Photon statistics of multimode even and odd coherent light

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Abstract

The even and odd coherent states are generalized for multimode case. The explicit forms for the photon distribution, Q-function and Wigner function are derived. In particular, it is shown that for two-mode case there exist strong correlations between these modes, under certain conditions, which are responsible for two-mode squeezing in case of even coherent states.

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1 Introduction

The even and odd coherent states of one mode oscillator have been introduced in [1] and they are studied in detail in Refs.[2-4]. The theoretical predictions of their possible generation have been discussed in Refs.[5-9]. The connection of these states with ‘Schrodinger cat’ states has been analysed specially because of the non-classical behaviour of these states [9]. An important possible application of these states has been predicted in interferometric detection of gravitational waves for reducing the optimal intensity of the input laser[10]. The properties of even coherent states are similar to the properties of squeezed vacuum states[11-14] and correlated vacuum states[15], because all such states are linear combination of the even photon number states. The only difference lies in the appearance of the normalization constants. These similarities allow us to investigate the possibilities to replace the squeezed light with even coherent light in the situations when the squeezed light shows some important applications. This problem has been posed and received a positive answer in Ref.[10], where the improvement for the sensitivity for the gravitational waves interferometric detection is suggested to achieve by using even coherent light as an alternative source to squeezed vacuum. Also it was shown that even coherent states may play an important role for enhancing the steady state squeezing in degenerate parametric oscillator when they are coupled to the cavity from outside by partially reflecting cavity end mirror [16].

Photon statistics of one-mode even and odd coherent states have been discussed in Ref.[2,3], where the photon distribution function has been obtained and nonclassical properties such as squeezing in quadrature component and higher-order squeezing phenomena are considered. Among the different theoretical predictions for the generation of such states, Gerry [5], has predicted that such single-mode even and odd coherent states can be produced in a Mach-Zender interferometer with Kerr medium and with the interaction of two-level atoms with the input coherent light. Recently, Gerry and Hach have shown another possibility of generating even and odd coherent light from long-time evolution of the competition between a two-photon absorption and two-photon parametric processes when the initial fields state is considered to be vacuum or one-photon state.
Gea-Banacloche [7], demonstrated the possibility for appearing such states in the resonant Jaynes-Cummings model (see also [8]). Brune et al. [9] discussed the creation of these states by dispersive atom-field couplings.

Till now the study of even and odd coherent states has been concentrated on single-mode. Such nonclassical light states corresponding to multimode field are also interesting because they in principle, may be generated in the similar situations which have been discussed for the single-mode case. Also the multimode even and odd coherent states may be considered as the simplest model of "Schrodinger cat" states for many degrees of freedom.

The aim of this paper is to consider the multimode generalization of the even and odd coherent states. We obtain the photon distribution function in multimode even and odd coherent states and calculated means and matrix of second order momenta for photon numbers corresponding to these distributions. We have shown that such distributions demonstrate essential differences with Poissonian photon distributions for multimode coherent states. Also we have evaluated the exact expressions for the Wigner function and Q-function for multimode even and odd coherent states. In addition we have studied the case of two modes in detail and have derived the exact expressions for the photon distribution function. We have also predicted the regions of different parameters for two-mode even and odd coherent states where the correlations between these two modes develop between them and which causes the basis for two-mode squeezing.

2 Single-mode even and odd coherent states

In this section we will review the properties of single-mode even and odd coherent states and on the ground of these calculations in the next section we will develop the formalism for multi-mode even and odd coherent states.

Usually a coherent state $| \alpha >$ can be defined as [17]

$$ | \alpha > = D(\alpha) | 0 >,$$ (1)
where the displacement operator $D(\alpha)$ is

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a].$$

\hspace{1cm} (2)

Also $\alpha$ is a complex number and $a(a^\dagger)$ being the annihilation and creation operators for the coherent field. In addition to normalized vacuum state $|0\rangle$, i.e., $<0 | 0> = 1$, we have decomposition of the normalized coherent states in terms of number states $|n\rangle$ as

$$|\alpha\rangle = e^{-\frac{\|\alpha\|^2}{2}} \sum_{n} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

\hspace{1cm} (3)

that corresponds to the photon distribution function for coherent states to be the Poissonian distribution function\[ 17,18\]

$$P(n) = e^{-\|\alpha\|^2} \frac{\|\alpha\|^{2n}}{n!}.$$

\hspace{1cm} (4)

