Hamiltonian formulation of $SL(3)$ Ur-KdV equation

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Abstract

We give a unified view of the relation between the $SL(2)$ KdV, the mKdV, and the Ur-KdV equations through the Fréchet derivatives and their inverses. For this we introduce a new procedure of obtaining the Ur-KdV equation, where we require that it has no non-local operators. We extend this method to the $SL(3)$ KdV equation, i.e., Boussinesq(Bsq) equation and obtain the hamiltonian structure of Ur-Bsq equation in a simple form. In particular, we explicitly construct the hamiltonian operator of the Ur-Bsq system which defines the poisson structure of the system, through the Fréchet derivative and its inverse.

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Integrable models[1] in two dimensions in terms of nonlinear differential equations have many interesting properties. In particular, they are bi-hamiltonian systems, which means there are at least two distinct hamiltonian structures[2]. Such a structure provides the integrability to the system through infinite number of conserved quantities in involution. Another interesting feature of the integrable system is the existence of the zero curvature condition, i.e., Lax equation which is a compatibility condition for the pair of auxiliary linear equations. Lax equation is crucial in solving the integrable system by the inverse scattering method. Recently relations between the integrable nonlinear differential equations and two dimensional conformal field theories drew a lot of attention. For example, the poisson brackets of the second hamiltonian structure of the KdV equation is the classical version of the Virasoro algebra[3]. More generally, it was shown[4] that the second hamiltonian structure of the generalized equations of KdV type[5] is the classical version of the extended conformal algebras, i.e., $W_N$ algebras[6]. Furthermore Polyakov[7] introduced a new class of $W_N$ algebras, denoted $W_N^{(l)}$, which was further investigated by Bershadsky[8], and it gives the poisson brackets of the second hamiltonian structure of the fractional KdV equations[9, 10, 11]. Another relation between the extended conformal algebras $W_N$ and the generalized KdV systems is that the classical bosonization rules for the $W_N$ algebras are given by the Miura map which is nothing but the relation between two different choices of gauge slice[12], and the free field representations for the $W_N^{(l)}$ algebras can be also obtained by the same method[11].

More recently, an action was constructed that gives the KdV or mKdV equations as equations of motion[13], and their hamiltonian structures appear as poisson bracket structures derived from this action. Since the second hamiltonian structure of the KdV equation is given by the local poisson brackets, the corresponding symplectic form must contain a non-local operator, and it follows that the action has to be non-local in the KdV variable. However, one can write a local action[13] for the so called “Ur-KdV equation”, which is related to the KdV equation through the antiplectic pair formalism of Wilson[14]. Such a local action constitutes of two parts; the kinetic term gives the evolution part of KdV equation that describes a free theory in an appropriate variable, and the potential term is written in terms of the infinite number of conserved quantities of KdV system.

In this paper, we first rederive the KdV equation using zero curvature formulation with
$SL(2)$ matrix valued Lax operators, and clarify the relation between the KdV, the mKdV, and the Ur-KdV equations via the Fréchet derivatives and their inverses. In doing so, we first introduce a one parameter family of Ur-KdV equation, and find that it becomes free of non-local pseudo-differential operators, only for a particular value of the parameter. We then extend this method to the $SL(3)$ KdV equation, called the Boussinesq (Bsq) equation, of which the second hamiltonian structure is the classical $W_3$ algebra. As a main result, we find the modified Bsq (mBsq) equation and the Ur-Bsq equation which is simpler in its form than one given in Ref.[13] and can be easily generalized for case of $SL(N)$ for $N > 3$. We also construct the hamiltonian operators of mBsq and Ur-Bsq equations through the Fréchet derivatives and their inverse. In particular, we see that the hamiltonian structure of Ur-Bsq equation is given by non-local poisson brackets.

