
\end{pmatrix}$$

(4.5)

is used to define the quadratic Casimir operator

$$C = Q_A h^{AB} Q_B$$

(4.6)

which commutes with all the generators $Q_A$.

Note that if we choose the arbitrary parameter $c$ appearing in $h^{AB}$ and $h_{AB}$ to be $c = \left( \frac{m}{\Lambda} \right)^2 / \left( 1 - \frac{\Lambda}{2} \left( \frac{m}{\Lambda} \right)^2 \right)$, the Casimir operator (4.6) will be proportional to the Hamiltonian (2.21). However, it is perfectly consistent (and more convenient) to set $c$ to zero. This is achieved by shifting $J$ with an appropriate multiple of $I$, and we make this choice henceforth. Furthermore, notice that by setting $\Lambda = 0$ the extended de Sitter algebra reduces to the extended Poincaré algebra, and the invariant inner product reduces accordingly. Setting $\overline{B} = 0$, we get back to the pure de Sitter case [6], in which case the fourth generator $I$ decouples and only a $3 \times 3$ sub-block of $h^{AB}$ is invertible, as expected.
IV.2 Extended de Sitter Gravity Action

We define a gauge field $A$ with values in the extended de Sitter algebra (2.10),

$$A = e^a P_a + \omega J + \overline{B}aI \quad (4.7)$$

The corresponding field strength is

$$F = dA + A^2$$
$$= (de^a + \epsilon^a b \omega \epsilon^b) P_a + (d\omega + \frac{\Lambda}{4} \epsilon^a \epsilon_{ab} \epsilon^b) J + \overline{B}(da + \frac{1}{2} \epsilon^a \epsilon_{ab} e^b) I \quad (4.8)$$

Using the above invariant inner product we define the pure gravity Lagrangian to be

$$L_{\text{grav}} = \frac{1}{4\pi G} \langle \eta | F \rangle = \frac{1}{4\pi G} \eta_A F^A$$
$$= \frac{1}{4\pi G} \left[ \eta_a (de^a + \epsilon^a b \omega \epsilon^b) + \eta_2 (d\omega + \frac{\Lambda}{4} \epsilon^a \epsilon_{ab} \epsilon^b) + \overline{B} \eta_3 (da + \frac{1}{2} \epsilon^a \epsilon_{ab} e^b) \right] \quad (4.9)$$

Then the equations of motion obtained by varying with respect to the field $\eta$ may be expressed as

$$F = 0 \quad (4.10)$$

Indeed, in a certain sense, the key to this type of gauge formulation of gravity is the fact that under an infinitesimal diffeomorphism $\delta x^\mu$ the gauge field changes according to the Lie derivative formula

$$\delta A_\mu = \delta x^\nu \partial_\mu A_\nu + (\partial_\mu \delta x^\nu) A_\nu \quad (4.11)$$

which may be rewritten as [13]

$$\delta A_\mu = \delta x^\nu F_{\nu\mu} + D_\mu (\delta x^\nu A_\nu) \quad (4.12)$$

so that “on shell”, where $F = 0$, the effect of a diffeomorphism is that of a gauge transformation (with field dependent transformation parameter).

The $P_a$ projection of $F = 0$,

$$de^a + \epsilon^a b \omega \epsilon^b = 0 \quad (4.13)$$

states that the torsion vanishes and defines the spin connection in terms of the Zweibein. The $J$ projection of $F = 0$

$$d\omega = -\frac{\Lambda}{4} \epsilon^a \epsilon_{ab} \epsilon^b \quad (4.14)$$

implies [see (1.4)] that the scalar curvature is constant:

$$R = \Lambda \quad (4.15)$$

Finally, the $I$ projection of $F = 0$ implies that $a$ is proportional to the spin connection, up to an arbitrary gauge transformation,

$$a = \frac{2}{\pi} \omega + d\theta \quad (4.16)$$
Using this last result we see that, “on shell”, the gauge field $A$ may be re-expressed as

$$A = e^a P_a + \omega (J + \frac{2\mathcal{F}}{\Lambda} I) + d\theta I$$

$$= e^a P_a + \omega \mathcal{J} + d\theta I$$

(4.17)

where replacing $J$ by the single generator $\mathcal{J} \equiv J + \frac{2\mathcal{F}}{\Lambda} I$ reduces the extended de Sitter algebra (2.10) to the standard de Sitter algebra. The corresponding field strength is

$$F = (de^a + \epsilon^a b \omega e^b) P_a + (d\omega + \frac{\Lambda}{4} e^a \epsilon_a e^b) \mathcal{J}$$

