Free and self-interacting scalar fields in the presence of conical singularities

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Free and self-interacting scalar fields in the presence of conical singularities are analyzed in some detail. The role of such a kind of singularities on free and vacuum energy and also on the one-loop effective action is pointed out using ζ-function regularization and heat-kernel techniques.

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I. INTRODUCTION

Quantum field theory in curved spacetime has been an important and interesting tool as a first step towards quantum gravity during the last two decades. Concentrating first on free scalar and higher spin fields (the main results may be found in Birrell and Davies [1]), afterwards also much progress has been made in self-interacting theories on curved spacetime (for a general review see [2]). For example the renormalizability of self-interacting $\phi^4$ scalar field theory in four dimensional curved spacetime has been shown (see for example [3, 4]). Furthermore, in connection with inflationary cosmological models the relevance of the gravitational background field for the effective potential has been considered in detail [5, 6, 2].

In all these considerations the so called heat-kernel coefficients play a central role because they determine the pole structure of the Green functions of the considered theory and thus the necessary counterterms in the corresponding effective Lagrangians (at least to one-loop) [7].

Recent progress on heat-kernel coefficients allowed the generalization of the mentioned considerations to curved spacetimes with boundary. In [8] (see also [9–13]) the magic $a_2$-coefficient has been determined for a second order elliptic operator, the leading symbol of which is the metric tensor for a general smooth manifold with smooth boundary. As an application the Casimir energy on this kind of manifolds has been analysed for free fields [14, 15] and the (one-loop) renormalization program has been performed in a self-interacting scalar field theory [16, 17].

Nowadays, quantum field theory on non-smooth manifolds like orbifolds is of increasing interest [18–21]. In general, it is very difficult to obtain the heat-kernel coefficients or the $\zeta$-functions of the relevant operators in such a kind of manifolds. Therefore, because at the moment there is no general scheme available, we concentrate in this article on the influence of the simplest non-smoothness a manifold may have, namely a conical singularity. It is realized by introducing a periodic coordinate ranging from 0 to $\beta$, which can be a real angle (as in a physical wedge) or imaginary time (as in the Rindler wedge). Some important work on quantum field theory of a free massless particle has already been done [22, 23]. We will generalize this work to the massive scalar field and to self-interacting scalar field theory on a cone. The influence of the conical singularity in the different contexts will be considered in this paper.

More detailed the organization of the paper is as follows. In Sec. II we briefly summarize the $\zeta$-function regularization technique as introduced by Hawking in Ref. [24] and use it in Sec. III in order to obtain the contribution to the vacuum energy due to the conical singularity, in the case of a massive scalar field, and in Sec. IV, in order to obtain finite temperature effects. In Sec. V we focus our attention on the interesting case of the Rindler wedge, in which the conical singularity is due to the appearance of a horizon. In Sec. VI we describe in detail how the presence of a singular point modify the one loop effective action for a self-interacting scalar field and finally, in Sec. VII, we conclude with some comments and possible applications of the result we
have obtained. The spectral geometry of the cone is given in the two Appendices, in which we also describe all mathematical apparatus we need.

Throughout the paper we will use units in which $\hbar = c = 1$.

II. $\zeta$-FUNCTION REGULARIZATION

In this section we will briefly introduce the $\zeta$-function regularization scheme [24, 25] used in order to define the relevant physical quantities. The considered spacetimes $\mathcal{M}$ will be $N$-dimensional ultrastatic ones, thus we present the basic equations needed in sections III, IV and V for a finite temperature quantum field theory of a free scalar field propagating in this kind of spacetimes. In Sec. VI we will have to add some part due to the self-interacting potential.

Finite temperature is incorporated using the Euclidean time formalism. Thus we perform a Wick rotation of the time variable $x^0$ to the imaginary time $\tau = i x^0$, requiring the scalar field $\phi$ to be periodic in imaginary time with period $\beta$, where $\beta = \frac{1}{T}$ is the inverse temperature.

The relevant second order elliptic differential operator describing the propagation of a free scalar field is given by $L_N = -\Delta_N + m^2 + \xi R$ together with appropriate boundary conditions for the field, where $\Delta_N$ is the Laplace-Beltrami operator on the manifold $\mathcal{M}$ and $m$ is the mass of the field. In terms of $L_N$ the partition function at temperature $T$ in the $\zeta$-function scheme is defined to be

$$Z_\beta = \int_{\phi(0,\vec{x})=\phi(\vec{x},\vec{y})} d[\phi] \exp \left( -\frac{1}{2} \int_{\mathcal{M}} \phi L_N \phi d^N x \right) = \exp \left( \frac{1}{2} \zeta'(0) |L_N/\mu|^2 \right)$$

(1)

where $\mu$ is an arbitrary renormalization parameter coming from the path integral measure. All regularized physical quantities then can be directly derived from Eq. (1) using the usual formulae of thermodynamics. As usual we call $Z_\beta$ partition function, but in fact it differs from the thermodynamical one for the presence of the vacuum energy. The corresponding zero-temperature results are obtained in the limit $\beta \to \infty$.

Let us mention that in the cases we shall consider in the paper, the manifold $\mathcal{M}$ will be the direct product between a $(N-2)$ dimensional manifold and a cone $C_\gamma$ with semi-angle $\sin^{-1}(\gamma/2\pi)$ with coordinates $\vec{r} \equiv (r, \varphi)$. In this way, the field operator separates in $L_N = -\Delta_{N-2} + \Delta_c + m^2 + \xi R$, $\Delta_c$ being the Laplacian on $C_\gamma$ (see Eq. (7)).