The KdV equation, $u_t = u_{xxx} + 6uu_x$, is a bi-hamiltonian system and is obtained by zero curvature condition parameterizing the two gauge potentials belonging to $SL(2)$ as

\[
U = \begin{pmatrix} 0 & 1 \\ -u + \lambda & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix},
\]

with a constant spectral parameter $\lambda$, and the corresponding Lax operator is $L = \partial_x - U$. The zero curvature condition $[\partial_x - U, \partial_t - V] = 0$ would lead to the dynamical equation and the constraint equations. Putting $B = \frac{\delta H}{\delta u}$, the dynamical equation can be expressed as a hamiltonian system:

\[
u_t = [\mathcal{D}^{(2)}(u) - 2\lambda\mathcal{D}^{(1)}(u)] \frac{\delta H}{\delta u},
\]

where the first hamiltonian operator, $\mathcal{D}^{(1)}(x)$, and the second hamiltonian operator, $\mathcal{D}^{(2)}(u)$, are given as

\[
\mathcal{D}^{(1)}[u] = \partial_x, \quad \mathcal{D}^{(2)}[u] = \frac{1}{2}\partial^3_x + 2u\partial_x + u_x.
\]

This equation expresses the feature encountered in the hamiltonian analysis of integrable hierarchies, i.e., the presence of two coordinated poisson structures expressed as the one-parameter family of brackets; $\{u(x), u(y)\} = \{u(x), u(y)\}_1 + \mu \{u(x), u(y)\}_2$ ($\mu$ is arbitrary). Taking $B = \frac{\delta H}{\delta u} = 2u$, the KdV equation is given as the second hamiltonian equation, \(u_t = \mathcal{D}^{(2)}[u] \frac{\delta H}{\delta u}\), and the corresponding poisson bracket is \(\{u(x), u(y)\} =\)
$D^{(2)}[u] \delta(x - y)$. This poisson structure represents the classical part of the Virasoro algebra [3].

To obtain the standard Miura map, we perform the gauge transformation $L \longrightarrow \Phi^{-1}L\Phi$ for the Lax operator $L$ with $\Phi(x,t)$ taking values in the strictly lower triangular matrices of $SL(2)$ with the diagonal elements set equal to one. Setting the transformed Lax operator as $\tilde{L} = \partial_x - \tilde{U}$, where $\tilde{U} = \Phi^{-1}U\Phi - \Phi^{-1}\Phi_x$, and choosing a gauge as $\tilde{U}_{11} = -j, \tilde{U}_{22} = j, \tilde{U}_{12} = 1$, and $\tilde{U}_{21} = \lambda$, we obtain the well-known Miura map; $u = j_x - j^2$. For the evolution equation of the variable $j$, that is modified KdV equation (mKdV), we introduce the Fréchet derivative of $u$ with respect to $j$, $\left[\frac{du}{dj}\right] = \partial_x - 2j$. In general, given a transformation $u = F[j, j_x, \cdots]$, $\left[\frac{du}{dj}\right]$ is the differential operator that implies $u_t = \left[\frac{du}{dj}\right] j_t$, and for any functional $H[u]$, we have $\frac{\delta H}{\delta j} = \left[\frac{du}{dj}\right]^* \frac{\delta H}{\delta u}$, where $\left[\frac{du}{dj}\right]^*$ is the formal adjoint of $\left[\frac{du}{dj}\right]$. The hamiltonian operator of mKdV equation, $D[j]$, which defines its hamiltonian equation, $j_t = D[j] \frac{\delta H}{\delta j}$, is related to the second hamiltonian operator of KdV equation through the above definition of the Fréchet derivative and their formal adjoint [14], as follows:

$$D^{(2)}[u] = \left[\frac{du}{dj}\right] D[j] \left[\frac{du}{dj}\right]^*$$. \hspace{1cm} (4)

In the above, we have $\left[\frac{du}{dj}\right]^* = -\partial_x - 2j$, and it is easy to show that $D[j] = -\frac{1}{2} \partial_x$ satisfies this equation. Therefore we can easily obtain the mKdV equation as follows:

$$j_t = D[j] \left[\frac{du}{dj}\right]^* \frac{\delta H}{\delta u} = j_{xxx} - 6j^2 j_x$$, \hspace{1cm} (5)

where we used $\frac{\delta H}{\delta u} = 2u$ of the KdV equation.