The coherent state $|\alpha\rangle$ itself is the eigenstate of the photon annihilation operator with eigenvalue $\alpha$, i.e.,

$$a |\alpha\rangle = \alpha |\alpha\rangle.$$

\hspace{1cm} (5)

The coherent states are the minimum uncertainty states. In Ref.\[ 1,18\)), it was proposed to consider the superposition of two coherent states $|\alpha\rangle$ and $|\beta\rangle$. One of the possible superposition describes the even coherent states

$$|\alpha_+\rangle = N_+(|\alpha\rangle + |\beta\rangle),$$

\hspace{1cm} (6)

Where the normalization constant $N_+$ is of the form

$$N_+ = \frac{e^{\frac{1}{2} |\alpha|^2}}{2\sqrt{\cosh |\alpha|^2}}.$$ 

\hspace{1cm} (7)

For odd coherent state the following relation holds

$$|\alpha_-\rangle = N_-(|\alpha\rangle - |\beta\rangle),$$

\hspace{1cm} (8)

and the normalization constant $N_-$ corresponding to the odd coherent states is

$$N_- = \frac{e^{\frac{1}{2} |\alpha|^2}}{2\sqrt{\sinh |\alpha|^2}}.$$ 

\hspace{1cm} (9)
These states can be generated from vacuum in the following way

\[ |\alpha_\pm > = D(\alpha_\pm) |0 >, \]

where displacement operators for the even and odd coherent states are related to the displacement operator of the coherent states as these states are the superposition of the coherent states and are following

\[
D(\alpha_+) = \cosh(\alpha a^\dagger - \alpha^* a), \\
D(\alpha_-) = \sinh(\alpha a^\dagger - \alpha^* a).
\]

Both states are the normalized eigenstates of the operator \(a^2\)

\[ a^2 |\alpha_\pm > = \alpha^2 |\alpha_\pm >, \]

where \(\alpha\) is an arbitrary complex number. It is important to mention here that when the operator \('a'\) operates on an even coherent state can generate an odd coherent state with eigenvalue \(\alpha\) and different normalization constant.

\[ a |\alpha_+ > = \alpha \sqrt{\tanh |\alpha|^2} |\alpha_- >, \]

and similarly

\[ a |\alpha_- > = \alpha \sqrt{\coth |\alpha|^2} |\alpha_+ >. \]

The decomposition of the even and odd coherent states in terms of number states can be obtained by using Eq.(3)

\[
|\alpha_+ > = N_+ e^{\frac{\alpha^2}{2}} \sum_n \frac{1 + (-1)^n}{\sqrt{n!}} \alpha^n |n >, \\
|\alpha_- > = N_- e^{\frac{\alpha^2}{2}} \sum_n \frac{1 - (-1)^n}{\sqrt{n!}} \alpha^n |n >.
\]

It can be easily verify from the above equation that even coherent state can only be expressed in terms of even number state and odd
coherent state in terms of odd number state. The photon distribution function can be obtained from Eq.(15)

\[
P_{(+)}(n) = \begin{cases} \frac{|\alpha|^n}{\cos|\alpha|^2} & \text{for } n=2k \\ 0 & \text{for } n=2k+1 \end{cases}
\]

\[
P_{(-)}(n) = \begin{cases} 0 & \text{for } n=2k \\ \frac{|\alpha|^{n+1}}{\sin|\alpha|^{2(2k+1)}} & \text{for } n=2k+1 \end{cases}
\]

The photon distribution function for the even coherent states exhibits the property that the probability of finding odd number of photon becomes zero. Similarly the probability of finding the even number of photons in case of odd coherent states becomes zero. Thus the probability distribution function for these states is highly oscillatory function.