For ultrastatic manifolds one obtains $L_N = -\partial_r^2 + L_{N-1}$ and after some algebraic manipulation using the Jacobi identity [26] one has [27]

$$\zeta_\beta(s|L_N/\mu^2) = \frac{\mu^2 s^{-1/2}}{4\pi \Gamma(s)} \zeta(s-1/2|L_{N-1}/\mu^2)$$

$$+ \sum_{n=1}^{\infty} \int_0^\infty t^{s-3/2} e^{-\{n\mu \beta^2/4\}} \text{Tr} e^{-t L_{N-1}/\mu^2} dt$$

(2)

from which, easily follows
\[
\ln Z_\beta = -\frac{\beta}{2} \left[ \tilde{\zeta}(-1/2|L_{N-1}) - \frac{\tilde{\mu}K_{N/2}(L_{N-1})}{\sqrt{4\pi}} \right] + \frac{\beta}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \int_{0}^{\infty} t^{-3/2} e^{-(n\beta)^2/4t} \text{Tr} e^{-tL_{N-1}} dt
\]

where \(\tilde{\mu} = 2\ln(\epsilon\mu/2), \) \(\tilde{\zeta}\) is the finite part, in the given point, of the spatial section of the \(\zeta\)-function \(\zeta\) and \(K_{N/2}(L_{N-1})\) is the coefficient of \(\sqrt{t}\) in the asymptotic expansion [28–30]

\[
\text{Tr} e^{-tL_{N-1}} \sim \sum_{n=0}^{\infty} K_{\frac{n}{2}}(L_{N-1}) \ t^{\frac{a-n-1}{2}}
\]

Obviously Eq. (3) separates into a zero-temperature contribution, the terms in brackets, and into contributions from the excited states. As it is well known, only the zero-temperature part needs renormalization, thus the finite temperature contribution is independent of the renormalization scale \(\mu\).

### III. THE ROLE OF THE CONICAL SINGULARITY: VACUUM ENERGY

Here we derive the expressions for the renormalized vacuum energy for a massive scalar field on \(\mathbb{R}^{N-2} \times C_\gamma\). Before doing that, let us first consider in some detail the heat-kernel and the \(\zeta\)-function of the relevant operator on the cone \(C_\gamma\). The corresponding quantities on the product manifold \(\mathbb{R}^{N-2} \times C_\gamma\) are then easily obtained.

Let us define the manifold \(C_\gamma\) using global coordinates \(\tilde{r} = (r, \varphi)\) with the range \(r \in [0, \infty), \varphi \in \mathbb{R}\), and with the metric

\[
ds^2 = dr^2 + r^2 \, d\varphi^2
\]

where we identify \(\varphi \sim \varphi + n\gamma, \ n \in \mathbb{N}\). The relevant operator for a free massive scalar field is

\[
L_c = -\Delta_c + m^2
\]

with periodic boundary conditions in the angular variable, where in the metric (5) the Laplace-Beltrami operator is given in the form

\[
\Delta_c = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\varphi^2
\]

A complete set of normalized eigenfunctions with

\[
L_c \psi_{n,\nu}(r, \varphi) = (\nu^2 + m^2) \psi_{n,\nu}(r, \varphi)
\]

may then be determined to be
\[ \psi_{n,\nu}(r, \varphi) = \frac{1}{\sqrt{\gamma}} e^{i \frac{2\pi m}{\gamma}} J_{\frac{m}{\gamma}}(r \nu) \]  

(9)

with \( J_k \) the regular Bessel function. Using this set of eigenfunctions it is then possible to determine the heat-kernel \( K_t^\gamma(\vec{r}, \vec{r}^\prime) \) and the local and global \( \zeta \)-function of the operator \( L_c \). Some details of the calculation are relegated to the appendix A. There we derive for the heat-kernel

\[
K_t^\gamma(\vec{r}, \vec{r}^\prime) = \frac{e^{-m^2 t - (\vec{r} - \vec{r}^\prime)^2 / 4t}}{4\pi t} \left\{ \sum_{-\pi < \mu \gamma \leq \pi} \exp \left[ -\frac{r r^\prime}{t} \sin^2 \frac{n\gamma + \varphi - \varphi^\prime}{2} \right] \right. 
\]

\[
- \sin \frac{2\pi^2}{\gamma} \int_{-\infty}^{+\infty} \frac{\cos(u)}{\cosh(2\pi u) - \cos \frac{2\pi^2}{\gamma}} du \right\} \]  

(10)

Considering coincident points \( \vec{r} = \vec{r}^\prime \), Eq. (10) simplifies to

\[
K_t^\gamma(\vec{r}, \vec{r}) = \frac{e^{-m^2 t}}{4\pi t} \left\{ \sum_{-\pi < \mu \gamma \leq \pi} \exp \left[ -\frac{r^2}{t} \sin^2 \frac{n\gamma}{2} \right] \right. 
\]

\[
- \int_{-\infty}^{+\infty} \frac{e^{-\frac{r^2}{t} \cosh^2 \frac{2\pi u}{\gamma}}}{F(u, \gamma)} du \right\} \]  

(11)

where we introduced

\[
F(u, \gamma) = \frac{1}{\gamma} \frac{\sin \frac{2\pi^2}{\gamma}}{\cosh \frac{2\pi u}{\gamma} - \cos \frac{2\pi^2}{\gamma}} \]  

(12)

Thus using a well known integral representation of the Kelvin function \( K_s \), one finds for the local \( \zeta \)-function

\[
\zeta^\gamma_{s, \vec{r}} = \frac{\Gamma(s - 1)}{4\pi \Gamma(s)} m^{2-2s} \]

\[
+ \frac{1}{2\pi \Gamma(s)} \sum_{-\pi < \mu \gamma \leq \pi} \frac{m}{r \sin \frac{m\gamma}{2}} K_{s-1} \left( 2r \left| m \sin \frac{n\gamma}{2} \right| \right) \]

\[
- \frac{1}{\pi \Gamma(s)} \int_{0}^{\infty} du \, F(u, \gamma) \frac{m}{r \cosh \frac{u}{2}} K_{s-1} \left( 2r m \cosh \frac{u}{2} \right) \]

(13)

where the prime indicates omission of the summation index \( n = 0 \). The first term is exactly the Minkowski space contribution. Normalizing our results to zero for the Minkowski space, we will neglect this part thus defining \( \zeta_{s, \vec{r}}^{\text{sing}} \). For \( \gamma \) rational, the third part is seen to vanish, thus showing clearly the contributions to \( \zeta_{s, \vec{r}}^{\text{sing}} \) resulting from \( C_\gamma \) not being an orbifold in that case. Furthermore, using the asymptotics
of $K_{L-1}$ one finds that $\zeta_{\gamma}^{\text{sing}}(s, \bar{r}) \sim r^{2s-2}$ for $r \to 0$. This corresponds to the results for a local $\zeta$-function $\zeta(s|A)$ associated with a second order elliptic differential operator on a manifold with a boundary, which behaves like $\delta^{2a-d}$ ($d$ being the spacetime dimension) when the geodesic distance $\delta$ to the boundary of the manifold is going to zero. This leads to the well known divergences in local quantum functions revealed in numerous situations (see for example [31–34]) and which may be renormalized away by suitable counterterms [35].

