Proper Field Quantization in Black Hole Spacetimes

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Canonical quantization of local field theories in classical black hole spacetimes with a single horizon leads to a particle number density with a thermal distribution in equilibrium at the Hawking temperature. A complete treatment including non-local quantum gravity effects has shown however that the full “thermal vacuum” of the theory is the false vacuum. In this work, we find the true vacuum consistent with the complete semiclassical analysis of quantum black holes. The theory is described by a “microcanonical” quantum field theory with fixed total energy \( E = M \), the mass of the black hole. Considerations making use of the microcanonical density matrix as well as the idempotency condition show that particles in black hole backgrounds are described by pure states, unlike the canonical formulation.

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I. INTRODUCTION

Canonical quantization of local field theories in curved spacetimes and especially spacetimes with horizons (such as black holes), was a very popular subject in the late seventies and early eighties.

In those years and until very recently, it was taken for a fact of life that the particle number density obtained in spacetimes with a single horizon was given by a thermal distribution with a temperature equal to the Gibbons-Hawking temperature.

Of course, such a result meant the loss of quantum coherence in those systems and it was very popular to assume that indeed this was the case.

A WKB semiclassical analysis of quantum gravity however suggested otherwise, as quantum black holes were essentially identifiable as excitations of p-brane theories and were shown never to achieve thermal equilibrium.

In recent papers [1–5], and especially the one of Ref. [6], the present authors presented a proof that the canonical (thermal) vacuum of local quantum field theories in black hole spacetimes is not the true vacuum of these theories. Crucial to the demonstration was the incorporation of all the quantum gravity (p-branes) excitation modes (back reaction effects) in the construction of such a vacuum. This incorporation fully takes into account the (quantum) non-locality of quantum gravity theories. It was then shown that the stability of the full thermal vacuum was tantamount to the existence of black hole solutions to the so-called Hagedorn self-consistency condition.

Of course, it is well known that string theories are the only solutions to Hagedorn’s condition and therefore, black hole solutions are excluded. The canonical vacuum is the false vacuum.

Our study of black hole statistical mechanics revealed however that an equilibrium state of a system of quantum black holes does exist and is describable with the use of the microcanonical ensemble.

This fact alone suggests immediately the proper route for quantization. The proper vac-
uum state should belong to an energy representation where the sum of the particle energies is fixed by the black hole mass (for a static black hole). The vacuum state of the theory is thus parametrized by the black hole mass and is, formally, the inverse Laplace transform of the canonical vacuum. We shall call it the microcanonical vacuum. All correlation functions can in principle be calculated with such a vacuum and an expression for the unperturbed microcanonical propagator is obtained in section III. It is essentially Weldon’s propagator [7].

Stability of the microcanonical vacuum is insured by the above mentioned existence of a microcanonical equilibrium state for quantum black hole systems. By the bootstrap property, a gas of black holes is equivalent to a single quantum black hole and therefore we expect the microcanonical formulation to describe pure states. In section III arguments based on density matrix calculations corroborate this expectation.

In the following section, in order to make the paper self-contained, we first present a short review of the semiclassical treatments of the quantum black hole problem.

II. BLACK HOLES AS P-BRANES

In Ref. [1] we demonstrated that the model of black holes as p-branes is free of the logical inconsistencies and paradoxes of the thermodynamical interpretation [8,9] of black hole physics. In this section we review the semiclassical methods used to obtain information from quantum gravity theory and discuss the rationale for treating black holes as p-branes.

A. The WKB Method

The WKB method is one of the earliest nonperturbative methods used in ordinary quantum mechanics. In the path integral formulation of quantum field theory the WKB method amounts to finding the classical solutions to the Euclidean equations of motion and then functionally integrating over quantum fluctuations around these classical solutions. The
Euclidean solutions, or instantons, describe the tunneling of particles through the effective potential of the theory.

