A PROBLEM WITH NON-ABELIAN DUALITY?

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ABSTRACT

We investigate duality transformations in a class of backgrounds with non-Abelian isometries, i.e. Bianchi-type (homogeneous) cosmologies in arbitrary dimensions. Simple duality transformations for the metric and the antisymmetric tensor field, generalizing those known from the Abelian isometry (Bianchi I) case, are obtained using either a Lagrangian or a Hamiltonian approach. Applying these prescriptions to a specific conformally invariant $\sigma$-model, we show that no dilaton transformation leads to a new conformal background. Some possible ways out of the problem are suggested.
1 Introduction

Duality transformations on conformal string backgrounds have recently attracted considerable attention. In a restricted sense duality transformations connect two (or more) apparently different, but actually equivalent, string theories. In the generalized sense used in this paper, duality transformations are simply meant to connect a given conformal background to other, generally inequivalent, conformal backgrounds. Examples of the latter type of dualities are the $O(d, d; R)$ Narain [1] transformations connecting all possible toroidal compactifications in $d$ dimensions. By contrast, just an $O(d, d; Z)$ subgroup relates physically equivalent theories [2].

An interesting feature of restricted duality transformations in the case of toroidal compactifications is the necessity to accompany the change in the metric and torsion fields by a suitable change of the dilaton. Only then can strict duality hold to all orders in the string-loop expansion [3].

In the case of homogeneous, Bianchi I cosmological backgrounds, a generalization of Narain’s group can be defined [4, 5, 6, 7]. Interestingly enough, it relates cosmologies of the standard kind (FRW Universe undergoing a decelerating expansion) to inflationary cosmologies offering some hope that, in string theory, inflation can be incorporated naturally in a pre-Big-Bang phase [8]. In this case, even the weaker form of duality requires, for the maintenance of conformal invariance, a non-trivial dilaton transformation. In fact, it is the very presence of a time-dependent dilaton that makes it possible for inflationary solutions to exist.

The presence of an $O(d, d; R)$ group is not at all confined to cosmological backgrounds. It was indeed shown [7] that it is a general property of backgrounds possessing $d$ Abelian isometries. In the case of Abelian isometries it is also possible to understand [5, 9] why and how the dilaton is to be transformed in order that the quantum equivalence of duality-related theories holds or, more generally, that the original conformal invariance is not lost.

Generalizing the above construction to backgrounds with non-Abelian isometries is an obvious mathematical challenge [10]. At the same time, such a problem is of great physical interest in a cosmological context since some of the most interesting cosmological models have non-Abelian rather than Abelian isometries. One of the fundamental
problems cured by inflation, the flatness problem, cannot be even addressed without considering cosmologies with non-Abelian isometries (Bianchi IX and V).

Some time ago, in a very stimulating paper [11], de la Ossa and Quevedo (DOQ) proposed a possible way to implement duality transformations for backgrounds with non-Abelian isometries (“non-Abelian duality”, for short). Besides giving a method for computing the transformation of the metric and antisymmetric tensors, these authors also gave a recipe for determining the dilaton transformation, checking its validity through $\beta$-function calculations in some cases.

In this paper we shall apply the DOQ method to the case of general Bianchi-type models (most general homogeneous cosmologies in arbitrary dimensions) double-checking their transformations of $G$ and $B$ by a completely different (Hamiltonian) approach. However, and to our surprise, when we add to these the dilaton transformation as given by DOQ, we fail to satisfy, in a specific example, the $\beta$-function constraints. What is worse, we find that no other dilaton transformation is capable of restoring the vanishing of all $\beta$-functions. Some possible interpretations of this surprising result are given at the end.