Whereas the local quantities are all well defined, the global quantities will diverge due to the infinite volume of the cone. This divergence is equal to the Minkowski space divergence and is given by the $n = 0$ contribution in Eq. (11). Subtraction of the Minkowski space contribution leads to the regularized global $\zeta$-function

$$\zeta_{\gamma}^{\text{sing}}(s) = \frac{m^{-2s}}{12} \left( \frac{2\pi}{\gamma} - \frac{\gamma}{2\pi} \right)$$  \hspace{1cm} (14)

Now everything is prepared to calculate the vacuum energy for a massive scalar field on $\mathbb{R}^{N-2} \times C_\gamma$. Defining the vacuum energy as the limit $\beta \to \infty$ of the internal one, Eq. (3) directly yields [27]

$$E_v = -\partial_\beta \ln Z_\beta \bigg|_{\beta \to \infty}$$

$$= \frac{1}{2} \left[ \zeta(-1/2|L_{N-1}) - \frac{\bar{\mu} K_{N/2}(L_{N-1})}{\sqrt{4\pi}} \right]$$  \hspace{1cm} (15)

This definition is in agreement with the considerations in Ref. [36], where also the relevant remarks concerning the necessary renormalization may be found.

In order to determine the vacuum energy (15) we need to calculate $\zeta(-1/2|L_{N-1})$. Using Eq. (14) this is easily done. The massless case has to be treated separately, but here we do not consider it because it was long discussed in the literature [37]. Calculational details for the massive case are relegated to the appendix A.

It is convenient to distinguish between the odd and even dimensional case. Using Eqs. (A9) and (A10) we see that for odd $N \geq 3$, $K_{N/2}(L_{N-1})$ vanishes and $\zeta_{\gamma}^{\text{sing}}(s|L_{N-1})$ is finite for $s = -1/2$, thus the regularized vacuum energy is simply given by

$$E_{\text{sing}} = \frac{1}{2} \zeta_{\gamma}^{\text{sing}}(-1/2|L_{N-1}) = -\frac{R_0}{2} \frac{V_{N-2} \Gamma(1 - N/2)}{(4\pi)^{N/2}} m^{N-2}$$  \hspace{1cm} (16)

where $V_n$ is the volume of a large box in $\mathbb{R}^n$, we introduced

$$R_0 = \frac{\pi}{3} \left( \frac{2\pi}{\gamma} - \frac{\gamma}{2\pi} \right)$$  \hspace{1cm} (17)

and by the suffix $\text{sing}$ we indicate the contribution due to the singularity of those quantities obtained by subtracting the corresponding quantities on $\mathbb{R}^N$. Of course
they are vanishing for $\gamma = 2\pi$, thus renormalizing the vacuum energy for the Minkowski spacetime to zero. As is clearly seen, the sign of the vacuum energy alternates with the dimension $N$.

In the even case $N \geq 4$, a careful application of Eq. (15) gives

$$E_{\text{sing}} = \frac{R_0}{2} \left( \frac{(-1)^{N/2} V_{N-3}}{(4\pi)^{N/2} \Gamma(N/2)} \right) m^{N-2} \left( C_{N-1} - \ln \frac{m^2}{\mu^2} \right)$$

(18)

where $C_n = \sum_{k=1}^{n} \frac{1}{k}$. We note that the vacuum energy vanishes in the massless limit.

IV. THE ROLE OF CONICAL SINGULARITY: FINITE TEMPERATURE

To see how conical singularities modify the finite temperature theory, we consider as an example a massive, non interacting scalar field on the manifold $\mathcal{M} = S^1 \times \mathbb{R}^{N-3} \times C_\gamma$ with periodic boundary conditions on the factor $S^1$. Denoting by $L_{S^1} = -\partial_x^2 + m^2$ the massive Laplacian on $S^1$, we have $L_N = L_{S^1} - \triangle_{N-3} - \triangle_c$ and the heat kernel is the product of the three kernels on $S^1$, $\mathbb{R}^{N-3}$ and $C_\gamma$. So, using Eq. (A9) we have

$$\text{Tr} e^{-t L_N} = \frac{V_{N-3}}{\Gamma(s)(4\pi)^{N/2}} \left( \frac{V(C_\gamma) f(s - \frac{N-1}{2}) + R_0 f(s - \frac{N-3}{2})}{4\pi t} \right)$$

(19)

where $M_\alpha^2(\beta) = m^2 + (2\pi n/\beta)^2$ has been put.

Taking the Mellin transform of Eq. (19) we have

$$\zeta_\beta(s|L_N/\mu^2) = \frac{\mu^2 s V_{N-3}}{\Gamma(s)(4\pi)^{N/2}} \left[ V(C_\gamma) f(s - \frac{N-1}{2}) + R_0 f(s - \frac{N-3}{2}) \right]$$

(20)

The function

$$f(s) = \Gamma(s) \zeta(s|L_{S^1}) = \sum_{n=-\infty}^{\infty} \int_0^{\infty} dt \ t^{s-1} \exp \left\{-m^2 t - \left(\frac{2\pi n}{\beta}\right)^2 t \right\}$$

(21)

has simple poles at $s = \frac{1}{2} - k$ with residues $(-1)^k \beta m^{2k}/\sqrt{4\pi k!}$ ($k = 0, 1, \ldots$) and it has the asymptotic expansion for $\beta \to \infty$ $f(s) \sim m \beta \Gamma(s-1/2)/\sqrt{4\pi m^2 s}$.

For the logarithm of the finite temperature partition function then we get

$$\ln Z_\beta = \frac{V_{N-3}}{2(4\pi)^{N/2}} \left[ V(C_\gamma) f\left(\frac{1-N}{2}\right) + R_0 f\left(\frac{3-N}{2}\right) \right]$$

(22)

valid for odd $N \geq 3$, while for even $N \geq 4$ it reads

$$\ln Z_\beta = \frac{V_{N-3}}{2(4\pi)^{N/2}} \left\{ V(C_\gamma) \left[ f\left(\frac{1-N}{2}\right) + \beta m^N (\gamma + \ln \mu^2) \frac{(-1)^{N/2}}{\sqrt{4\pi \Gamma(1+N/2)}} \right] + R_0 \left[ f\left(\frac{3-N}{2}\right) - \beta m^{N-2} (\gamma + \ln \mu^2) \frac{(-1)^{N/2}}{\sqrt{4\pi \Gamma(N/2)}} \right] \right\}$$

(23)
$\tilde{f}(s)$ being the finite part of $f(s)$ in the given point. Taking the derivative with respect to $\beta$ and the limit $\beta \to \infty$ we again recover Eqs. (16) and (18).