For a black hole the instanton solutions allow us to calculate the semiclassical probability for a particle to tunnel through the horizon. The nature of the instanton and the resulting expression for the Euclidean action depend upon the characteristics of the black hole. For example the Euclidean spacetime for a \( D \)-dimensional Schwarzschild black hole has a conical singularity, which is removed by requiring that the imaginary time dimension be circular with circumference \( \beta_H \), the Hawking inverse “temperature”. The gravitational instantons in this case are periodic instantons. The Euclidean action determined from the Euclidean metric

\[
ds^2 = e^{2\Phi} \, dr^2 + e^{-2\Phi} \, dr^2 + r^2 d\Omega_{D-2}^2 , \tag{2.1}
\]

where

\[
e^{2\Phi} = 1 - \left( \frac{r_+}{r} \right)^{D-3} , \tag{2.2}
\]

and \( r_+ \) is the horizon radius, is given by

\[
S_E = \frac{A_{D-2}}{16\pi} \beta_H r_+^{D-3} . \tag{2.3}
\]

In this expression \( A_D \) is the area of a unit \( D \)-sphere and the vanishing of the conical singularity relates \( \beta_H \) to the horizon radius \( r_+ \)

\[
\beta_H = \frac{2\pi}{[e^\Phi \partial_r e^\Phi]_{r=r_+}} = \frac{4\pi r_+}{D-3} . \tag{2.4}
\]

The horizon radius is determined by the black hole mass \( M \)

\[
M = \frac{(D-2)}{16\pi} A_{D-2} r_+^{D-3} . \tag{2.5}
\]

In terms of the mass the Euclidean action can be written as

\[
S_E(M) = \frac{\beta_H M}{D-2} = C(D)M \frac{\beta_H}{D-2} , \tag{2.6}
\]
where $C(D)$ is defined as,

$$C(D) = \frac{4^{\frac{D-1}{2}} \pi ^{\frac{D-2}{2}}}{(D - 3)(D - 2)^{\frac{D-2}{2}} A^{\frac{D-2}{2}}_{E_{D-2}}}.$$  \hspace{1cm} (2.7)

The tunneling probability of a single particle escaping the black hole in the WKB approximation,

$$P \sim e^{-S_{E(M)}/\hbar},$$  \hspace{1cm} (2.8)

is essentially the inverse of the quantum degeneracy of states, so

$$\sigma \sim e^{C(D)M^{\frac{D-2}{2}}}.$$  \hspace{1cm} (2.9)

This expression for the degeneracy of states when compared to those of known non-local quantum theories shows that the $D$-dimensional Schwarzschild black holes are quantum excitation modes of a $\frac{D-2}{2}$-brane. The implication of this result is that black holes are elementary particles.

We have studied [1–4] a gas of such particles in the microcanonical ensemble and have shown that they form a conformal theory in the sense that this ensemble obeys the statistical bootstrap condition and that the S-matrix is dual. For a gas of $N$ black holes the equilibrium state is the one for which there is one very massive black hole and $(N - 1)$ massless black holes.

**B. Mean-Field Theory**

In this approximation [12] the S-matrix elements are calculated using in and out states which are assumed to be free in the distant past and distant future. In the present case we wish to consider quantum fields scattering off the black hole horizon. There is a doubling of the number of degrees of freedom in this case because the horizon divides space into two causally disconnected regions, requiring two different Fock spaces. The mathematical structure used to describe this situation is the same as that for field theories at finite temperature, e.g. the thermofield dynamics formalism [13]. The states describing the system
are direct products of the basis vectors of the Fock spaces of the two disconnected regions. For example, the thermal vacuum state for outgoing particles can be written as

$$|\text{out}, 0> = Z^{-1/2}(\beta) \sum_{n=0}^{\infty} e^{-\beta n \omega/2} |n> \otimes |\bar{n}>,$$

(2.10)

where $|n>$ and $|\bar{n}>$ are the Fock spaces of the two disconnected regions. An observer outside the horizon sees only the $|n>$ states directly. The partition function $Z(\beta)$ is given by

$$Z = \sum_{n=0}^{\infty} e^{-\beta n \omega},$$

(2.11)

and for any observable operator $\mathcal{O}$ the vacuum expectation value is

$$<\text{out}, 0 | \mathcal{O} | \text{out}, 0> = \frac{1}{Z(\beta)} \sum_{n=0}^{\infty} e^{-\beta n \omega} <n | \mathcal{O} | n>.$$

(2.12)

The inverse temperature $\beta$ in this case is determined by the surface gravity $\kappa$ of the black hole

$$\beta = \frac{2\pi}{\kappa}.$$

(2.13)

This is the same as the expression for $\beta_H$, which we encountered in the WKB approximation.