2 Strings in homogeneous cosmological backgrounds

The two-dimensional $\sigma$-model action describing string propagation in a generic background of its massless modes can be written (in orthonormal gauge) as:

$$S = \frac{1}{4\pi} \int d^2 z \{ \partial X^M[G_{MN}(X) + B_{MN}(X)] \partial X^N + \frac{\sqrt{|g|}}{2} R^{(2)}(X) \phi(X) \}, \quad M, N = 0, \ldots, d,$$

where $X^M \equiv (t, X^m) \ (m = 1, \ldots, d)$ are the string coordinates, $R^{(2)}$ is the scalar curvature of the 2-dimensional world-sheet and the fields $G_{MN} = G_{NM}, B_{MN} = -B_{NM}$ and $\phi$ are general functions of $X$. Having in mind possible cosmological applications, we shall restrict ourselves to the particular case of spatially homogeneous cosmological backgrounds, for which a synchronous frame can be defined in which

$$G_{MN}(X) = \begin{pmatrix} -1 & 0 \\ 0 & G_{mn}(t, \vec{X}) \end{pmatrix}, \quad B_{MN}(X) = \begin{pmatrix} 0 & 0 \\ 0 & B_{mn}(t, \vec{X}) \end{pmatrix}.$$
The requirement of spatial homogeneity implies that the $d$-dimensional spatial submanifold is invariant under the action of a $d$-parameter transitive Lie group of isometries $G_d$. The generators $T_\alpha$ of the corresponding Lie algebra are expressible in terms of a set of $d$-dimensional Killing vectors $\xi^m_\alpha$, which can be taken to depend only on $\vec{X}$:

$$T_\alpha = \xi^m_\alpha(\vec{X}) \partial_m, \quad \alpha = 1, \ldots, d. \quad (3)$$

Lie algebras corresponding to different groups are fully characterized by their structure constants $C^\gamma_{\alpha\beta}$, defined as usual by

$$[T_\alpha, T_\beta] = C^\gamma_{\alpha\beta} T_\gamma. \quad (4)$$

A complete classification of the allowed algebras exists for the phenomenologically interesting case of $d = 3$ (see, for instance, [12]). Accordingly, all four-dimensional spatially homogeneous spacetimes, also known as Bianchi models, fall into one of nine classes. Bianchi I, characterized by an Abelian isometry group isomorphic to the three-dimensional translation group (all $C^\gamma_{\alpha\beta} = 0$), coincides with the spatially flat anisotropic Friedmann-Robertson-Walker (FRW) Universe, while Bianchi V (respectively IX) contains, as a special case, the FRW open (respectively closed) isotropic Universe.

The existence of isometries allows one to factorize the background fields $G$ and $B$ in the form:

$$G_{mn}(t, \vec{X}) = e^\alpha_m(\vec{X}) \gamma_{\alpha\beta}(t) e^\beta_n(\vec{X}), \quad \gamma_{\alpha\beta} = \gamma_{\beta\alpha}, \quad (5)$$

$$B_{mn}(t, \vec{X}) = e^\alpha_m(\vec{X}) \beta_{\alpha\beta}(t) e^\beta_n(\vec{X}), \quad \beta_{\alpha\beta} = -\beta_{\beta\alpha}, \quad (6)$$

where all dependence on the spatial coordinates is contained in the “triads” $e^\alpha_m$. The specific form of the latter is fixed, up to space diffeomorphisms, by the particular isometry group involved. The linear differential operators

$$e^m_\alpha \partial_m, \quad e^m_\alpha e^n_\alpha \equiv \delta^m_n, \quad (7)$$

satisfy the same commutation relations as the generators $T_\alpha$. Thus, eq. (4) can be equivalently rewritten as [13]

$$e^m_\alpha e^n_\beta (\partial_n e^\gamma_m - \partial_m e^\gamma_n) = C^\gamma_{\alpha\beta}. \quad (8)$$
Combining (2), (5) and (6) into (1) we obtain

\[ S = \frac{1}{4\pi} \int d^2 z \left\{ -\partial_t \tilde{\phi} + \partial X^m \epsilon_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) \epsilon_n^\beta \tilde{\partial} X^n + \frac{\sqrt{g}}{2} R^{(2)} \phi \right\}, \]  

which will be our starting point for discussing non-Abelian duality.