(4.18)

IV.3 Canonical Structure of the Gravity Sector

To study the canonical structure of the gauge invariant gravity Lagrangian (4.9), we first write the action in Hamiltonian form

$$I_{grav} = \frac{1}{4\pi G} \int d^2 x \epsilon^{\mu \nu} \eta_A F_{\mu \nu}^A$$

$$= \frac{1}{4\pi G} \int dt dx (\eta_A \partial_0 A_1^A - A_0^A \mathcal{G}_A) - \frac{1}{4\pi G} \int dt dx \partial_1 (\eta_A A_0^A)$$

(4.19)

where the constraints $\mathcal{G}_A$ are just the spatial covariant derivative of $\eta_A$

$$\mathcal{G}_A = - (\partial_1 \eta_A + f_{ABC} A_1^B \eta^C)$$

(4.20)

Here $f_{AB}^C = f_{ABD} h^{DC}$ are the structure constants of the algebra. Then using the symplectic structure deduced from the action

$$[\eta_A(x), A_1^B(y)] = 4\pi G \delta^B_A \delta(x-y)$$

(4.21)

it is straightforward to check that the constraint algebra satisfies the algebra

$$[\mathcal{G}_A(x), \mathcal{G}_B(y)] = f_{AB}^C \mathcal{G}_C(x) \delta(x-y)$$

(4.22)

as is usual in gauge theories.

IV.4 Gauge Solution Of The Equations Of Motion

In the gauge theoretical language it is particularly easy to solve the equations of motion. This is because the general solution for the gravitational fields, see eq. (4.10), is

$$A = \text{pure gauge}$$

$$= U^{-1} dU$$

(4.23)

for some group element $U$. A simple computation shows that by taking

$$U = e^{x^+ P_+ e^{- \log(1 + \frac{\Lambda}{4} x^2) \mathcal{J}}} e^{-P_-}$$

(4.24)
one finds that

\[ A = U^{-1}dU \]
\[ = \epsilon^a P_a + \omega J + \overline{\mathbf{a}} I \]  
(4.25)

where \( \epsilon^a \), \( \omega \) and \( a \) coincide with the Zweibein, spin connection and gauge field \( a \) given in (3.6).

The equations of motion for the Lagrange multiplier fields \( \eta \) may be succinctly expressed as

\[ D\eta = 0 \]  
(4.26)

where \( D \) refers to the covariant derivative with respect to the gauge field \( A \) in (4.17). Thus, with \( A = U^{-1}dU \), the solution for \( \eta \) is just

\[ \eta = U^{-1}\eta(0)U \]  
(4.27)

where \( \eta(0) \) is some constant element of the extended de Sitter algebra. Indeed, expanding an arbitrary constant algebra element as

\[ \eta(0) = \varphi^+ P_+ + \varphi^- P_- + \alpha J + \beta I \]  
(4.28)

(note that for algebraic convenience we use \( \mathcal{J} \equiv J + \frac{2\beta}{\Lambda} I \) instead of \( J \)) one finds that

\[ \eta = U^{-1}\eta(0)U \]
\[ = \left( \frac{\varphi^+ + \frac{\Lambda}{4}(x^+)^2 \varphi^- - \alpha x^+}{1 + \frac{\Lambda}{8} x^2} \right) P_+ + \left( \frac{\varphi^- + \frac{\Lambda}{4}(x^-)^2 \varphi^+ + \alpha x^-}{1 + \frac{\Lambda}{8} x^2} \right) P_- \]
\[ - \frac{\Lambda}{2} \left( \frac{x^+ \varphi^- - x^- \varphi^+ - \frac{\alpha}{2} (1 - \frac{\Lambda}{8} x^2)}{1 + \frac{\Lambda}{8} x^2} \right) J + \beta I \]  
(4.29)

If we write the arbitrary integration constant \( \beta \) as \( -2\lambda_0/\Lambda \), then it is straightforward to use \( \eta_A = h_{AB}\eta^B \) to show that this solution for \( \eta \) coincides with that obtained before in (3.10) by integrating directly the equations of motion.