Now we note that there should be an inverse operator of $\left[\frac{du}{dj}\right]$ in order to define $D[j]$ in eq.(4). This suggests that we must take the form of transformation with respect to $j$ as $j = a \rho_x \rho^{-1}$ for the factorization of the Fréchet derivative $\left[\frac{du}{dj}\right]$, where $a$ is a constant. If we reset $\rho = q_x$, we have a transformation called the Cole-Hopf map; $j = a q_{xx} q_x^{-1}$. Using this transformation we can factorize the Fréchet derivative and their formal adjoint, in terms of variable $q$, as $\left[\frac{du}{dj}\right] = q_x^{2a} \partial_x q_x^{-2a}$, $\left[\frac{du}{dj}\right]^* = -q_x^{2a} \partial_x q_x^{2a}$, respectively, where we will be
using pseudo-differential operator, \( \partial_x^{-1} \), which satisfies \( \partial_x \partial_x^{-1} f = \partial_x^{-1} \partial_x f = f \). Therefore the inverses of these differential operators are written by

\[
\begin{align*}
\left[ \frac{du}{dj} \right]^{-1} &= \left[ \frac{dj}{du} \right] = q_x^{2a} \partial_x^{-1} q_x^{-2a} , &\left( \left[ \frac{du}{dj} \right]^{*} \right)^{-1} &= \left[ \frac{dj}{du} \right]^{*} = -q_x^{-2a} \partial_x^{-1} q_x^{2a} . \\
\end{align*}
\]

(6)

Through this Cole-Hopf map and the Miura map, we have also a relation between the variables \( u \) and \( q \) as follows:

\[
u = a \left\{ q_{xxx} q_x^{-1} - (a + 1)q_x^{2} q_x^{-2} \right\}.
\]

(7)

Using this relation, the second hamiltonian operator of KdV system can also be expressed in terms of variable \( q \):

\[
D^{(2)}[u] = \frac{1}{2} q_x^{2a} \partial_x q_x^{-2a} \partial_x q_x^{-2a} \partial_x q_x^{2a}.
\]

From this observation, we can easily find the hamiltonian operator of mKdV equation.

Now we can obtain the evolution equation of variable \( q \), through the Fréchet derivative of \( j \) with respect to \( q \),

\[
q_t = \left[ \frac{dq}{dj} \right] j_t = q_{xxx} - \partial_x^{-1} \left\{ 3q_{xxx} q_x^{-1} + 2(a^2 - 1)q_x^{3} q_x^{-2} \right\}.
\]

(8)

Taking \( a^2 = \frac{1}{4} \), we can eliminate the pseudo-differential operator, \( \partial_x^{-1} \), which is not adequate in an evolution equation. Then the above equation is the Ur-KdV equation given in Ref.[14]. The constant factor \( a \) of the Cole-Hopf map can take two values, \( \frac{1}{2}, -\frac{1}{2} \), but two choices are not independent; they are related by setting \( j \) to \( -j \). When we take \( a = \frac{1}{2} \), the transformation eq.(7) is the well-known Schwartzian derivative of \( q \) to \( 2u \). Using the Fréchet derivative of \( j \) with respect to \( q \) (or \( u \) to \( q \)) and their inverses, we can find the hamiltonian operator, \( D[q] \), which defines the hamiltonian equation of Ur-KdV equation, as follows:

\[
D[q] = \left[ \frac{dq}{dj} \right] D[j] \left[ \frac{dq}{dj} \right]^{*} = \left[ \frac{dq}{du} \right] D[u] \left[ \frac{dq}{du} \right]^{*} = -2 \partial_x^{-1} q_x \partial_x^{-1} q_x \partial_x^{-1} ,
\]