The results Eqs. (22) and (23) are exact and clearly show the contribution due to the conical singularity vanishing if $\gamma = 2\pi$. Approximations valid for low or high temperature may be obtained by using well known techniques. We will not do so explicitly, but content ourselves with some remarks.

First of all, performing a Poisson resummation in Eq. (21) the low temperature expansion of Eqs. (22) and (23) is obtained.

For the discussion of the high-temperature limit let us utilize the general results of [38] (see also [39, 40, 13, 15, 41]). There it has been shown, that the relevant quantities for the high temperature expansion are the heat-kernel coefficients of the operator (for the considered case) $-\Delta_{N-3} - \Delta_0 + m^2$. In Appendix B we showed, that contributions only due to the existence of the conical singularity arise, thus altering all orders of the expansion apart from the leading Planckian term. Thus, in the same way a boundary changes the high-temperature behaviour of the theory, also a non-smoothness of a manifold changes this behaviour as exemplified by the cone.

In addition to finite temperature one may also be interested to introduce finite densities into the theory thus considering a charged scalar field. Of course it is once more possible to derive the equations corresponding to Eqs. (22) and (23). However, in that context one is especially interested in the phenomenon of Bose-Einstein condensation at high temperature, which may be discussed already using the general formalism given recently by Toms [42, 43]. Utilizing his results, it may be shown that the critical temperature at which the gas of a free charged scalar fields in the given spacetime will condensate is given by

$$T_C = \left( \frac{\frac{\pi}{4} Q}{2 \zeta R (N-2) \Gamma \left( \frac{N}{2} \right) m^2} \right)^{\frac{1}{N-2}} \tag{24}$$

(with the thermal average of the charge density $Q$), thus not changing the Minkowski spacetime result derived in [44]. This is essentially due to the fact, that the smallest eigenvalue of the Laplacian on the cone and in Minkowski spacetime are equal.

Another important consequence of the additional contributions due to conical singularities is connected with the conformal anomaly. In flat Minkowski spacetime the trace of the energy-stress tensor for a free massless particle is of course vanishing. As is well known [1] for a general spacetime it is proportional to the coefficient $K \frac{\omega^4}{2}$ in Eq. (4). As we have shown in Appendix B, $K_1$ depends on the angle $\gamma$ and is nonvanishing for $\gamma \neq 2\pi$, thus we find an anomalous trace for the scalar field on the cone. Having in mind, that the angle $\gamma$ in the presence of horizons may present temperature (see Sec. V), we explain the temperature dependence of the anomaly recently found in [20] in a static de Sitter spacetime. As we will explain more detailed in the Conclusions, this result is to be expected because such spacetimes may be
interpreted to have temperature dependent curvature tensors [45] and consequently also a temperature dependent stress tensor anomaly.

Both Eqs. (22) and (23) are valid for massive fields. In the massless case, disregarding the null eigenvalue (here it is equivalent to omit the zero temperature contribution), one has $f(s) = 2 \Gamma(s)(\beta/2\pi)^{2s} \zeta_R(2s)$, $\zeta_R(s)$ being the usual Riemann $\zeta$-function, and so one obtains the more explicit formulae for the logarithm of the thermodynamic partition function

$$\ln Z_\beta = \frac{V_{N-3}}{(4\pi)^{N-2}} \left[ \frac{V(C_\gamma)}{\Gamma(\frac{N+1}{2})} \left( \frac{2\pi}{\beta} \right)^{N-1} \zeta_R'(1 - N) \right.$$

$$\left. + \frac{R_0}{\Gamma(\frac{N-1}{2})} \left( \frac{2\pi}{\beta} \right)^{N-3} \zeta_R'(3 - N) \right]$$

(25)

$$\ln Z_\beta = \frac{V_{N-3}}{(4\pi)^{N-2}} \left[ V(C_\gamma) \left( \frac{2\pi}{\beta} \right)^{N-1} \Gamma(\frac{1-N}{2}) \zeta_R(1 - N) \right.$$

$$\left. + R_0 \left( \frac{2\pi}{\beta} \right)^{N-3} \Gamma(\frac{3-N}{2}) \zeta_R(3 - N) \right]$$

(26)

for odd and even $N$ respectively. Here comparing with the flat spacetime results [46], the comments of the influence of the conical singularity on the high-temperature behaviour are made explicit at subleading order.

As a simple application of the latter equation, we compute the free energy for a massless scalar field on the manifold $\mathcal{M} = S^1 \times \mathbb{R} \times C_\gamma$. By using definition $F = -\ln Z_\beta/\beta$ we directly get

$$F = -V_1 \left[ \frac{V(C_\gamma)\pi^2}{90\beta^4} + \frac{R_0}{24\beta^2} \right]$$

(27)

where once more we see that the singularity gives a contribution at subleading order proportional to $T^2$ to the free energy. A formula very close to the previous one is valid for the internal energy. In fact one has

$$E = V_1 \left[ \frac{V(C_\gamma)\pi^2}{30\beta^4} + \frac{R_0}{24\beta^2} \right]$$

(28)

showing the same structure as Eq. (27).

V. THE RINDLER WEDGE

Now we consider the space-time region $\mathcal{R}$, which is causally accessible to a uniformly accelerated observer with acceleration $a$. The metric can be taken in the form
\[ a^2 ds^2 = -a^2 \xi^2 dt^2 + d\xi^2 + dy^2 + dz^2 \]  

(29)

where \( 0 < \xi < \infty \). Although at each point there are ten Killing vector fields, only six of them generate a global group of isometries of \( \mathcal{R} \). The set \( \Sigma \) defined by \( \xi = \xi_0 \) and \( t = 0 \) is a space-like submanifold which is left fixed by the one parameter group of isometries generated by the Killing field

\[ K = \sinh t \partial_\xi + \left( \frac{1}{\xi_0} - \frac{1}{\xi} \cosh t \right) \partial_t \]  

(30)

This is unique up to a scale and choice of \( \xi_0 \). In such a situation the set of all future directed null geodesics which intersect \( \Sigma \) form a so called bifurcate Killing horizon. One can easily see this is the limit of vision for an observer with uniform acceleration \( 1/\xi_0 \). This is thus the surface gravity of the horizon. The Rindler metric has such a bifurcate horizon and precisely for that reason its Euclidean brother, with \( \tau = it \) compactified on \( S^1 \), describes the product manifold \( \mathcal{M} = \mathcal{C}_\gamma \times \mathbb{R}^2 \), where now \( \gamma = a\beta \) is proportional to the inverse temperature. The theory of a quantum field on this manifold should describe a thermal state in the Fock space built with the Rindler mode functions, and we should clarify its relation with the corresponding Minkowski state. At \( \beta = \beta_H = 2\pi/a \) we know this is just the Minkowski vacuum while at \( \beta = \infty \) it is the Rindler one (\( \beta_H \) represents the Hawking inverse temperature).