V. GAUGE COVARIANT MATTER-GRAVITY COUPLING

V.1 de Sitter Gauge Invariant Action

To discuss the gauge covariant form of the matter-gravity coupling it is most convenient to work with the first order formalism of Section II.2. To facilitate comparison with earlier papers [5], we redefine the fields \( p_a \) in the action (2.13) as \( -\epsilon_a^b p_b \), so that the first order action reads

\[ I_{\text{matter}} = - \int d\tau \left\{ (p_a \epsilon_a^b \epsilon^b_\mu + A_\omega_\mu + B a_\mu) \dot{x}^\mu + \frac{1}{2} N (p^2 + m^2) \right\} \]  
(5.1)
This is Lorentz invariant, but not de Sitter invariant (due to the lack of translational invariance of the Zweibein $e^a_{\mu}$). It is, however, possible to write this action in a de Sitter invariant form using the gauge formulation. We shall use a variant of a technique due to Grignani and Nardelli [14] in which the factor $-e^a_{\mu}e^b_{\nu}\dot{x}^\mu$ is replaced by the gauge covariant derivative (with respect to $\tau$) of a set of auxiliary variables $q^a$. The idea is to introduce these extra degrees of freedom in a gauge covariant manner and so that the action (5.1) is obtained by a particular gauge choice. A similar idea has been studied in the context of four dimensional de Sitter gravity theories by Stelle and West [15].

Specifically, we define elements $q$ and $p$ of the extended de Sitter algebra. Thus $p$ and $q$ transform as

$$q \rightarrow U^{-1}qU$$
$$p \rightarrow U^{-1}pU$$

Furthermore, we choose $q$ and $p$ to be of the form

$$q \equiv T^{-1}q(0)T$$
$$p \equiv T^{-1}p(0)T$$

where $q(0)$ is a constant algebra element with components just in the $J$ and $I$ directions

$$q(0) = J + AI,$$  \hspace{1cm} (5.4)

where $p(0)$ has components just in the de Sitter translation directions

$$p(0) = p^aP_a,$$  \hspace{1cm} (5.5)

and where $T$ is a generator of a de Sitter translation

$$T = e^{-\xi^a\epsilon^b_aP_b}$$  \hspace{1cm} (5.6)

We shall refer to the parameters $\xi^a$ which characterize $T$ as “de Sitter coordinates”. The algebra elements $q$ and $p$ in Eq. (5.3) depend upon the four degrees of freedom characterized by $\xi^a$ and $p^a$. These relations, (5.2) to (5.6), define a nonlinear action [15] of the de Sitter group on the de Sitter coordinates $\xi^a$. That is, under the transformation (5.2) we deduce that $T$ transforms as

$$T \rightarrow T' \equiv e^{-\xi'^a\epsilon^b_aP_b}$$  \hspace{1cm} (5.7)

where $\xi'^a$ is a nonlinear function of the original de Sitter coordinates $\xi^a$ and the parameters of the transformation group element.

Now consider the action

$$I = \int d\tau \{ \langle p|D_{\tau}q \rangle + \overline{q}pA - T^{-1}\dot{T} - \frac{1}{2}N(\langle p|p \rangle + m^2) \}$$  \hspace{1cm} (5.8)
where \( A = A_\mu \dot{x}^\mu \). Under a gauge transformation,
\[
A \rightarrow U^{-1}AU + U^{-1}U
\]
the action (5.8) is gauge invariant (with surface terms taken to vanish). Moreover, if we transform to the gauge in which
\[
q = q_0
\]
this has the effect of removing the de Sitter coordinates. In this gauge, \( p = P_0 \), and
\[
A = A_0 = (\epsilon^a \rho P_a + \omega + B \partial \nu \nu \nu) \dot{x}^\mu
\]
Then
\[
\langle p | D_\tau q \rangle = \langle p_0 | \dot{q}_0 + [A_0, q_0] \rangle = -p_0 \epsilon^a \rho \dot{x}^\mu
\]
and
\[
\overline{B} \langle q | A - T^{-1} \hat{T} \rangle = \overline{B} \langle q_0 | A_0 \rangle = -(\hat{A} \omega + B \partial \nu \nu \nu) \dot{x}^\mu
\]
so that the action (5.8) reduces to the original action (5.1).