3 Non-Abelian duality

Starting from (9), a “dual” σ-model action can be defined with respect to the full non-Abelian isometry group \( G_d \), in strict analogy with the Abelian case. We shall follow, successively, a Lagrangian and a Hamiltonian approach, showing that both yield identical results.

3.1 Lagrangian approach

In the Lagrangian approach [9, 14, 11] duality transformations for the background tensor fields are obtained from a chain of formal manipulations on the functional integral which defines the partition function of the initial theory:

\[ \mathcal{Z} = \int [dt][dX^m] e^{-S[t,X^m]}. \]  

As a first step, one gauges the global symmetry corresponding to \( G_d \) by introducing a set of pure-gauge potentials \( A^\gamma, \bar{A}^\gamma \), which are minimally coupled to the string coordinates:

\[ \partial X^m \rightarrow \partial X^m + A^\gamma \xi_m^\gamma (\bar{X}), \quad \bar{\partial} X^m \rightarrow \bar{\partial} X^m + \bar{A}^\gamma \xi_m^\gamma (\bar{X}). \]  

The new total action \( S' \) reads:

\[ S' = S + \frac{1}{4\pi} \int d^2 z \left\{ A^\gamma \xi_m^\gamma e_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\beta \tilde{\partial} X^n + \right. \]
\[ \bar{A}^\delta \partial X^m e_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\beta \xi_n^\delta + \]
\[ A^\gamma \bar{A}^\delta \xi_m^\gamma e_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\beta \xi_n^\delta + \tilde{X}_\gamma F^\gamma \right\}, \]  

where \( F^\gamma \) is the field-strength corresponding to \( A^\gamma \) and \( \bar{A}^\gamma \) and the Lagrange multipliers \( \tilde{X}_\gamma \) are used to enforce the constraint \( F^\gamma = 0 \). In terms of the action \( S' \) we have:

\[ \mathcal{Z} = \int [dt][dX^m][dA^\alpha][d\bar{A}^\beta][d\tilde{X}_\gamma] \frac{1}{V_{G_d}} e^{-S'[t,X^m,A^\alpha,\bar{A}^\beta,\tilde{X}_\gamma]}, \]  

4
where $V_{\mathcal{G},d}$ stands for the formal gauge group volume, it being understood that a Faddeev-Popov gauge-fixing procedure must be performed to render the path-integral well-defined. The original model is recovered from (13) by first integrating over the Lagrange multipliers and then fixing the potentials to zero. Alternatively, the fact that the action is quadratic in the gauge potentials allows one to obtain the dual theory by integrating first over $A$ and $\tilde{A}$ and by subsequently fixing the residual gauge symmetry in a suitable way. It is convenient to rewrite $S'$ in the compact form:

$$S' = S + \frac{1}{4\pi} \int d^2z \left( A^\gamma \tilde{u}_\gamma + \tilde{A}^\delta u_\delta + A^\gamma m_{\gamma\delta} \tilde{A}^\delta \right),$$

where

$$u_\delta = -\partial \tilde{X}_\delta + \partial X^m e_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\delta,$$

$$\tilde{u}_\gamma = \tilde{\partial}X_\gamma + \xi_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\delta,$$

$$m_{\gamma\delta} = C_{\gamma\delta}^\lambda X_\lambda + \xi_m^\alpha (\gamma_{\alpha\beta} + \beta_{\alpha\beta}) e_n^\delta.$$  

After (classical) integration over the gauge potentials one gets:

$$S'' = S - \frac{1}{4\pi} \int d^2z \left[ u_\gamma (m^{-1}) \gamma^\delta \tilde{u}_\delta \right].$$