The expectation values of the first order moments for the annihilation and creation operators for the field in even and odd coherent states is zero which can be verified by using Eqs.(13),(14)

\[
< a >_+ = < \alpha_+ | a | \alpha_+ > = \frac{\alpha_+}{N_+} < \alpha_+ | \alpha_- > = 0.
\]

The above result can be obtained because even and odd coherent states are orthogonal states[1]. The similar result is also true for the odd coherent states. Also the expectation values of the second order moments are

\[
< a^2 >_{(+)} = < \alpha_\pm | a^2 | \alpha_\pm > = \alpha^2,
\]

\[
< a^\dagger a >_{(+)} = < \alpha_+ | a^\dagger a | \alpha_+ > = | \alpha |^2 \tanh | \alpha |^2,
\]

\[
< a^\dagger a >_{(-)} = | \alpha |^2 \coth | \alpha |^2.
\]

Then by defining the quadratures of the electromagnetic field mode as

\[
X_1 = \frac{a + a^\dagger}{2},
\]

\[
X_2 = \frac{a - a^\dagger}{2i}.
\]
The variances in the two quadratures of even coherent state are

\[
\Delta X_1^2 = \frac{2 |\alpha|^2 \tanh |\alpha|^2 + 2 |\alpha|^2 \cos 2\theta + 1}{4}, \\
\Delta X_2^2 = \frac{2 |\alpha|^2 \tanh |\alpha|^2 - 2 |\alpha|^2 \cos 2\theta + 1}{4},
\]  

(20)

where we have expressed the complex quantity as \( \alpha = |\alpha| e^{i\theta} \) and \( \theta \) be the phase of the coherent state amplitude. Eq.(20) shows that for \( \theta = \pi/2 \) the first quadrature shows some amount of squeezing for small values of \( |\alpha| \) and for \( \theta = 0 \) second quadrature is squeezed. In Fig.(1) we have plotted \( \Delta X_1^2 \) against \( |\alpha| \), for \( \theta = \pi/2 \). The figure illustrates the region for squeezing. This nonclassical behavior of even coherent states is also shown in Refs.[2,3], where \( \alpha \) is considered to be real and is predicted that squeezing occurs in second quadrature. We, on the other hand, have predicted the possibility of squeezing alternatively in both the quadratures depending upon the phase of the complex amplitude \( |\alpha| \). For odd coherent states we get the same expressions as in Eq.(20), but \( \tanh |\alpha|^2 \) should be replaced by \( \coth |\alpha|^2 \). Also odd coherent states do not exhibit the property of second-order squeezing.

The variances into the photon number operators for even coherent states can also be calculated by using Eq.(16).

\[
\sigma_{n_+} = <\alpha_+ | (a^\dagger a)^2 | \alpha_+ > - <\alpha_+ | a^\dagger a | \alpha_+ >^2, \\
= |\alpha|^4 + |\alpha|^2 \tanh |\alpha|^2 - |\alpha|^4 \tanh^2 |\alpha|^2, 
\]  

(21)

and for odd coherent states

\[
\sigma_{n_-} = <\alpha_- | (a^\dagger a)^2 | \alpha_- > - <\alpha_- | a^\dagger a | \alpha_- >^2, \\
= |\alpha|^4 + |\alpha|^2 \coth |\alpha|^2 - |\alpha|^4 \coth^2 |\alpha|^2. 
\]  

(22)

For large values of \( |\alpha| \), the photon number variances for even and odd coherent states become equal and we have

\[
\sigma_{n_\pm} = |\alpha|^2. 
\]  

(23)
3 multimode even and odd coherent states

In this section we will discuss the properties of the multimode even and odd coherent states. We define the multimode even and odd coherent states as

\[ | A_\pm \rangle = N_\pm (| A \rangle \pm -A \rangle), \]  

(24)

where the multimode coherent state \( | A \rangle \) is

\[ | A \rangle = | \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \rangle = D(A) | 0 \rangle, \]  

(25)

and the multimode coherent state is created from multimode vacuum state \( | 0 \rangle \) by the multimode displacement operator \( D(A) \) which is the exact analog of one mode displacement operator (Eq.(2)).

The normalization constants for multimode even and odd coherent states become

\[ N_+ = \frac{e^{\frac{|A|^2}{2}}}{2\sqrt{\cosh |A|^2}}, \]
\[ N_- = \frac{e^{\frac{|A|^2}{2}}}{2\sqrt{\sinh |A|^2}}. \]  

(26)

where \( \mathbf{A}=\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \) is a complex vector and its modulus is

\[ | A |^2 = | \alpha_1 |^2 + | \alpha_2 |^2 + ... + | \alpha_n |^2 = \sum_{m=1}^{n} | \alpha_m |^2. \]  

(27)

Such multimode even and odd coherent states can be decomposed into multimode number states as

\[ | A_+ \rangle = N_+ \sum_{n} \frac{e^{-|A|^2}}{\sqrt{n_1!...\sqrt{n_n!}}} (1 + (-1)^{n_1+n_2+...+n_n}) | n \rangle, \]
\[ | A_- \rangle = N_- \sum_{n} \frac{e^{-|A|^2}}{\sqrt{n_1!...\sqrt{n_n!}}} (1 - (-1)^{n_1+n_2+...+n_n}) | n \rangle, \]  

(28)

where \( | n \rangle = | n_1, n_2, ..., n_n \rangle \) being the multimode number state.