If we take $\mathcal{O}$ in Eq.(2.12) to be the number operator, the particle number density for a given mass $m$ is

$$n_k(m; \beta_H) = \frac{1}{e^{\beta_H \omega(m)} - 1}.$$

(2.14)

In a local field theory, this expression would present a problem because it implies a loss of quantum coherence in the scattering process. The $\text{in}$ state is a pure state

$$|\text{in}, 0> = |0> \otimes |\bar{0}>,$$

(2.15)

but the expression in Eq.(2.14) clearly arises from a mixture of states, resulting in a failure of the unitarity principle during the scattering process

$$|\text{out}, 0> = S^{-1}(\beta)|\text{in}, 0> \quad S^{-1} \neq S^\dagger.$$

(2.16)
Since black holes are, in our point of view, p-brane quantum excitations, Eq.(2.14) cannot be the final result. Quantum non-local effects (back reactions) must be taken into account. The expression obtained from the thermal vacuum for a single mass state must be summed over in order to take into account all possible mass states

\[ n_k(\beta_H) = \int_0^\infty dm \sigma(m) n_k(m; \beta_H). \]  

(2.17)

The expressions for the thermal vacuum and the canonical partition function are then given by

\[ |\text{out}, 0 \rangle = Z^{-1/2(\beta)} \prod_{m,k}^{\infty} \sum_{n_k,m=0}^{\infty} \prod_{m,k} e^{-\frac{\beta}{2} n_k m \omega_{k,m}} |n_k, m \rangle \otimes |\hat{n}_k, m \rangle , \]

(2.18)

and

\[ Z(\beta_H) = \exp \left( -\frac{V}{(2\pi)^{D-1}} \int_{-\infty}^{\infty} d^{D-1} k \int_0^\infty dm \sigma(m) \ln [1 - e^{-\beta_H \omega_k(m)}] \right), \]

(2.19)

where \( \omega_k(m) = \sqrt{k^2 + m^2} \). The partition function can also be written in terms of the statistical mechanical density of states

\[ Z(\beta_H) = \int_0^\infty dE e^{-\beta_H E} \Omega(E) . \]

(2.20)

Equating the latter expression to that in Eq.(2.19), we obtain the self-consistency condition first written down by Hagedorn [14] as a model of strong interactions at high energy. \( \Omega(E) \) depends upon \( \sigma(E) \), and we have a statistical bootstrap requirement, which can be stated as

\[ \frac{\sigma(E)}{\Omega(E)} \to 1 ; \quad (E \to \infty) . \]

(2.21)

The unique solution of Hagedorn’s condition together with the bootstrap requirement is of the form

\[ \sigma(m) \sim e^{bm} ; \quad (m \to \infty) , \]

(2.22)

with the restriction that \( \beta_H > b \), where \( b^{-1} \) is the Hagedorn temperature. This expression for \( \rho(m) \) is the same as that obtained from string theories.
Our analysis of the tunneling probability in the WKB approximation for a particle to escape a Schwarzschild black hole showed that black holes are p-branes with \( p = \frac{D-3}{D-1} \) (see Eq. (2.9)). Since \( p > 1 \) unless \( D \rightarrow \infty \), black holes do not satisfy the self-consistency condition because of the divergence of the canonical partition function. The black hole system is not in thermal equilibrium, therefore there is no self-consistent solution for the quantum density of states under the assumption of thermal equilibrium. The thermal vacuum is the false vacuum for this system.

In order to properly quantize fields in black hole backgrounds we must start from the true vacuum. In the next section we lay the foundation for the solution of this problem by starting from the proper microcanonical description and developing a relatively simple expression for the true vacuum.

III. THE MICROCANONICAL FORMULATION

The review of the preceding section clearly exhibited the fact that, taking into account the full non-locality of quantum gravity theories (p-brane theories), the “thermal vacuum” of the traditional canonical quantization of fields in black hole spacetimes is not stable. It is the false vacuum. In this section, with the help of our knowledge of black hole statistical mechanics, we shall find the true vacuum of the theory as well as an expression for the free particle propagator, the basic object in perturbation theory.

To make rapid progress, let us re-express the canonical (thermal) vacuum (2.18) as follows,

\[
|O(\beta)\rangle = \hat{\rho}^\dagger(\beta; \mathbf{H}) |\Omega\rangle,
\]  

(3.1)

where \( \hat{\rho}(\beta) \) is the normalized canonical density matrix operator (acting solely on the \( |n\rangle \) subspace). It is given by,

\[
\hat{\rho}(\beta, \mathbf{H}) = \frac{\rho(\beta, \mathbf{H})}{\langle \Omega | \rho (\beta, \mathbf{H}) | \Omega \rangle},
\]  

(3.2)

where,
\[ \rho(\beta, \mathbf{H}) = e^{-\beta \mathbf{H}}, \]  

in which \( \mathbf{H} \) is the Hamiltonian operator and,

\[ |\mathcal{S} > = \prod_{k,m} \sum_{n_{k,m}} |n_{k,m} > \otimes |\tilde{n}_{k,m} >. \]  

Notice that,

\[ \text{tr} \mathcal{O} = <\mathcal{S}|\mathcal{O}|\mathcal{S}> , \]

for any operator \( \mathcal{O} \) acting on the physical subspace (as seen by the observer outside the horizon).