A convenient gauge choice, whose viability is locally guaranteed by the transitiveness of the group $\mathcal{G}_d$, turns out to be $X^m \equiv C^m$, with $C^m$ a suitable (possibly group-dependent) constant vector. Under this choice (18) simplifies considerably, owing to the property $e_n^\delta e_n^\delta |_{X=C} = \delta^\delta$. This yields the following general form for the dual action:

$$\tilde{S} = \frac{1}{4\pi} \int d^2z \left\{ -\partial \partial \tilde{\partial} t + \partial \tilde{\partial} X_\gamma (\gamma + \beta + \kappa)^{-1} \gamma^\delta \tilde{\partial} X_\delta + \frac{\sqrt{g}}{2} R^{(2)} \phi \right\},$$

where $\kappa$ stands for the antisymmetric matrix defined by

$$\kappa_{\alpha\beta} \equiv C^\gamma_{\alpha\beta} \tilde{X}_\gamma.$$  

From eq. (19), the following prescription for the dual backgrounds $\tilde{G}$ and $\tilde{B}$ can be inferred:

$$\tilde{G} + \tilde{B} = (\gamma + \beta + \kappa)^{-1},$$

5
or, using the symmetry properties of $\gamma$, $\beta$ and $\kappa$,

$$
\begin{align*}
\tilde{G} &= (\gamma - \beta - \kappa)^{-1}\gamma(\gamma + \beta + \kappa)^{-1}, \\
\tilde{B} &= -(\gamma - \beta - \kappa)^{-1}(\beta + \kappa)(\gamma + \beta + \kappa)^{-1}.
\end{align*}
$$

(22)  
(23)

We note that the above transformations correctly reduce to the Abelian ones when $\kappa = 0$.

### 3.2 Hamiltonian approach

The transformation rules (22) and (23) for the background tensor fields can be alternatively inferred from a Hamiltonian framework. The total Hamiltonian density for a string in the backgrounds (2) can be written as:

$$
H_T = \frac{1}{2} X^0 \zeta_0 X^{00} + \frac{1}{2} P_0 G^{00} P_0 + H, \quad H = \frac{1}{2} Z^I M_I J Z^J,
$$

(24)

where

$$
Z^I \equiv (P_i, X^i), \quad I = 1, \ldots, 2d, \quad i = 1, \ldots, d
$$

(25)

are 2d-dimensional phase-space coordinates and $M$ is the $2d \times 2d$ matrix [2]

$$
M = \begin{pmatrix}
G^{-1} & -G^{-1} B \\
BG^{-1} & G - BG^{-1} B
\end{pmatrix}.
$$

(26)

We shall try to follow the strategy of [2] in the case of $X^i$:dependent, spatially homogeneous backgrounds (5), (6). In this case $M$ reads:

$$
M_{IJ} = \begin{pmatrix}
e^{i}_\gamma \alpha \beta \xi^j & e^{i}_{\gamma} \alpha \lambda \beta \xi^j \\
e^{i}_{\alpha} \beta \lambda \gamma \beta \xi^j & e^{i}_{\alpha} (\gamma_{\alpha \beta} - \beta_{\alpha \lambda} \gamma_{\beta \mu} \beta_{\mu \beta}) \xi^j
\end{pmatrix}.
$$

(27)

We now perform two successive classical canonical transformations. The first one is induced by the generating functional:

$$
F = - \int d\sigma \, d\tau \, (X^m e^\alpha_m \delta_{\alpha \beta} \xi^\beta_n X_n),
$$

(28)

where $e^\alpha_m$ is the same set of functions of the new coordinates $\tilde{X}^i$ as $e^\alpha_m$ is of $X^i$.