Also from Eq.(24), we can derive an important relation for the mul-
timode even and odd coherent states, i.e.,

\[
\begin{align*}
    a_i | A_+ > &= \alpha_i \sqrt{\tanh | A |^2} | A_+ >, \\
    a_i | A_- > &= \alpha_i \sqrt{\coth | A |^2} | A_+ >.
\end{align*}
\]  

(29)

From above equation it can be easily verified that the expectation value of the first order moments of the annihilation and creation operators for the ith mode for the multimode even and odd coherent states become zero. The probability of finding \( n \) photons in multimode even and odd coherent states can be worked out with the help of Eq.(28).

\[
\begin{align*}
P_+(n) &= \frac{[\alpha_1 |^{2n_1}| \alpha_2 |^{2n_2} ... | \alpha_n |^{2n_n}}{(n_1)! (n_2)! ... (n_n)! \cosh | A |^2}, \quad n_1 + n_2 + ... + n_n = 2k; \\
P_-(n) &= \frac{[\alpha_1 |^{2n_1}| \alpha_2 |^{2n_2} ... | \alpha_n |^{2n_n}}{(n_1)! (n_2)! ... (n_n)! \sinh | A |^2}, \quad n_1 + n_2 + ... + n_n = 2k + 1.
\end{align*}
\]  

(30)

Multimode coherent states are the product of independent coherent states of each mode and photon distribution function is the product of independent Poissonian distribution functions. But in the present case of multimode even and odd coherent states we cannot factorize their multimode photon distribution functions due to the presence of the nonfactorizable \( \cosh | A |^2 \) and \( \sinh | A |^2 \). This fact implies the phenomenon of statistical dependences of different modes of these states on each other.

In order to describe the properties of the distribution functions from Eq.(30), we will calculate the symmetric \( 2N \times 2N \) dispersion matrix for multimode field’s quadrature components. For even and odd coherent states we have

\[
< A_\pm | a_i a_k | A_\pm > = \alpha_i \alpha_k,
\]  

(31)

and complex conjugate values of the above equation for \( < A_\pm | a_i^\dagger a_k^\dagger | A_\pm >. \) Since the quantity \( < A_\pm | a_i | A_\pm > \) is equal to zero the above equation gives two \( N \times N \) blocks of the dispersion matrix
For other two $N \times N$ blocks of this matrix, we have for the multimode even coherent states
\[
\sigma^+_{(a^\dagger_i a_k)} = \langle A_+ \mid \frac{1}{2}(a_i^\dagger a_k + a_k^\dagger a_i) \mid A_+ \rangle = \alpha^*_i \alpha_k \tanh |A|^2 + \frac{1}{2} \delta_{ik},
\]
and for multimode odd coherent states
\[
\sigma^-_{(a^\dagger_i a_k)} = \langle A_- \mid \frac{1}{2}(a_i^\dagger a_k + a_k^\dagger a_i) \mid A_- \rangle = \alpha^*_i \alpha_k \coth |A|^2 + \frac{1}{2} \delta_{ik}.
\]

For the dispersion matrix, the mean values of the photon numbers $n_i = a_i^\dagger a_i$ for multimode even and odd coherent states are following
\[
\langle A_+ \mid n_i \mid A_+ \rangle = \left| \alpha_i \right|^2 \tanh |A|^2,
\]
\[
\langle A_- \mid n_i \mid A_- \rangle = \left| \alpha_i \right|^2 \coth |A|^2 .
\]

Taking into account the above equation, the symmetric $N \times N$ dispersion matrices for photon number operators can be obtained from the above given distribution functions for multimode even and odd coherent states. By defining
\[
\sigma_{ik}^\pm = \langle A_\pm \mid n_i n_k \mid A_\pm \rangle
\]
the corresponding expressions in such states are
\[
\sigma^+_{ik} = \left| \alpha_i \right|^2 \left| \alpha_k \right|^2 \text{sech}^2 |A|^2 + \left| \alpha_i \right|^2 \tanh |A|^2 \delta_{ik},
\]
\[
\sigma^-_{ik} = -\left| \alpha_i \right|^2 \left| \alpha_k \right|^2 \text{cosech}^2 |A|^2 + \left| \alpha_i \right|^2 \coth |A|^2 \delta_{ik}.
\]

As the nondiagonal matrix elements of the dispersion density matrix are not equal to zero so we can predict an important feature of multimode even and odd coherent states that different modes of these states are correlated with each other. In other words, as we have mentioned before, there exist some statistical dependences of different modes on each other.