(9)

where,

\[
\begin{align*}
\left[ \frac{dq}{dj} \right] = 2 \partial_x^{-1} q_x \partial_x^{-1} , &\left[ \frac{dq}{dj} \right]^{*} = 2 \partial_x^{-1} q_x \partial_x^{-1} , \\
\left[ \frac{dq}{du} \right] = 2 \partial_x^{-1} q_x \partial_x^{-1} q_x \partial_x^{-1} q_x^{-1} , &\left[ \frac{dq}{du} \right]^{*} = -2 \partial_x^{-1} \partial_x^{-1} q_x \partial_x^{-1} q_x \partial_x^{-1} ,
\end{align*}
\]
and we have taken $a = \frac{1}{2}$. The non-local poisson brackets of the Ur-KdV equation are given by the hamiltonian operator as follows: $\{q(x), q(y)\} = -2\partial_x^{-1}q_x\partial_x^{-1}q_x\delta(x - y)$. Finally we can see the exact relation between three hamiltonian operator $D^{(2)}[u], D[j]$ and $D[q]$ using the Fréchet derivatives and their inverses.

As a generalization of the above procedure, we can find the $SL(3)$ Ur-KdV equation by requiring that it has no non-local operators, and also find its hamiltonian structure. The two gauge potentials belonging to $SL(3)$ is parametrized as

\[
U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_1 + \lambda & u_2 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A(x,t) & B(x,t) & C(x,t) \\ D(x,t) & E(x,t) & F(x,t) \\ G(x,t) & H(x,t) & I(x,t) \end{pmatrix}, \quad Tr \ V = 0, \quad (10)
\]

with the constant parameter $\lambda$, and the corresponding Lax operator is $L = \partial_x - U$. Using the zero curvature condition $[\partial_x - U, \partial_t - V] = 0$, we obtain the dynamical equations of $u_1, u_2$ and the constraint equations[15]. These equations are expressed only in terms of the variables of upper triangular part of $V$, i.e. $B, C$ and $F$. Redefining $u = u_2, \quad v = u_2x - 2u_1$, we recover the standard expression[9, 10]. Using one of the constraint equations, $C_x + B - F = 0$, and setting $\tilde{F} = F - \frac{1}{2}C_x, \tilde{C} = -\frac{1}{2}C$, we can write the hamiltonian equation of $u, v$ variables in a matrix form:

\[
(u_t, \quad v_t)^{\dagger} = \left(D^{(2)}[u,v] - 6\lambda D^{(1)}[u,v]\right) \left(\tilde{F}, \quad \tilde{C}\right)^{\dagger}, \quad (11)
\]

where the subscript $t$ denotes the time derivative. Only non-vanishing component of the first hamiltonian operator, $D^{(1)}[u,v]$, are $D^{(1)}_{12} = D^{(1)}_{21} = \partial_x$. The second hamiltonian operator, $D^{(2)}[u,v]$, is given as follows:

\[
D^{(2)}[u,v] = \begin{pmatrix} -2\partial_x^3 + 2u\partial_x + u_x & 3v\partial_x + 2v_x \\ 3v\partial_x + v_x & 2\partial_x^2/3 - 10u\partial_x^2/3 - 5u_x\partial_x^2 \\ \end{pmatrix}.
\]

When we take the upper triangular part of $V$ as the dual of the upper triangular part of $U$, i.e. $B = F = 0, C = 1 (\tilde{F} = 0, \tilde{C} = -\frac{1}{2})$, we obtain the Bsq equation, $u_t = -v_x, \quad v_t = \frac{1}{3}u_{xxx} - \frac{1}{3}uu_x$. The other elements of $V$ are obtained in terms of $u, v$ variables by the constraint equation. The above hamiltonian operators define the poisson structure of Bsq.
equation. The poisson brackets represented by the second hamiltonian operator are the classical version of $W_3$ algebra\[4\].