Introducing the new variables $E^\alpha_m \equiv \partial_m (e^\alpha_n X^n)$ and $\tilde{E}^\alpha_m \equiv \tilde{\partial}_m (\tilde{e}^\alpha_n \tilde{X}^n)$ we can write, after use of (8) and some algebra,

$$
\begin{align*}
X^m e^\alpha_n &= \tilde{P}_m \tilde{E}^\alpha_m, \\
P_n e^\alpha_m &= \tilde{E}_m \tilde{X}^m - \tilde{G}^\alpha_\gamma \tilde{P}_m \tilde{E}^{\alpha \beta} \tilde{e}^\beta_\gamma \tilde{X}^\gamma.
\end{align*}
$$

(29)  
(30)
The second canonical transformation is just a general coordinate transformation:

\[ \tilde{X}_\alpha = \hat{\epsilon}_{\alpha \gamma} \tilde{X}^\gamma, \quad \tilde{P}_\alpha = \hat{P}_{\alpha \gamma} \tilde{E}^\gamma. \]  

(31)

Combining the two transformations, we can luckily re-express the quantities appearing in the Hamiltonian entirely in terms of the final phase-space coordinates \( \tilde{X}' \) and \( \tilde{P} \):

\[ X^n_{\alpha} e_{\alpha \alpha} = \tilde{P}_\alpha \]  

(32)

\[ P^n_{\alpha} e_{\alpha \alpha} = \tilde{X}'_\alpha - C_{\alpha \beta} \hat{P}_\beta \tilde{X}_\gamma. \]  

(33)

Substituting eqs. (32) and (33) into the expression for \( H \), we can reinterpret the new Hamiltonian as one defining a new background matrix \( \tilde{M} \) given by:

\[ \tilde{M} = \begin{pmatrix} \gamma + \beta + \kappa & (\gamma - \beta - \kappa) & (\beta + \kappa) \gamma^{-1} \\ \gamma^{-1} (\beta + \kappa) & \gamma^{-1} \end{pmatrix}. \]  

(34)

After use of eq. (26), one finds that (34) defines the same transformations on \( \tilde{G} \) and \( \tilde{B} \) as the one obtained in the Lagrangian approach (eqs. (22) and (23)).

We close this section by noticing that the dual backgrounds (22) and (23) do not share in general the same isometries as the original ones. This result is in agreement with the general conclusions of [11].

4 Transformation of the dilaton and a puzzle

As we discussed already in the introduction, the maintenance of conformal invariance requires, even in the Abelian case, a non-trivial transformation of the dilaton. The necessity of such a transformation is easily inferred from the computation of \( \beta \)-functions in the original and duality-related \( \sigma \)-models. In the Lagrangian approach, a more direct method for determining how the dilaton has to transform runs as follows [5, 9].

In the formal functional-integral manipulations used to go from one theory to its dual we have been cavalier about functional determinants, in particular about those coming from integration over the gauge potentials. Indeed, since the coefficient of the term quadratic in the potentials is not an elliptic differential operator, the corresponding functional integral is ill-defined and needs to be regulated.
The functional determinant can then be explicitly calculated using heat kernel techniques \cite{5, 9}. In particular, its Weyl anomaly part provides an additional term proportional to $R^{(2)}$ to the dual action. This is naturally interpreted as a shift of the original dilaton field.

In the case of Abelian isometries one finds:

$$\tilde{\phi} = \phi - \ln \det(G). \quad (35)$$

Eq. (35) can be also obtained in the context of the dimensionally-reduced string effective action \cite{4, 5, 6}. The same quantum contribution to the dilaton transformation is also expected to follow in the Hamiltonian approach, provided one correctly takes into account the problems of operator ordering when writing down the quantum version of canonical transformations. This has not yet been done explicitly, to our knowledge.

The following question naturally arises at this point: is there a transformation of the dilaton which guarantees conformal invariance in the non-Abelian case? And if yes, which is it?