The \( \xi \)-function for a scalar field on such a manifold is given by Eq. (A10). Here we need the density, then we consider Eq. (A8) in the coincidence limit and replace \( \gamma \) by \( a\beta \) and \( r \) by \( \xi \). In the massless case, for the free energy we easily get

\[ F_{\text{sing}} = -\frac{1}{2} \xi_{\text{sing}}(0) = -\int_{\mathcal{M}} \frac{1}{1440 \xi^4 \pi^2} \left( \frac{\beta_H^4}{\beta^4} + \frac{10 \beta_H^2}{\beta^2} - 11 \right) d^4 x \]  

(31)

where only the contribution due to the singularity has been written.

Now in a thermal state we expect there is no flux of energy-momentum with respect to the observers defined by the unique time-like Killing field \( \mathcal{K} = (1, 0, 0, 0) \). So the energy-momentum tensor must obey \( T_{0i}(\gamma, \xi) = 0 \). Moreover it must be traceless since there are no anomalies in Minkowski space. Hence, in any coordinate system the stress tensor will be

\[ T_{ab}(\beta, \xi) = \frac{1}{1440 \xi^4 \pi^2} \left( \frac{\beta_H^4}{\beta^4} + \frac{10 \beta_H^2}{\beta^2} - 11 \right) \frac{1}{\mathcal{K}^2} \left( g_{ab} - \frac{4 \mathcal{K}_a \mathcal{K}_b}{\mathcal{K}^2} \right) \]  

(32)

where \( \mathcal{K}^2 = \mathcal{K}_a \mathcal{K}^a \). Of course, at the Hawking temperature \( T = a/2\pi \) it vanishes since the \( \beta = \beta_H \) thermal state corresponds to Minkowski vacuum. One can say there is no conical singularity, in this case. At \( \beta = \infty \) it correspond to the vacuum energy in the Rindler wedge.
VI. RENORMALIZATION OF $\lambda \phi^4$ ON $\mathcal{M} = \mathbb{R}^2 \times C_\gamma$

As the last point of our paper let us concentrate on self-interacting $\lambda \phi^4$ theory. To ensure renormalizability we restrict to a four dimensional manifold, namely $\mathcal{M} = \mathbb{R}^2 \times C_\gamma$.

We will be especially interested in the one-loop effective action of the theory. Because the effective action is a concept well discussed in the literature (see for example [2]) the description of its evaluation will be very brief. We will mainly follow [47].

In the functional integral perturbative approach, the effective action is expanded in powers of $\hbar$ as

$$\Gamma[\hat{\phi}] = S[\hat{\phi}] + \Gamma^{(1)} + \Gamma'$$ (33)

where $S[\hat{\phi}]$ is the classical action and the one-loop contribution $\Gamma^{(1)}$ to the action is given by

$$\Gamma^{(1)} = \frac{1}{2} \ln \det \frac{A}{\mu^2} = \frac{1}{2}\zeta'(0)|A/\mu^2|$$ (34)

with the $\zeta$-function $\zeta(s|A/\mu^2)$ of the operator $A = -\Delta + M^2$, where we introduced the effective mass $M^2 = m^2 + \frac{\Delta}{2} \phi^2$. Furthermore, $\Gamma'$ represents higher loop corrections which we are not going to discuss.

Assuming constant background field $\hat{\phi}$, the analysis of Sec. III may be used. Nearly without any further calculation one finds the representation

$$\zeta(s, \bar{\tau}|A) = \frac{V_2}{4\pi(s-1)} \zeta_{\gamma}(s, \bar{\tau})$$ (35)

with $\zeta_{\gamma}(s, \bar{\tau})$ given in Eq. (13) with the replacement $m \rightarrow M$. Using Eq. (14) the integrated version of Eq. (35) reads

$$\zeta(s|A) = \frac{V_2 V(C_{\gamma})}{16\pi^2(s-2)(s-1)} M^{2(2-s)}$$ (36)

$$+ \frac{V_2 R_0}{16\pi^2(s-1)} M^{2(1-s)}$$

where the first term corresponds to the result of Coleman and Weinberg [48] and the second term is a correction due to the singularity of the cone. As is well known by now, all terms appearing in $\zeta(0|A) \sim K_2(x, x)$ need counterterms in order to perform the renormalization [5, 49, 47, 7]. So we see already here, that due to

$$\zeta(0|A) = \frac{V_2 V(C_{\gamma})}{32\pi^2} M^4 - \frac{V_2 R_0}{16\pi^2} M^2$$ (37)

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in addition to the Minkowski space counterterms, additional counterterms are forced by the conical singularity, which will be determined in the following. First using Eq. (36) the effective action is found to be

\[
\Gamma(\phi) \sim \int \left[ -\frac{\phi \hat{\Delta} \phi}{2} + \frac{\lambda \phi^4}{24} + \frac{m^2 \phi^2}{2} \right] d^4x + \frac{1}{64\pi^2} \int_\mathcal{M} M^4(\phi) \left( \ln \frac{M^2(\phi)}{\mu^2} - \frac{3}{2} \right) d^4x - \frac{1}{64\pi^2} \int_\mathcal{M} 2R_0 M^2(\phi) \left( \ln \frac{M^2(\phi)}{\mu^2} - 1 \right) \delta(\bar{r}) d^4x
\]

(38)

The last term in Eq. (38) is due to the presence of the singularity, \( \delta(\bar{r}) \) being the Dirac delta function on the cone. The result agrees with the adiabatic expansion in terms of heat-kernel coefficients presented in [47]. This is easily seen by using the heat-kernel coefficients given in Eq. (B4). For the example of the cone, the integrated coefficients \( K_i \) are vanishing for \( l > 2 \) and the adiabatic expansion terminates leading to Eq. (38).

In order to renormalize all coupling constants, to the effective Lagrangian density \( \mathcal{L} \) in Eq. (38) we have to add the counterterms

\[
\delta \mathcal{L} = \delta \Lambda + \frac{\delta \lambda \phi^4}{24} + \frac{\delta m^2 \phi^2}{2} + \delta \epsilon_0 R + \frac{1}{2} \delta \xi R \phi^2
\]

(39)

where we introduced \( R = 6R_0 \delta(\bar{r}) \).