To look for the Miura map of the Bsq equation, that is the free field representation of the classical $W_3$ algebra, we perform the gauge transformation $L \rightarrow \Phi^{-1} L \Phi$ for the Lax operator $L$ with $\Phi(x,t)$ taking values in the strictly lower triangular matrices of $SL(3)$ with the diagonal elements set equal to one. Setting the transformed Lax operator as $\tilde{L} = \partial_x - \tilde{U}$, where $\tilde{U} = \Phi^{-1} U \Phi - \Phi^{-1} \Phi_x$, and choosing a gauge as

$$
\tilde{U} = \begin{pmatrix}
    j_1 & 1 & 0 \\
    0 & -(j_1 + j_2) & 1 \\
    \lambda & 0 & j_2
\end{pmatrix},
$$

we obtain the Miura map of $u$ and $v$ with respect to $j_1$ and $j_2$,

$$
u = (j_1 - j_2)_x + j_1^2 + j_1 j_2 + j_2^2,
$$

$$
v = -(j_1 + j_2)_{xx} - (2 j_1 - j_2) j_{1x} + (2 j_2 - j_1) j_{2x} + 2(j_1 + j_2) j_1 j_2.
$$

The modified Bsq equation ($mBsq$) for $j_1, j_2$ and its hamiltonian structure can be exhibited by the Fréchet derivative and their formal adjoint of the above transformation, which is, in a matrix form,

$$
\begin{bmatrix}
    \frac{d(u,v)}{d(j_1,j_2)}
\end{bmatrix}
= \begin{pmatrix}
    \partial_x + 2 j_1 + j_2 & -\partial_x + 2 j_2 + j_1 \\
    -\partial_x^2 + (j_2 - 2 j_1) \partial_x - 2 j_{1x} & -\partial_x^2 + (2 j_2 - j_1) \partial_x + j_{1x} \\
    -j_{2x} + 2 j_2^2 + 4 j_1 j_2 & +2 j_{2x} + 4 j_1 j_2 + 2 j_1^2
\end{pmatrix},
$$

$$
\begin{bmatrix}
    \frac{d(u,v)}{d(j_1,j_2)}
\end{bmatrix}^\dagger
= \begin{pmatrix}
    \partial_x + 2 j_1 + j_2 & -\partial_x^2 - (j_2 - 2 j_1) \partial_x - j_{2x} + 2 j_2^2 + 4 j_1 j_2 \\
    \partial_x + 2 j_2 + j_1 & -\partial_x^2 - (2 j_2 - j_1) \partial_x + 2 j_{1x} + 4 j_1 j_2 + 2 j_1^2
\end{pmatrix}.
$$

The hamiltonian operator of mBsq equation, $\mathcal{D}[j_1,j_2]$, which defines its hamiltonian equation, and the second hamiltonian operator of Bsq equation, $\mathcal{D}^{(2)}[u,v]$, are related by, through the Fréchet derivative and their formal adjoint,

$$
\mathcal{D}^{(2)}[u,v] = \left[ \frac{d(u,v)}{d(j_1,j_2)} \right] \mathcal{D}[j_1,j_2] \left[ \frac{d(u,v)}{d(j_1,j_2)} \right]^\dagger.
$$
Now we come to our main point. As mentioned before, we need to have a transformation by which we can obtain the inverses of the Fréchet derivative and their formal adjoint, eq.(15), in order to find $D[j_1, j_2]$. Therefore we write the transformation which is the generalized Cole-Hopf map as follows:

$$ j_1 = \alpha \psi_{xx} \psi_x^{-1} + \beta \phi_{xx} \phi_x^{-1}, \quad j_2 = \gamma \psi_{xx} \psi_x^{-1} + \delta \phi_{xx} \phi_x^{-1}, $$