It has been argued \cite{11} that the correct prescription is simply:

$$\tilde{\phi} = \left[ \phi + \frac{1}{2} \ln \frac{\det \tilde{G}}{\det G} - \ln \det \left( \frac{\delta \mathcal{F}}{\delta \omega} \right) \right]_{\mathcal{F} = 0}. \quad (36)$$

In eq. (36) $\mathcal{F}$ is the gauge fixing function appearing in the path-integral representation of the partition function and $\omega$ are the parameters of the isometry transformation. Eq. (36) was shown \cite{11} to correctly reinstate one-loop conformal invariance in the case of $\sigma$-models with maximal $SO(d)$ isometry symmetry. This is a somewhat special case, however, since no $B$ field is introduced by the duality transformation if it was initially zero.

Since no rigorous argument exists ensuring the general validity of eq. (36), we made an explicit check of that recipe in the case of a particular, Bianchi-type conformal background. Consider the $\sigma$-model defined by (9) with $d = 3$, $B_{mn} = \phi \equiv 0$, and

$$G_{mn}(t, x, y, z) = \text{diag}(a^2(t), a^2(t)e^{-2x}, a^2(t)e^{-2z}). \quad (37)$$

The metric (37) is of the form (5) with

$$e^e_m = \text{diag}(1, e^{-x}, e^{-z}), \quad \gamma_{\alpha\beta} = a^2(t)\delta_{\alpha\beta}. \quad (38)$$
Using (8) one obtains, for the non-vanishing structure constants:

\[ C_{12}^2 = -C_{21}^2 = C_{13}^3 = -C_{31}^3 = 1, \quad \text{zero otherwise.} \quad (39) \]

showing that the model is of the Bianchi V type.

It is easy to see that, by choosing \( a(t) \equiv t \), the above background fulfills the \( \beta \)-function equations trivially (after adding the right number of “spectator” dimensions). Indeed the whole Riemann tensor of the model vanishes identically, implying that \( G_{MN} \) is flat (it provides, indeed, an unconventional parametrization of the “Milne” [15] portion of Minkowski space-time). For an analogous abelian case see [16].

The new tensor backgrounds \( \tilde{G} \) and \( \tilde{B} \), when calculated by means of eqs. (22) and (23), read (hereafter we drop the tilde for the new spatial coordinates):

\[
\tilde{G} = \frac{1}{D} \begin{pmatrix} t^4 & 0 & 0 \\ 0 & t^4 + z^2 & -yz \\ 0 & -yz & t^4 + y^2 \end{pmatrix}, \quad \tilde{B} = \frac{1}{D} \begin{pmatrix} 0 & -t^2y & -t^2z \\ t^2y & 0 & 0 \\ t^2z & 0 & 0 \end{pmatrix},
\]

(40)

where \( D = t^2(t^4 + y^2 + z^2) \). The prescription (36), adapted to our case, gives:

\[
\tilde{\phi} = -\ln(\gamma + \kappa) = -\ln(D).
\]

(41)

By direct computation one can see that eqs. (40) and (41) do not satisfy the \( \beta \)-function equations, in particular the one for the \( B \) field (\( H_{MAB} = \partial_M B_{AB} \))

\[
\partial_M \left( e^{-\tilde{\phi}} \sqrt{\det \tilde{G}} \tilde{H}^{MAB} \right) = 0.
\]

(42)

One finds, indeed, for the transformed backgrounds,

\[
\tilde{H}^{201} = -\frac{4y}{t}, \quad \tilde{H}^{301} = -\frac{4z}{t}, \quad e^{-\tilde{\phi}} \sqrt{\det \tilde{G}} = t^3,
\]

(43)

so that the \( A = 0, B = 1 \) component of eq. (42) is clearly not satisfied.