Before determining the counterterms explicitly let us give the motivation for the chosen scheme. Apart from the conical singularity, which is normally excluded from the manifold, the curvature of the cone is obviously vanishing. However, in a recent article [45], the notion of tensors on smooth manifolds has been generalized to tensor-distributions on manifolds with singular points thus finding a curvature distribution in these singular points, for example for the cone this distribution is given by the \( \bar{R} \) introduced above (from that point of view, we are studying the minimally coupled case). The very nice feature in regarding \( 6R_0 \delta(\bar{r}) \) as the curvature of the cone is now, that one can take over the renormalization scheme from smooth manifolds [48, 5] without changes. Thus the necessary counterterms are given as in Eq. (39), the last two terms being concentrated on the tip of the cone. Higher order terms in the curvature are not needed for the present case due to the results presented in Appendix B.

After these remarks, going on as usual, one imposes the renormalization conditions [48, 5]

\[
0 = \mathcal{L} \bigg|_{\phi=\phi_0,R_0=0}
\]

\[
\lambda = \frac{\partial^4 \mathcal{L}}{\partial \phi^4} \bigg|_{\phi=\phi_1,R_0=0}
\]
\[ m^2 = \frac{\partial^2 \mathcal{L}}{\partial \phi^2} \bigg|_{\phi=0, R_0=0} \]
\[ 0 = \frac{\partial^3 \mathcal{L}}{\partial \phi^2 \partial R} \bigg|_{\phi=0, R_0=0} \]
\[ \epsilon = \frac{\partial \mathcal{L}}{\partial R} \bigg|_{\phi=0, R_0=0} \]

and explicitly gets the counterterms

\[ 64 \pi^2 \delta \Lambda = -64 \pi^2 \left( \frac{m^2 \varphi_0^2}{2} + \frac{\lambda \varphi_0^4}{24} \right) + \frac{\lambda^2 \varphi_0^4}{3M_1^4} (M_1^2 - m^2)(2M_1^2 + m^2) \]
\[ + M_0^4 \left( \ln \frac{M_1^2}{M_0^2} + \frac{3}{2} \right) - 2m^2 M_0^4 \left( \ln \frac{M_1^2}{m^2} + 1 \right) \]
\[ + m^4 \left( \ln \frac{M_1^2}{m^2} - \ln \frac{m^2}{\mu^2} + 2 \right) \]
\[ 64 \pi^2 \delta \lambda = \lambda^2 \left( \frac{8m^4}{M_1^4} + \frac{8m^2}{M_1^2} - 16 - 6 \ln \frac{M_1^2}{\mu^2} \right) \]
\[ 64 \pi^2 \delta m^2 = -2\lambda m^2 \left( \ln \frac{m^2}{\mu^2} - 1 \right) \]
\[ 64 \pi^2 \delta \epsilon_0 = 2m^2 \left( \xi - \frac{1}{6} \right) \left[ 1 - \ln \frac{m^2}{\mu^2} \right] \]
\[ 64 \pi^2 \delta \xi = -2\lambda \left( \xi - \frac{1}{6} \right) \left[ \ln \frac{M_1^2}{\mu^2} + \frac{\lambda \varphi_0^2}{M_1^2} \right] \]

where \( M_1^2 = m^2 + \frac{1}{2} \varphi_0^2 \) has been put. For the sake of generality, we chose different values \( \varphi_i \) for the definition of the physical coupling constants. This is due to the fact that in general they are measured at different scales, the behaviour with respect to a change of scale being determined by the renormalization group equations. In particular, \( \varphi_0 \) is the true minimum of the potential and classically it is equal to zero.

Let us stress once more, that due to the conical singularity additional counterterms concentrated on the tip of the cone are necessary and have to be included in the classical Lagrangian. This is a direct consequence of the heat-kernel expansion presented in appendix B. As is seen, the heat-kernel coefficients contain contributions concentrated in the singularity \( r = 0 \) of the cone. Now, the one-loop renormalization counterterms are completely determined by the coefficient \( K_2(x, x) \) \([47, 7]\), thus leading to counterterms of the mentioned type. As we have shortly described, this terms may be seen as resulting from the curvature of the cone concentrated in the singular point.

After some tedious calculations one finds the renormalized one-loop contribution to the effective Lagrangian density in the form
\[ 64\pi^2 \mathcal{L}_r^{(1)} = -32\pi^2 m^2 \varphi_0^2 - \frac{8\pi^2 \lambda \varphi_0^4}{3} + \lambda m^2 \varphi_0^2 \left( \ln \frac{m^2}{M_0^2} + \frac{1}{2} \right) \]

\[ + m^4 \ln \frac{M^2}{M_0^2} - 2m^2 R_0 \delta(\tilde{r}) \ln \frac{M^2}{m^2} \]

\[ - \frac{\lambda^2 \varphi_0^4}{4} \left[ \ln \frac{M_0^2}{M_1^2} - \frac{3}{2} - \frac{4(M_1^2 - m^2)(2M_1^2 + m^2)}{3M_1^4} \right] \]

\[ + \left\{ R_0 \delta(\tilde{r}) \left[ \ln \frac{M_1^2}{M_2^2} + 1 + \frac{\lambda \varphi_0^2}{M_2^2} \right] + m^2 \left[ \ln \frac{M^2}{m^2} - \frac{1}{2} \right] \right\} \lambda \phi^2 \]

\[ + \left\{ \ln \frac{M^2}{M_1^2} - \frac{25}{6} + \frac{4m^2(m^2 + M_1^2)}{3M_1^4} \right\} \frac{\lambda^2 \phi^4}{4} \]

Eq. (42) clearly shows the influence of the singular point. Of course, in the limit in which \( R_0 \) (effectively the curvature) and \( m \) go to 0, we obtain the well known Coleman-Weinberg result [48].

VII. CONCLUSION

In this paper we considered different aspects of the quantum field theory of a scalar field in the presence of a conical singularity. Especially we calculated the vacuum energy and the free energy of a non-interacting scalar field and the effective potential in a self-interacting \( \phi^4 \)-theory. The presented differences to the corresponding Minkowski spacetime results may clearly be led back to the presence of the conical singularity. As was exemplified especially clear in Sec. VI, in general the influence of the singularity may be well understood by attaching non-vanishing curvature tensors to the cone \( C_\gamma \) as proposed in [45]. Then as a natural consequence these curvature terms lead to additional contributions in the physical quantities we calculated.