(17)

where, $\alpha, \beta, \gamma$ and $\delta$ are arbitrary constants. We see that this form of Cole-Hopf map can be easily generalized to the $SL(N), N > 3$ case. Since the calculation involving arbitrary values of $\alpha, \beta, \gamma, \delta$ is quite lengthy, we sketch the steps here. It turns out that only particular values of $\alpha, \beta, \gamma, \delta$ are picked out. We later go through the steps more explicitly with a particular solution of $\alpha, \beta, \gamma, \delta$. Using this map, we can factorize the Fréchet derivatives, eq.(15), and find their inverses, denoted $[\frac{d(j_1, j_2)}{d(u,v)}], [\frac{d(j_1, j_2)}{d(u,v)}]^\dagger$. After a lengthy calculation, the Hamiltonian operator of mBsq equation, $D[j_1, j_2]$, is obtained by eq.(16), through expression of the second Hamiltonian operator of Bsq equation in terms of $\psi, \phi$. Then, it is easy to obtain the mBsq equation. The Hamiltonian operator of Ur-Bsq equation (the evolution equation of variables $\psi, \phi$), $D[\psi, \phi]$, is related by the Hamiltonian operator of mBsq equation, through the Fréchet derivatives of the above transformation as the eq.(16). It is not difficult to find the inverse Fréchet derivatives of $j_1, j_2$ with respect to $\psi, \phi$ and its formal adjoint, $[\frac{d(\psi, \phi)}{d(j_1, j_2)}], [\frac{d(\psi, \phi)}{d(j_1, j_2)}]^\dagger$, and we can obtain the non-local Hamiltonian operator of Ur-Bsq equation. Finally, we can find the Ur-Bsq equation through the mBsq equation as follows:

$$ \psi_t = \alpha \psi_{xx} + \partial_x^{-1}(b \psi_x \phi_{xxx} \phi_x^{-1} + c \psi_x \phi_{xx} \phi_x^{-2} + d \psi_x \phi_{xx} \phi_x^{-1} + e \psi_{xxx} \phi_x^{-1}), $$

$$ \phi_t = f \phi_{xx} + \partial_x^{-1}(g \phi_x \psi_{xxx} \psi_x^{-1} + h \phi_x \psi_{xx} \psi_x^{-1} + k \phi_x \psi_{xx} \psi_x^{-2} + l \phi_{xxx} \psi_x^{-1}), $$

(18)

where,

$$ a = \beta(2\alpha + \gamma) + \delta(\alpha + 2\gamma), \quad b = 2(\beta^2 + \beta \delta + \delta^2), $$

$$ c = \beta(2\alpha^2 - 2\alpha + 2\alpha \gamma - \gamma - \gamma^2) + \delta(\alpha^2 - \alpha - 2\alpha \gamma - 2\gamma - 2\gamma^2), $$

$$ d = \beta(2\beta^2 - 2\beta + 2\beta \delta - \delta - \delta^2) + \delta(\beta^2 - \beta - 2\beta \delta - 2\delta - 2\delta^2), $$

$$ e = 2(2\alpha \beta \delta - 2\beta \gamma \delta - 2\alpha \beta^2 - 2\gamma \delta^2 - \alpha \delta^2 + \beta^2 \gamma), $$

$$ f = -\alpha(2\beta + \delta) - \gamma(\beta + 2\delta), \quad g = -2(\alpha^2 + \alpha \gamma + \gamma^2), $$

(19)
These evolution equations include the pseudo-differential operator, \( \partial_x^{-1} \), which is not physically adequate in evolution equation. This can be avoided if we require following conditions for \( b = -d = e, c = 0 \) and \( g = -k = l, h = 0 \). There are six nontrivial solution sets of \((\alpha, \beta, \gamma, \delta)\) which satisfy these conditions. They are \((\alpha, \beta, \gamma, \delta) = (\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}), (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})\). In fact, these six sets do not give independent Cole-Hopf map of eq.(17), but they are related to each other through the adequate linear combination of \( j_1, j_2 \). The Ur-Bsq equations and its hamiltonian operators given by six solutions are different from only constant factors respectively. We will pick up one of the six solution sets, i.e., \((\alpha, \beta, \gamma, \delta) = (\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3})\), and derive the above-sketched steps. Firstly, using the transformation eq.(17) the elements of the \( \text{Fréchet derivatives} \) eq.(15) are factorized in terms of \( \psi, \phi \) as