In order to see whether conformal invariance can be recovered by just changing the definition of the transformed dilaton, we treated \( \tilde{\phi} \) as an unknown function in eq. (42). The general solution for \( \tilde{\phi} \) turns out to be:

\[
\tilde{\phi} = -\ln \frac{t^4 + y^2 + z^2}{yz} + f \left( \frac{y}{z} \right),
\]

(44)
with \( f \) an arbitrary function of \( y/z \). The dilaton \( \beta \)-function equation requires \( f \) to satisfy a Riccati-type differential equation. Unfortunately, independently of the choice of \( f \), eq. (44) does not fulfil the remaining \( \beta \)-function equations

\[
\ddot{H}^B_A + \nabla_A \nabla^B \phi - \frac{1}{4} \dot{H}_{AMN} \dot{H}^{BMN} = 0.
\] (45)

Consider for instance the \( A = 0, B = 0 \) component, which does not depend on \( f \). Using the expressions (40) and (44) for the background fields one finds

\[
\ddot{R}_0^0 = 2\left( \frac{t^2(z^2 + 2y^2) - 3t^8}{t^2(z^2 + y^2 + t^4)^2} \right),
\] (46)

\[
\ddot{\phi}^0 = -t^2 4\left( \frac{4t^4 - 12(y^2 + z^2)}{(z^2 + y^2 + t^4)^2} \right),
\] (47)

\[
-\frac{1}{4} \ddot{H}_{0AB} \ddot{H}^{0AB} = \frac{y^2 + z^2}{(z^2 + y^2 + t^4)^2},
\] (48)

which add up to \( 2/t^2 \neq 0 \). Thus no choice for the transformed dilaton appears to restore conformal invariance.

Before discussing the possible implications of this result, we have to dismiss the possibility that the violation of conformal invariance we found can be fixed by higher order terms in the \( \beta \)-functions. The following argument shows that this is impossible: let us change the time coordinate from cosmic time \( t \) to \( \tau = \sqrt{t} \). It is easy to check that the transformed backgrounds become homogeneous of degree \( -1 \) in the new coordinates. As a result, one can show that the contributions to the \( \beta \)-functions of each extra \( \sigma \)-model loop contain an extra power of \( X^{-1} \) (where \( X \) stands for any one of the new coordinates \( y, z \) or \( \tau \)). Since the violation of conformal invariance that we found is of \( O(X^{-1}) \), it cannot be cancelled by higher order terms.

5 Discussion

Barring some trivial computational mistake, what could be the meaning of our counterexample? Obviously, we have no definite answer to this question. All we can offer at the moment are some conjectures which we list now in order of decreasing (subjective) appeal.
1. In the non-Abelian case, the functional determinant encountered in going from one theory to its dual appears to be more complicated than in the Abelian case. As already observed by Schwarz and Tseytlin [9], counterterms of types other than the dilaton’s can be induced a priori. In the Abelian case these happen to vanish. Unfortunately, even in our case, they do not seem to be of much help since, as shown by a simple dimensional counting, they would be of higher order in $\alpha'$. By contrast the non-Abelian correction to the duality transformations, in spite of involving derivatives of the backgrounds through $C^\varphi_{\beta\gamma}$, are easily shown not to contain any extra factors of $\alpha'$.

2. Entirely new counterterms could be generated, i.e. operators that do not correspond to any of the massless backgrounds. Possible examples could be a tachyonic background or an excited massive background. If this is the case, then presumably more and more massive fields will be brought in by the duality transformation as one goes to higher and higher orders in $\alpha'$.

3. The determinant could generate non-local counterterms, in which case the dual theory would have no standard “string-in-a-background” interpretation but only a conformal field theory formulation.

4. Finally, there could simply be no conformal field theory which is dual to one with a particularly complicated non-abelian isometry.

In conclusion, and independently of what the solution of our puzzle will eventually be, we do feel that a full understanding of non-Abelian duality represents a very interesting physical and mathematical challenge.

We are grateful to J. Maharana for an independent check of some of our calculations and for discussions. We also acknowledge useful conversations with L. Alvarez Gaumé, X. de la Ossa, A. Giveon, E. Kiritsis, F. Quevedo, E. Rabinovici, A. Sagnotti and A. A. Tseytlin. One of us (R. R.) wishes to thank the Dipartimento di Fisica, Università di Roma “Tor Vergata” for partial financial support.
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