Realizing that the curvature tensors depend on the angle \( \gamma \) which, in spacetimes with horizons like the Rindler wedge, represents temperature, one is naturally led to the temperature dependence of the stress-energy tensor anomaly in the presence of horizons [20], which is, however, not in conflict with new results presented for example in [50].

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APPENDIX A: HEAT KERNEL ON THE CONE

In this appendix we briefly derive two representations for the heat kernel of the Laplace operator \(-\Delta_c\) on the cone, for a detailed discussion see ref. [51].
The problem was already studied a long time ago by Sommerfeld [52] and later by Carslaw [53] in connection with heat diffusion and more recently by Dowker [22] and Brown and Ottewill [37] in connection with quantum field theory in Rindler space (for an exhaustive list of references on the subject we refer the reader to Ref. [22]).

Let $\mathcal{M}_\infty$ be the infinitely sheeted Riemann surface with global coordinates $\tilde{r} \equiv (r, \varphi)$ running inside $[0, \infty) \times (-\infty, +\infty)$ and metric $ds^2 = dr^2 + r^2 d\varphi^2$. Let us further consider the 2-dimensional manifold, $C_\gamma$, which results from the identification $\varphi \sim \varphi + n\gamma$. If $\gamma = 2\pi / j$ ($j \geq 1$) it is a true orbifold with fundamental domain the infinite wedge $0 < r < \infty$, $0 < \varphi < \gamma$, otherwise it is not, but the metric still describe a cone with semi-angle $\sin^{-1}(\gamma / 2\pi)$. We are interested in the two basic objects for quantum field theory, namely the heat-kernel and the $\zeta$-function of the massive Laplace operator on the cone, $L_c = -\Delta_c + m^2$, and we choose periodic boundary conditions. To simplify the formulae we set $m = 0$. The presence of mass simply gives rise to a trivial exponential factor which we shall take into account when necessary.

Let us study first the heat kernel for $\gamma = \infty$.

A complete set of normalized eigenfunctions is easily found to be

$$
\psi = \frac{1}{(2\pi)^{1/2}} e^{ik\varphi} J_k(\nu r)
$$

(A1)

together with its complex conjugate. Here $k \geq 0$ and $\nu^2 \geq 0$ is the eigenvalue corresponding to $\psi$ and $\psi^*$, while $J_k$ is the regular Bessel function. These states have a density measure $d\mu = \nu^{-1}\delta(\nu - \nu')$, so that the heat kernel reads

$$
K_t^\infty(\tilde{r}, \tilde{r}' \mid -\Delta_c) = \int_0^\infty \frac{dk}{\pi} e^{i k(\varphi - \varphi')} \int_0^\infty e^{-\nu^2 t} J_k(\nu r) J_k(\nu r') \nu d\nu
$$

(A2)

This may from a known integral representation of the function $I_k(r)$ [54] as

$$
K_t^\infty(\tilde{r}, \tilde{r}' \mid -\Delta_c) = \frac{1}{2\pi t} e^{-\frac{r^2 + r'^2}{4t}} \int_0^\infty e^{i k(\varphi - \varphi')} I_k \left( \frac{rr'}{2t} \right) dk
$$

(A3)

The kernel at finite $\gamma$ can now be obtained either using the periodicity sum over the frequencies $2\pi n / \gamma$ or using the eigenfunction expansion. The result is

$$
K_t^\gamma(\tilde{r}, \tilde{r}' \mid -\Delta_c) = \frac{1}{2\gamma t} e^{-\frac{r^2 + r'^2}{4t}} \sum_{n = -\infty}^{+\infty} e^{i \frac{2\pi n}{\gamma} (\varphi - \varphi')} I_{\frac{r + r'}{2t}} \left( \frac{rr'}{2t} \right)
$$

(A4)

The latter expression for the heat kernel of the Laplace operator on the cone, which we obtain as the natural expansion in terms of eigenfunctions, can be written in more convenient forms by using known integral representations for $I_n$ [54]. In particular here we use two representations. The first one clearly isolate the effects related to the (possible) irrationality of $\gamma$, which causes our cone to be not a good quotient space (indeed, from this point of view Scott [55] describes the irrational-$\gamma$ cone as an
extremely nasty space), while the second is very convenient from the computational point of view.

The representations read

\[
I_v(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{z \cos \theta} \cos \nu \theta d\theta - \frac{\sin \nu \pi}{\pi} \int_{0}^{\infty} e^{-z \cosh \nu \pi} dx
\]  
(A5)

valid for \( \Re z > 0 \) and \( \Re \nu \geq 0 \), and

\[
I_v(z) = \frac{e^{-\nu \pi / 2}}{2\pi} \int_{\Gamma} e^{i\nu s} \sin s ds
\]  
(A6)

valid for \( \Re z > 0 \) and \(-\pi < \arg z \leq \pi / 2\). In the latter expression, \( \Gamma \) is an arbitrary open contour integral in the complex plane from \( \varepsilon - \pi + i \infty \) to \( \varepsilon + \pi + i \infty \).

If we plug Eq. (A5) into Eq. (A4) then we get the kernel as a sum of two different terms which arise from the two integrals in Eq. (A5). Then using Poisson summation formula we finally obtain

\[
K_{\gamma}^{\gamma}(\tilde{r}, \tilde{r}'; - \Delta_\gamma) = e^{-\|\tilde{r} - \tilde{r}'\|^2 / 4t} \frac{1}{4\pi t} \left\{ \sum_{\frac{\pi - (\varphi - \varphi')}{\gamma} \leq n \leq \frac{\pi - (\varphi - \varphi')}{\gamma}} \exp \left[ -\frac{rr'}{t} \sin \frac{n \gamma + \varphi - \varphi'}{2} \right] \right. 
\]

\[ - \sin \frac{2\pi^2}{\gamma} \int_{-\infty}^{\infty} e^{-\frac{rr'}{t} \cosh^2 \frac{\pi - \gamma}{2}} \cosh(2\pi u) \cos \frac{\pi u}{\gamma} du \} \]  
(A7)

where the maximum value of the \( n \) summation index is equal to \( \frac{\pi - (\varphi - \varphi')}{\gamma} \) when this quantity is an integer.