\[
\left[ \frac{d(u, v)}{d(j_1, j_2)} \right] = \left( \begin{array}{cc}
\psi_x \partial_x \psi_x^{-1} & -\phi_x \partial_x \phi_x^{-1} \\
-\psi_x^{-1} \partial_x \psi_x^{-1} & -\phi_x^{-1} \partial_x \phi_x^{-1}
\end{array} \right),
\]

\[
\left[ \frac{d(u, v)}{d(j_1, j_2)} \right]^\dagger = \left( \begin{array}{cc}
-\psi_x^{-1} \partial_x \psi_x^{-1} & -\phi_x^{-1} \partial_x \phi_x^{-1} \\
\phi_x^{-1} \partial_x \phi_x^{-1} & -\phi_x^{-1} \partial_x \phi_x^{-1}
\end{array} \right).
\]

By the definition of the \( \text{Fréchet derivatives} \) and their formal adjoint, the inverse of these differential operator are

\[
\left[ \frac{d(j_1, j_2)}{d(u, v)} \right] = \frac{1}{2} \left( \begin{array}{cc}
\psi_x \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} & -\psi_x \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} \\
-\phi_x \partial_x^{-1} \psi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} & -\phi_x \partial_x^{-1} \psi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1}
\end{array} \right),
\]

\[
\left[ \frac{d(j_1, j_2)}{d(u, v)} \right]^\dagger = \frac{1}{2} \left( \begin{array}{cc}
-\psi_x^{-1} \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} & -\psi_x^{-1} \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} \\
\phi_x^{-1} \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1} & \phi_x^{-1} \partial_x^{-1} \phi_x \partial_x^{-1} \phi_x^{-1} \psi_x^{-1}
\end{array} \right).
\]

Using the above operators and expressing the second hamiltonian operator of Bsq equation in terms of \( \psi, \phi \) through the transformation, eq.(17), we can find the hamiltonian operator
of mBsq equation through eq.(16), as follows:

\[
D[j_1, j_2] = \frac{1}{3} \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix} \partial_x. \tag{22}
\]

Then the mBsq equation [13] is

\[
\begin{pmatrix}
j_1 \\
j_2
\end{pmatrix}_t = D[j_1, j_2] \left[ \frac{d(u, v)}{d(j_1, j_2)} \right] \left( \frac{\delta}{\delta u} \right) H = \frac{1}{3} \begin{pmatrix}
(j_1 x + 2j_2 x + j_1^2 - 2j_1 j_2)_x \\
(-2j_1 x - j_2 x - 2j_1^2 + j_2^2 - 2j_1 j_2)_x
\end{pmatrix}, \tag{23}
\]

where we used \( \frac{\delta H}{\delta u} = \tilde{F} = 0 \), \( \frac{\delta H}{\delta v} = \tilde{C} = -\frac{1}{2} \) in the case of the Bsq equation. The corresponding poisson brackets is defined by the hamiltonian operator of mBsq equation. Setting \( J_1 = \sqrt{\frac{3}{2}}(j_1 + j_2), J_2 = \frac{1}{2}(j_1 - j_2) \), we obtain the poisson brackets in terms of two bosons \( J_1, J_2 \) that express the free field realization of W_3 algebra.

We now find the Ur-Bsq equation. Through the Fréchet derivative and its formal adjoint of the transformation eq.(17), the hamiltonian operator of Ur-Bsq equation which defines its hamiltonian system and the poisson structure is related with the hamiltonian operator of mBsq equation as eq.(16). The inverse of Fréchet derivative and its formal adjoint of the transformation eq.(17) are easily obtained by the definition of the Fréchet derivative:

\[
\begin{pmatrix}
d(\psi, \phi) \\
d(j_1, j_2)
\end{pmatrix} = \begin{pmatrix}
-2\partial_x^{-1}\psi_x\partial_x^{-1} & -\partial_x^{-1}\psi_x\partial_x^{-1} \\
-\partial_x^{-1}\phi_x\partial_x^{-1} & 2\partial_x^{-1}\phi_x\partial_x^{-1}
\end{pmatrix},
\]

\[
\begin{pmatrix}
d(\psi, \phi) \dagger \\
d(j_1, j_2)
\end{pmatrix} = \begin{pmatrix}
-2\partial_x^{-1}\psi_x\partial_x^{-1} & -\partial_x^{-1}\phi_x\partial_x^{-1} \\
-\partial_x^{-1}\psi_x\partial_x^{-1} & 2\partial_x^{-1}\phi_x\partial_x^{-1}
\end{pmatrix}. \tag{24}
\]