Another interesting property for the multimode even and odd coherent states is the Q-function and it can be obtained in the following manner. First of all the density matrices for the multimode even and odd coherent states are
\[
\rho_{\pm} = | A_\pm > < A_\pm |,
\]
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\rho_{\pm} = | A_\pm > < A_\pm |,
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\[
\rho_{\pm} = | A_\pm > < A_\pm |,
then the Q-function can be calculated as

\[
Q_+(B, B^*) = \frac{1}{\pi} < B | \rho_+ | B >, \\
= \frac{4}{\pi} N_+^2 e^{-[(|A|^2 + |B|^2)]} | \cosh(AB^*) |^2, \\
Q_-(B, B^*) = \frac{1}{\pi} < B | \rho_- | B >, \\
= \frac{4}{\pi} N_-^2 e^{-[(|A|^2 + |B|^2)]} | \sinh(AB^*) |^2, 
\]

(38)

where \(| B > = | \beta_1, \beta_2, \ldots, \beta_n >\) is another multimode coherent state with multimode eigenvalue \( B = (\beta_1, \beta_2, \ldots, \beta_n) \). We call these functions for even and odd coherent states as the Q-functions for the Schrodinger cat states. In Fig.(2) we have plotted a three-dimensional plots of Q-function for single-mode even coherent states and for different choices of the quantity \(| \alpha |\) versus Re(\( \beta \)) and \( \text{Im}(\beta) \). Plots illustrate the fact that for small values of \(| \alpha |\), we get a single peak of the probability function and how this single peak begins to split into two peaks for the increasing values of \(| \alpha |\). Figs.(3) show Q-function for single-mode odd coherent state and it shows the crater type behaviour for the Q-function for small values of the quantity \(| \alpha |\) and for its larger values the Q-function begins to split into two peaks in a similar manners as in case of even coherent states. Hence we have shown that the behaviour of the plots for Q-functions of even and odd coherent states are essentially different for small amplitudes of \(| \alpha |\) and become very similar for its large values.

The Wigner function for the multimode coherent states is [19]

\[
W_{A,B}(q, p) = 2^N \exp[-2ZZ^* + 2AZ^* + 2BZ^* - AB^* - \frac{|A|^2}{2} - \frac{|B|^2}{2}],
\]

(39)

where

\[
Z = \frac{q + ip}{\sqrt{2}}.
\]

(40)

For even and odd coherent states the Wigner function will be the following different combinations of the above equation

\[
W_{A_+}(q, p) = | N_+ |^2 [W_{(A,B=A)}(q, p) + W_{(A,B=-A)}(q, p)],
\]

11
\[ W_{A^-}(q, p) = \begin{cases} +W_{(-A,B=A)}(q, p) + W_{(-A,B=-A)}(q, p) \big| N_+ \big|^2 - W_{(-A,B=A)}(q, p) + W_{(-A,B=-A)}(q, p), \end{cases} \]

(41)

where the explicit forms of \( N_{\pm} \) are given in Eq.(26). For multimode case we use the following notations

\[ A^* = \alpha_1 Z_1^* + \alpha_2 Z_2^* + \ldots + \alpha_n Z_n^*, \]
\[ Z^* = Z_1 Z_2^* + \ldots + Z_n Z_n^*. \]  

(42)

After describing some properties of the multimode even and odd coherent states, now in the following subsection we will discuss the case of two-mode even and odd coherent states.