If we introduce the so-called thermal doublet notation,

\[ \phi^a = \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix}, \]

the free particle causal propagator is now given as,

\[ -i <\mathcal{S}|T \hat{\rho}^{1-a}(\beta) \phi^a(x_1)\phi^b(x_2)\hat{\rho}^a(\beta)|\mathcal{S}> = \Delta^{ab}_{\beta,a}(x_1,x_2). \]

At equilibrium, the choice of the parameter \( \alpha \) is completely arbitrary. In other words, physical observables are \( \alpha \)-independent. This situation changes out of equilibrium however, but one need not be concerned by that here. The thermal vacuum of Eq.(2.18) corresponds to the choice \( \alpha = \frac{1}{2} \). The Fourier transform of the propagator (3.7) is now given as follows,

\[ \Delta^{ab}_{\beta,a}(k) = \frac{\tau_3}{k^2 + m^2 - i\epsilon \tau_3} + \frac{2\pi i \delta(k^2 + m^2)}{e^{\beta |k_0|} - 1} \begin{pmatrix} 1 & e^{\alpha \beta |k_0|} \\ e^{(1-\alpha) \beta |k_0|} & 1 \end{pmatrix} \]

where \( \tau_3 \) is the Pauli matrix in thermal doublet space.

For convenience, we now choose \( \alpha = 1 \). So all connected correlation functions are calculated as follows,

\[ G_{\beta}^{a_1a_2 \cdots a_N}(1,2,\cdots,N) = -i <\mathcal{S}|T \phi^{a_1}(1)\phi^{a_2}(2) \cdots \phi^{a_N}(N)|\beta>, \]

where,
\[ |\beta\rangle = \hat{\rho}(\beta, \mathbf{H})|\Omega\rangle . \quad (3.10) \]

Our previous work on black hole statistical mechanics made it clear that the canonical ensemble doesn’t exist for black holes and that the equilibrium state must be analyzed in the fundamental microcanonical ensemble. This piece of knowledge now strongly suggests the proper route for solving the present problem. We are then led to define (formally) the microcanonical vacuum \(|E\rangle\) as follows,

\[ |\beta\rangle = \int_0^\infty dE \ e^{-\beta E} |E\rangle . \quad (3.11) \]

Physical correlation functions are now defined as follows,

\[ G_E^{a_1 \cdots a_N}(1, 2, \cdots, N) = \langle \Omega | T\phi^{a_1}(1)\phi^{a_2}(2) \cdots \phi^{a_N}(N) |E\rangle . \quad (3.12) \]

Of course only those correlations with \(a_1 = a_2 = \cdots = a_N = 1\) are directly observable outside the black hole horizon.

Now since \(\langle \Omega | \beta \rangle = 1\), the microcanonical vacuum is normalized as follows,

\[ \langle \Omega | E \rangle = \delta(E) . \quad (3.13) \]

Recalling that,

\[ Z^{\pm 1}(\beta, V) = \int_0^\infty dE \ e^{-\beta E} \Omega(E, \pm V) , \quad (3.14) \]

and,

\[ \rho(\beta, \mathbf{H}) = \int_0^\infty dE \ e^{-\beta E} \rho(E, \mathbf{H}) , \quad (3.15) \]

where,

\[ \rho(E, \mathbf{H}) = \delta(E - \mathbf{H}) , \quad (3.16) \]

is the (un-normalized) microcanonical density matrix, and also making use of Eq.(3.10), we arrive at the following expression for the microcanonical vacuum \(|E\rangle\),

\[ |E\rangle = \Omega(E - \mathbf{H}, -V)|\Omega\rangle , \quad (3.17) \]
in which the density of states $\Omega(E, V)$ is given as,

$$
\Omega(E, V) = \delta(E) + \sum_{n=1}^{\infty} \left[ \frac{V}{(2\pi)^{D-1}} \right]^n \prod_{i=1}^{n} \left[ \int_{0}^{\infty} dm_i \sigma(m_i) \int_{-\infty}^{\infty} d^{D-1}k_i \right] \times \frac{1}{l_1 l_2 \cdots l_n} \delta(E - \sum_{i=1}^{n} l_i \omega_k(m_i)),
$$