Using these differential operators and the hamiltonian operator of mBsq equation, the hamiltonian operator of Ur-Bsq equation is

\[
D[\psi, \phi] = \begin{pmatrix}
2\partial_x^{-1}\psi_x\partial_x^{-1} & \partial_x^{-1}\psi_x\partial_x^{-1}\phi_x\partial_x^{-1} \\
-\partial_x^{-1}\phi_x\partial_x^{-1} & 2\partial_x^{-1}\phi_x\partial_x^{-1}\phi_x\partial_x^{-1}
\end{pmatrix}. \tag{25}
\]
This hamiltonian operator defines the poisson structure of Ur-Bsq equation;

\[
\{\psi(x), \psi(y)\} = 2\partial_x^{-1}\psi_x\partial_x^{-1}\psi_x\partial_x^{-1}\delta(x-y), \quad \{\psi(x), \phi(y)\} = \partial_x^{-1}\psi_x\partial_x^{-1}\phi_x\partial_x^{-1}\delta(x-y), \\
\{\phi(x), \phi(y)\} = 2\partial_x^{-1}\phi_x\partial_x^{-1}\phi_x\partial_x^{-1}\delta(x-y),
\]

where we note that \(\psi\) and \(\phi\) separately satisfy subalgebras, each of which is that of the Ur-KdV equation. We can see clearly that the symplectic form associated with the corresponding hamiltonian structure of the Ur-Bsq equation which is given by an inverse of the poisson structure is local. the Ur-Bsq equation is easily obtained, through the inverse Fréchet derivatives of \(\psi, \phi\) with respect to \(j_1, j_2\) and the transformation eq.(17), as follows:

\[
\psi_t = \frac{1}{3}\left(-\psi_{xx} - 2\phi_{xx}\phi_x^{-1}\psi_x\right), \quad \phi_t = \frac{1}{3}\left(\phi_{xx} + 2\psi_{xx}\psi_x^{-1}\phi_x\right).
\]

There are a relation between these equations and the Ur-Bsq equation given in Ref.[13]. That is, redefining of \(\tilde{\psi} = \phi, \tilde{\phi}_x = \phi_x\psi\), we can recover the Ur-Bsq equation given in Ref.[13] of \(\tilde{\psi}, \tilde{\phi}\). Using an obvious form of Cole-Hopf map for \(SL(N), N > 3\) cases, we expect to get quite a simple form for Ur equations for these cases. Of course details has to be worked out yet.

The work presented here clarifies the relation for \(SL(3)\) Ur-KdV equation corresponding to \(SL(3)\) KdV equation. While the poisson structure of the usual KdV equation defined by its hamiltonian operator contains highly local differential operators, that of the Ur-KdV equation is non-local operators. Because the symplectic form is associated with the inverse of poisson brackets in a classical mechanical system, the symplectic form of KdV system is non-local and that of Ur-KdV system local. Therefore, from the hamiltonian viewpoint, one say that the one flow takes place on a poisson manifold, the other on a symplectic manifold, and then the hamiltonian operators of two system (in the case of \(SL(2), (u, D^{(2)}[u]), (q, D[q])\)) is called the antiplectic pair by Wilson[14]. It was noted[13] that the above observation gives a clue to look for a local action of KdV system. In this paper, we found the Ur-KdV systems in the point of view what we must have a transformation by which can be obtained the inverse Fréchet derivatives of the Miura map. In addition, this view point clarifies the relation between the hamiltonian operators of KdV, mKdV and Ur-KdV equation.

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References