### 3.1 Two mode even and odd coherent states

For two mode even and odd coherent states, Eq.(24) can be redefined as

\[ | A_{\pm} \rangle = N_{\pm}( | \alpha_1 + \alpha_2 \rangle \pm | -\alpha_1 - \alpha_2 \rangle, \]

(43)

and the normalization constants becomes

\[ N_+ = e^{\frac{1}{2} | \alpha_1 |^2 + | \alpha_2 |^2}, \]
\[ N_- = e^{\frac{1}{2} | \alpha_1 |^2 - | \alpha_2 |^2}. \]

(44)

Also from Eq.(29) we can define the relation

\[ a_i | A_+ \rangle = \alpha_i \sqrt{tanh(| \alpha_1 |^2 + | \alpha_2 |^2)} | A_- \rangle, \]
\[ a_i | A_- \rangle = \alpha_i \sqrt{coth(| \alpha_1 |^2 + | \alpha_2 |^2)} | A_+ \rangle, \quad i = 1, 2. \]

(45)

Above relation is an important result and with the help of this expression it is very simple to evaluate the first and higher order expectation values of different operators. For instance for even coherent
states the first and second order moments are
\[
< a_i >_+ = < A_+ | a_i | A_+ > , \\
= \frac{\alpha_i}{4e^{-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)}} \sinh(\frac{\alpha_1^2 + \alpha_2^2}{2}) < A_+ | A_- > = 0,
\]
(46)

as even and odd coherent states are orthogonal states. Also
\[
< a_i^2 >_+ = \alpha_i^2 < A_+ | A_+ > = \alpha_i^2 , \\
< a_i^\dagger a_i > = \alpha_i^2 \left( \frac{|N_+|^2}{|N_-|^2} < A_+ | A_+ > \right), \\
= \alpha_i^2 \tanh(|\alpha_1|^2 + |\alpha_2|^2),
\]
for \( i = 1, 2 \)
\[
< a_1^\dagger a_2 >_+ = \alpha_1^* \alpha_2 \tanh(|\alpha_1|^2 + |\alpha_2|^2).
\]
(47)

Such expectation values of first and second order moments will be used to evaluate squeezing in two mode even coherent states.

For two mode even coherent states we may define the quadratures of the field modes as
\[
d_1 = \frac{d + d^\dagger}{2}, \\
d_2 = \frac{d - d^\dagger}{2i},
\]
(48)

where \( d \) is the superposition of the two field modes
\[
d = \frac{\alpha_1 + \alpha_2}{\sqrt{2}}.
\]
(49)

The variances into the two quadratures are
\[
\Delta d_{1,2}^2 = < (d_1 \pm d_2)^2 > - < d_1 \pm d_2 >^2 ,
\]
(50)

where ‘+’ sign corresponds to \( \Delta d_1 \) and ‘-’ for \( \Delta d_2 \). After substituting the values of second order moments from Eq.(47) into Eq.(50) we have
\[
\Delta d_{1,2}^2 = \frac{1}{4} \left[ (|\alpha_1|^2 + |\alpha_2|^2 + 2 |\alpha_1| |\alpha_2| \cos(\theta_1 - \theta_2)) \times \tanh(|\alpha_1|^2 + |\alpha_2|^2) \pm (|\alpha_1|^2 + |\alpha_2|^2 + 2 |\alpha_1| |\alpha_2| \cos(\theta_1 + \theta_2)) \right], 
\]
(51)
where we have defined the complex quantity as $\alpha_i = \alpha_i e^{i\theta_i}$. In Fig.(4), we have plotted such variance in the second quadrature versus $|\alpha_1|$ and $|\alpha_2|$, for $\theta_1 = \theta_2 = 0$, plot shows the region where we can get maximum amount of squeezing in two-mode even coherent states.

The photon distribution function gives the probability of finding $2k$ photons for two-mode even coherent state and is defined as

$$P_+(2k) = \frac{(|\alpha_1|^2 + |\alpha_2|^2)^k}{(2k)! \cosh(|\alpha_1|^2 + |\alpha_2|^2)},$$

where $2k = n_1 + n_2$, for both $n_1$ and $n_2$ to be even or odd numbers. Similarly for two-mode odd coherent state it gives the probability of finding $2k + 1$ photons

$$P_-(2k + 1) = \frac{(|\alpha_1|^2 + |\alpha_2|^2)^{(2k+1)}}{(2k + 1)! \sinh(|\alpha_1|^2 + |\alpha_2|^2)}.$$  (53)