(3.18)

where $\sigma(m)$ is the quantum black hole degeneracy of states and,

$$
\omega_k(m) = \sqrt{k^2 + m^2}.
$$

(3.19)

It may be useful to define a “normalized” microcanonical vacuum as follows,

$$
|O(E) > \equiv \frac{|E >}{\delta(0)},
$$

(3.20)

so that,

$$
< \exists |O(E) > = 1 \ (E = 0) .
$$

(3.21)

The set of Eqs. (3.12), (3.13), (3.16)-(3.20) therefore properly describes our quantum theory of fields in black hole spacetimes. Of course, we neglected here the interaction effects. These can be taken into account in perturbation theory with the use of the following microcanonical causal propagator,

$$
\Delta_{\epsilon, 1}^{a b} = \frac{1}{\delta(0)} \left( \frac{\tau_3 \delta(E)}{k^2 + m^2 - i\epsilon \tau_3} + 2\pi i \delta(k^2 + m^2) \right)
$$

$$
\times \left[ \sum_{l=1}^{\infty} \delta(E - l|k_0|) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \delta(E) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]
$$

(3.22)

in which use has been made of Eq.(3.21).

The physical (1,1)-component of the above propagator matrix is essentially Weldon’s propagator.

As opposed to the thermal propagator of Eq.(3.8), the above microcanonical propagator yields a particle number density (Hawking radiation) seemingly describing that of a pure state,
\[ n_{k,m} = \sum_{i=1}^{\infty} \frac{\delta(E - l\omega_{k}(m))}{\delta(0)}. \] (3.23)

A calculation supportive of this argument is based on the following expression for the “normalized” microcanonical density of states \( \hat{\rho}(E, \mathbf{H}) \),

\[ \hat{\rho}(E, \mathbf{H}) = \frac{\delta(E - \mathbf{H})}{\delta(0)}. \] (3.24)

Therefore we have,

\[ \hat{\rho}_{E_{1}E_{2}}(E) = \frac{\delta(E - E_{1})\delta(E_{1} - E_{2})}{\delta(0)}, \] (3.25)

and,

\[ \int_{0}^{\infty} dE \ \hat{\rho}_{EE} = 1. \] (3.26)

It is well known that a necessary and sufficient condition for a density matrix to describe a pure state is the following idempotency condition,

\[ \int_{0}^{\infty} dE' \ \hat{\rho}_{E_{1}E'} \hat{\rho}_{E'E_{2}} = \hat{\rho}_{E_{1}E_{2}}. \] (3.27)

It is immediate to show that the form (3.25) for the density matrix does satisfy the requirement (3.28). The eigenvalue of the density matrix (3.25) is 1.

Of course, in the usual context of ordinary statistical mechanics, the form (3.24) for the microcanonical density matrix is an idealization, an expression valid only approximatively at macroscopic scales such that the mesoscopic statistical fluctuations \( \delta E \) are large compared to the quantum fluctuations \( \Delta E \) (\( \delta E \gg \Delta E \)), but nevertheless small when seen from the macroscopic viewpoint at which statistical mechanics describes the system. In this way the microcanonical ensemble still describes mixed states.

The situation here is quite different as there is no statistical fluctuation concept to start with. The present system is a \textit{bona fide} quantum system embedded in a classical background. There is no ensemble theory here. Consequently, the form (3.24) for the density matrix is \textbf{not} an idealization but a precise statement, valid right down to the level of the quantum fluctuations. In this sense, the present microcanonical description is in a pure state.
IV. CONCLUSION

In this work, we presented a consistent quantization scheme for field theories in black hole spacetime backgrounds.

Results from the study of black hole statistical mechanics strongly suggested a fixed energy (static black hole mass) representation basis for the Hilbert space of the theory, instead of the usual “thermal state”. Such a representation is formally constructed by taking the inverse Laplace transform of the thermal description. Of course, because of the form for the quantum black hole degeneracy of states, only the microcanonical (E-) representation is well defined and leads to a stable vacuum.

We have further argued that, in the microcanonical representation, particle states in the black hole background (Hawking’s radiation) are pure states, unlike the traditional thermal description. This conclusion was reached on the basis of the idempotency condition obeyed by pure states density matrices.

Of course, ever deeper understanding of the results presented here will be at the core of our future endeavors.

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