From Eq. (A6), after some calculations, we get the known integral representation in the complex plane [56]

\[
K_{\gamma}^{\gamma}(\tilde{r}, \tilde{r}'; - \Delta_\gamma) = e^{-\|\tilde{r} - \tilde{r}'\|^2 / 4t} \frac{1}{4\pi t} \left[ 1 + \frac{1}{\gamma} \int_{\Gamma'} e^{-\frac{rr' / t \cosh^2 (z / 2)}{2}} dz \right]
\]  
(A8)

Here \( \Gamma' \) is an arbitrary contour integral composed of two branches, the first one from \( \pi + i \infty \) to \( \pi - i \infty \), intersecting the positive real axis very close to the origin and the second being the specular copy of the previous one. For later convenience, in Eq. (A8) we have emphasized the contribution due to the conical singularity.

By integrating the latter representation on the cone, we easily obtain the trace of the kernel in the form

\[
\text{Tr} e^{-t \Delta_\gamma} = \frac{V(C_\gamma)}{4\pi t} + \frac{1}{12} \left( \frac{2\pi}{\gamma} - \frac{\gamma}{2\pi} \right)
\]  
(A9)

where \( V(C_\gamma) \) is the (infinite) volume of the cone. Of course, only Weyl [57] and Kac [58] contributions are present.
If we consider the more general case of a massive theory on the manifold $M = \mathbb{R}^{N-2} \times C$, then the previous expression has to be multiplied by the factor $(4\pi t)^{-(N-2)/2} V_{N-2} \exp(-tm^2)$, the exact heat-kernel of $-\Delta + m^2$ in $\mathbb{R}^{N-2}$. So the contribution to the $\zeta$-function due to the singularity, that means after subtraction of the Minkowskispace part, can be computed by a Mellin transform, the result being

$$\zeta_{\text{sing}}(s|L_N) = \frac{\pi}{3} \left(\frac{2\pi}{\gamma} - \frac{\gamma}{2\pi}\right) V_{N-2} \Gamma(s - \frac{N-2}{2}) \frac{m^{2(s-\frac{N-2}{2})}}{(4\pi)^{N/2} \Gamma(s)}$$  \hspace{1cm} (A10)

### APPENDIX B: SINGULAR HEAT KERNEL EXPANSION

Here we derive the asymptotic expansion for the heat kernel derived in the previous Appendix A. Of course we expect such an expansion to be a distribution concentrated on the tip of the cone (apart from the first term which corresponds to the kernel on $\mathbb{R}^2$). For the kernel we use the second integral representation, Eq. (A8).

In order to obtain the parametrix for the trace, we expand $\exp(-\sigma r^2)$ in terms of distributions on the cone. We consider test functions $\phi(r, \varphi)$ with $\gamma$ periodicity with respect to $\varphi$, and define [59]

$$\tilde{\phi}(r) = \frac{1}{\gamma} \int_0^\gamma \phi(r, \varphi) d\varphi$$  \hspace{1cm} (B1)

The Taylor expansion for $\tilde{\phi}(r)$ is given by the Pizzetti formula [59]

$$\tilde{\phi}(r) = \sum_k \frac{\Delta_k \phi(0, 0)}{(2^k k!)^2}$$  \hspace{1cm} (B2)

Using the latter relation we easily get

$$\frac{\alpha \sigma}{\pi} e^{-\sigma r^2} = \sum_k \frac{\Delta_k \delta(\vec{r})}{k!(2\sigma)^k} = \exp\left(\frac{\Delta \sigma}{2\sigma}\right) \delta(\vec{r})$$  \hspace{1cm} (B3)

where $\alpha = 2\pi/\gamma$ has been set. If we use this expression in Eq. (A8) for $\vec{r} = \vec{r}'$ we can make the complex integral by the residue method. The final result for the asymptotic behaviour for $t \to 0$ of the heat-kernel reads

$$K_t^\gamma(\vec{r}, \vec{r} - \Delta_c) = \frac{1}{4\pi t} \left[1 + \sum_{k=0}^{\infty} \frac{R_k \Delta_k \delta(\vec{r})}{2^k k!} \right] t^{k+1}$$  \hspace{1cm} (B4)

where $R_k$ is the residue in zero of the function $-i\pi [\sin(\varphi/2)]^{-2(k+1)} (1 - e^{-iz})^{-1}$. In particular we have

$$R_0 = \frac{\pi (\alpha^2 - 1)}{3\alpha}; \quad R_1 = \frac{\alpha^2 + 11}{15} R_0; \quad R_2 = \frac{2\alpha^4 + 23\alpha^2 + 191}{315} R_0$$  \hspace{1cm} (B5)
It is seen that by integrating (B4) in order to obtain the integrated heat-kernel coefficient, only the term \( k = 0 \) survives. Thus in the calculated physical quantities only \( R_0 \) is present.

Now we want to extend the expansion in Eq. (B4) to a more general operator \( L_4 = -\Delta + m^2 + X(x) \), on \( \mathcal{M} = \mathbb{R}^2 \times \mathcal{C} \). To this aim we make the ansatz

\[
K^\gamma_i(x, x'|L_4) \sim K^\gamma_i(x, x'|-\Delta) e^{-tM^2} \sum_n b_n(x, x') t^n
\]

where \( M^2 = m^2 + X \) is a positive scalar function and \( K^\gamma_i(x, x'|-\Delta) \) is the kernel of the Laplacian on \( \mathcal{M} \), that is the product of the two kernels \( K_{\mathbb{R}^2} K_{\mathcal{C}} \). It has to be noted that the diagonal part \( K^\gamma_i(x, x|-\Delta) \) of this kernel differs from the corresponding one on \( \mathbb{R}^4 \) for the addition of an exponentially vanishing function as \( t \to 0 \). This is also true for the derivatives of any orders with respect to \( x \). This means that the coefficients \( b_n \), in the coincidence limit \( x' \to x \), are not modified in the presence of the singularity (the whole contribution of that being in \( K^\gamma_i(\tilde{r}, \tilde{r}|-\Delta_c) \)). Then we have

\[
\begin{align*}
b_0(x, x) &= 1; \\
b_1(x, x) &= 0; \\
b_2(x, x) &= -\frac{\Delta X}{6}; \\
b_3(x, x) &= \frac{(\nabla X)^2}{12} - \frac{\Delta^2 X}{60}
\end{align*}
\]

which finishes the summary of our results on the heat-kernel expansion on the cone.
REFERENCES

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[45] Balasin H and Nachbagauer H. On the distributional nature of the energy momentum
tensor of a black hole or what curves the schwarzschild geometry?