This is analogous [5] to the Higgs phenomenon in vector gauge theories. There, given the gauge noninvariant Lagrange density
\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{2} \partial \mu \rho \partial \nu \rho + \frac{1}{2} \rho^2 A_\mu A^\mu - V(\rho)
\]
one may introduce an additional variable \( \theta \) (the analogue of the de Sitter coordinates \( \xi^a \)), define \( \varphi \equiv \frac{1}{\sqrt{2}} \rho e^{i \theta} \) and then consider the gauge invariant Lagrange density
\[
\mathcal{L}' = -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + (\partial \mu - i A_\mu) \varphi^* (\partial \nu + i A^\nu) \varphi - V(2 \sqrt{\varphi^* \varphi})
\]
The original Lagrange density may be regained either by a choice of gauge (set \( \theta \) to zero and define \( \varphi \equiv \frac{1}{\sqrt{2}} \rho e^{i \theta} \)), or alternatively, in any gauge, by redefining variables in \( \mathcal{L}' \) as \( \varphi = \frac{1}{\sqrt{2}} \rho e^{i \theta} \), \( \dot{A}_\mu = A_\mu + \partial \nu \theta \). In the example considered in this section, we only need to introduce additional degrees of freedom \( \xi^a \) corresponding to the de Sitter translation generators \( P_a \) since the action (5.1) is already Lorentz invariant.
V.2 Gauge Equations of Motion

Including both the matter and gravity contributions, the total gauge theoretical action is

\[ I_{\text{total}} = \int d\tau \left\{ (p | D_\tau q) + \mathcal{B} (q | A - T^{-1} \dot{T}) - \frac{1}{2} N ((p | p) + m^2) \right\} + \frac{1}{4\pi G} \int \langle \eta | F \rangle \]  

(5.15)

Since \( q \equiv T^{-1} q_{(0)} T \) we can write the first term in the action as \( \langle [q, p] | A - T^{-1} \dot{T} \rangle \) so that the action may be expressed as

\[ I_{\text{total}} = \int \left\{ (\mathcal{B} q + [q, p] | A - T^{-1} \dot{T}) - \frac{1}{2} N ((p | p) + m^2) \right\} + \frac{1}{4\pi G} \int \langle \eta | F \rangle \]  

(5.16)

This second, equivalent, form of the action is useful for obtaining some of the equations of motion. Variation with respect to the fields \( \eta, N, p, \xi^a \) and \( A_\mu \) lead to the equations of motion:

\[ F = 0 \]  

(5.17a)

\[ \langle p | p \rangle + m^2 = 0 \]  

(5.17b)

\[ D_\tau q - N p = 0 \]  

(5.17c)

\[ \mathcal{B} D_\tau q - [D_\tau p, q] = 0 \]  

(5.17d)

\[ \epsilon^{\mu\nu} D_\nu \eta = 4\pi G \int d\tau \left\langle \mathcal{B} q(\tau) + [q(\tau), p(\tau)] \right\rangle \dot{x}^\mu \delta^2(x - x(\tau)) \]  

(5.17e)

Note that in the gauge \( \xi^a = 0 \) Eq. (5.17d), obtained by varying with respect to the de Sitter coordinates \( \xi^a \), does not lead to an independent equation of motion (see Ref. [5]). This statement is in fact true in any gauge.