For this case $n_1$ is even and $n_2$ is odd number or vise versa. The dispersion matrices for the number operators in case of two-mode even and odd coherent states may be defined in terms of $2 \times 2$ matrices as

$$\sigma^{\pm}_{n_1, n_2} = \begin{pmatrix} \sigma^{\pm}_{11} & \sigma^{\pm}_{12} \\ \sigma^{\pm}_{21} & \sigma^{\pm}_{22} \end{pmatrix},$$

where

$$\sigma^{\pm}_{11} = |\alpha_1|^4 \sech^2(|\alpha_1|^2 + |\alpha_2|^2) + |\alpha_1|^2 \tanh(|\alpha_1|^2 + |\alpha_2|^2),$$

$$\sigma^{\pm}_{12} = |\alpha_1|^2 |\alpha_2|^2 \sech^2(|\alpha_1|^2 + |\alpha_2|^2),$$

$$\sigma^{\pm}_{21} = |\alpha_1|^4 \coth^2(|\alpha_1|^2 + |\alpha_2|^2) + |\alpha_1|^2 \coth(|\alpha_1|^2 + |\alpha_2|^2),$$

$$\sigma^{\pm}_{22} = -|\alpha_1|^2 |\alpha_2|^2 \cosech^2(|\alpha_1|^2 + |\alpha_2|^2).$$  (55)

For single-mode case it has been shown [2] that the photon distribution functions demonstrate super and sub-Poissonian properties for even and odd coherent states respectively. It can be easily shown on the basis of the above expressions that same conclusions may be drawn for the two-mode (and multimode) even and odd coherent states. In order to show the correlations between these two modes of
even and odd coherent states, we define the correlation coefficients as
\[
R_{\pm} = \frac{\sigma_{12}^\pm}{\sqrt{\sigma_{11}^\pm \sigma_{22}^\pm}},
\] (56)
and by using Eq.(55), we obtain the expression
\[
R_{\pm} = \frac{\pm |\alpha_1| |\alpha_2|}{\sqrt{(|\alpha_1|^2 \pm \frac{1}{2} sinh 2(|\alpha_1|^2 + |\alpha_2|^2))( |\alpha_2|^2 \pm \frac{1}{2} sinh 2(|\alpha_1|^2 + |\alpha_2|^2))}}. \] (57)

It is clear from the above equation that for large values of $|\alpha_1|$ and $|\alpha_2|$ this correlation tends to zero, but for small values of these quantities there exists an essential correlations between these two modes of even and odd coherent states. In Fig.(5) we have plotted $R_{+}$ versus $|\alpha_1|$ for different values of $|\alpha_2|$. It is also evident from the graph that the correlations between two-mode even coherent states occur for small values of $|\alpha_1|$ and $|\alpha_2|$ and they disappear for large values. For such small values these correlations are also responsible for two-mode squeezing as shown in Fig.(4).

In conclusions, we would emphasize that the suggested multimode generalization of even and odd coherent states might be extended for other superpositions of single-mode coherent states discussed by Schleich et al. [20,21]. These states are superpositions of coherent states with arbitrary relative phase factors. The photon distributions for such multimode states should demonstrate essential dependences of correlations between different modes on the phase factors. Such dependences produce the structure of the photon distribution functions similar to the structure in case of two-mode squeezed vacuum state [22], with oscillatory behaviour like for single-mode squeezed light, obtained by Schleich and Wheeler [23].
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Figure Captions

**Fig.(1).** Variance in the first quadrature of the field mode $\Delta X_1^2$ versus $|\alpha|$ for $\theta = \pi/2$.

**Fig.(2a).** Three dimensional plot of Q-function for single-mode even coherent light $Q_+(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=0.5$.

**Fig.(2b).** $Q_+(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=1.3$.

**Fig.(2c).** $Q_+(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=3$.

**Fig.(3a).** Three dimensional plot of Q-function for single-mode even coherent light $Q_-(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=0.5$.

**Fig.(3b).** $Q_-(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=1.3$.

**Fig.(3c).** $Q_-(\beta, \beta^*)$ versus Re($\beta$) and Im($\beta$) for $|\alpha|=3$.

**Fig.(4).** Three dimensional plot of the variance in the second quadrature of the field modes in case of two-mode even coherent states $\Delta d_2^2$ versus $|\alpha_1|$ and $|\alpha_2|$, for $\theta_1 = \theta_2 = 0$.

**Fig.(5) Correlation coefficient $R_+$ for two-mode photon numbers of even coherent states versus $|\alpha_1|$ for $|\alpha_2|=0.5, 1$ and 1.5.