The first three of these equations (5.17a – c) are easily solved, as before, to give

\[ A = U^{-1} dU \]  

\[ N = \frac{1}{m} \sqrt{-\langle D_\tau q | D_\tau q \rangle} \]  

\[ p = \frac{m D_\tau q}{\sqrt{-\langle D_\tau q | D_\tau q \rangle}} \]  

(5.18)

In the special gauge (5.10), (5.11), \( U \) is given by (4.24) and \( D_\tau q = -\epsilon^{a\nu} \epsilon^{\mu}_b \dot{x}^\mu P_a \), so that

\[ N = \frac{1}{m} \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} \]  

\[ p = -\frac{\epsilon^{a\nu} \epsilon^{\mu}_b \dot{x}^\mu P_a}{N} \]  

(5.19)

as is seen directly from the action (5.1). Equation (5.17e) is a little more subtle. Off the world line, where the delta function vanishes, we have \( D_\mu \eta = 0 \), whose solution is presented
in (4.29). Then, in the parametrization in which \( x^0(\tau) = \tau = t \), we may integrate across the world-line to obtain the solution

\[
\eta = U^{-1}(x) \left( \eta(0) + \Delta \eta(0) \varepsilon(x^1 - x^1(t)) \right) U(x)
\]

\[
\Delta \eta(0) = 4\pi GU(x(t)) \left( \overline{B}_q(t) + [q(t), p(t)] \right) U^{-1}(x(t))
\]

(5.20)

where \( \eta(0) \) is the matter-free solution in (4.28) and \( \Delta \eta(0) \) is a \textbf{constant} depending on the parameters (3.2) of the particle trajectory (this is easily shown using the identity \( D_\tau(\overline{B}_q + [q, p]) = 0 \)). We can then eliminate the inessential \( \xi^a \) degrees of freedom by going to the the gauge (5.10), (5.11) in which case

\[
\overline{B}_q(t) + [q(t), p(t)] = -\frac{\varepsilon^a \dot{x}^a P_a}{N} + \overline{B}(J + AI)
\]

(5.21)

We see thus that the constants \( \phi^a, \alpha, \) and \( \beta \) in (4.28), (4.29) get merely shifted when one crosses the particle trajectory.

**VI. CONCLUDING REMARKS**

We conclude by briefly summarizing our results. We have given a gauge formulation for gravitational forces experienced by point particles in two dimensional spacetime. For the maximally symmetric case of point particles interacting with a constant (in general, nonzero) curvature gravitational background there are three conserved generators (corresponding to the three spacetime Killing vectors), which satisfy the de Sitter symmetry algebra. When we include the novel matter-gravity interaction in which the matter is minimally coupled to a gauge field with gauge curvature correlated with the spacetime metric curvature, the symmetry generators are modified and the symmetry algebra acquires a central extension. Formally, this is analogous to the idea of “magnetic translation” symmetry, generalized to curved surfaces. The resulting theory, based on this “extended de Sitter algebra”, encompasses previous gauge theoretical models of two dimensional gravitational theories. In the limit in which the new matter-gravity coupling is removed one regains the pure de Sitter theory of Ref. [6]. And in the limit of vanishing spacetime curvature, the extended de Sitter model contracts smoothly to the extended Poincaré theory [4,5].

In these theories, one effectively trades spacetime diffeomorphism invariance for on shell gauge invariance. The significance of the gauge formulation has been illustrated here by studying the symmetry generators and the equations of motion. First, one sees that in a purely geometrical formulation of this model (in terms of metric quantities), the extended de Sitter algebra arises as a Noether symmetry algebra. Then one notices that it is possible to define a gauge field with values in this same algebra in such a way that a simple gauge action yields the original geometric theory. Being a first order (in derivatives) action, the gauge theory equations of motion are easier to solve and, moreover, the gauge structure provides a natural and concise algebraic set of fields with which to analyze the model. This should also play an important role in the quantization of this model, a question to
be addressed in future work. In this study we have focused our attention on the point particle interaction. The coupling of matter fields, like fermions, is also possible and its gauge formulation can be carried over following the same ideas than in Ref. 5.

Contact with the geometric formulation of dilaton gravity [2] is achieved by considering the “stringy” metric $\bar{g}_{\mu\nu} = e^a_\mu h_{ab} e^b_\nu / \eta$. In the extended Poincaré case ($\Lambda = 0$), one then recovers the well-known black-hole configuration, where the quadratic expression (3.7) for $\eta$ describes the location of the horizon and the “mass” of the black-hole [4]. The generalization of this connection to the extended de Sitter case ($\Lambda \neq 0$) is an interesting question which also deserves future attention.

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