Coherent States of $SU(l, 1)$ groups

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Abstract

This work can be considered as a continuation of our previous one (J.Phys., 26 (1993) 313), in which an explicit form of coherent states ($CS$) for all $SU(N)$ groups was constructed by means of representations on polynomials. Here we extend that approach to any $SU(l, 1)$ group and construct explicitly corresponding $CS$. The $CS$ are parametrized by dots of a coset space, which is, in that particular case, the open complex ball $CD^l$. This space together with the projective space $CP^l$, which parametrizes $CS$ of the $SU(l + 1)$ group, exhausts all complex spaces of constant curvature. Thus, both sets of $CS$ provide a possibility for an explicit analysis of the quantization problem on all the spaces of constant curvature. That is a reason
why $CS$ of the $SU(N)$ and $SU(l, 1)$ groups are of importance in connection with the quantization theory. The $CS$ constructed form an overcompleted system in the representation space and, as quantum states, possess of a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of symbols of operators; the limit of the so called star commutator of the symbols generates the Poisson bracket in $CD^l$, the latter plays the role of the phase space for the corresponding classical mechanics.

1 Introduction

For a long time coherent states ($CS$) are widely being utilized in quantum physics [[1, 2, 3, 4, 5]]. On account of the fact that they are parametrized by points of the phase space of a corresponding classical mechanics, they present themselves as a natural and convenient tool for establishing of a correspondence between the classical and quantum description. The $CS$ introduced by Schrödinger and Glauber were mainly used in this context. From mathematical point of view $CS$ form a continuous basis in Hilbert space (general description of Hilbert spaces with basis vectors labelled by discrete, continuous, or a mixture of two types of indices is given in [6]). As it is well known, it is possible to connect quantum mechanical $CS$ with orbits of Lie groups [7]. In particular, ”ordinary” $CS$ of Schrödinger and Glauber turned out to be orbits of the Heisenberg-Weyl group. A connection between $CS$ and a quantization of classical systems, in particular, systems with a curved phase space, was also established [8]. From that point of view the case of the flat phase space corresponds to the Heisenberg-Weyl group and to the Schrödinger-Glauber $CS$. Kahlerian symplectic manifolds of constant holomorphic curvature can serve as the simplest example of a curved phase space. Such spaces are, for positive curvature, the projective spaces $CP^l$, and, for negative curvature, the open complex balls $CD^l$ [9]. The groups $SU(N)$, $N = l + 1$ and $SU(l, 1)$ are groups of movements for the spaces $CP^l$ and $CD^l$ correspondingly, and the latter are the coset spaces $SU(N)/U(l)$ and $SU(l, 1)/U(l)$. The quantization on the former is connected with a construction of $CS$ of the groups $SU(N)$, and on the latter with the one of the groups $SU(l, 1)$. The circumstances mentioned, besides all others
arguments, stress the importance of the investigation of $CS$ for that groups as a first and necessary step in a systematic construction of quantization theory for systems with curved phase spaces. One ought to say the investigation of $CS$ of these groups has another motivation as well. As for the group $SU(N)$, their importance for the physics is well known and does not need to be explained here. As to the $SU(l, 1)$ ones, they arise often in quantum mechanics as groups of the dynamical symmetry. For example, the group of the dynamical symmetry of a particle in the magnetic field is $SU(2, 1)$ [2], the same is the group of dynamical symmetry of Einstein-Maxwell equations for axial-symmetric field configurations [10] and so on.

An explicit form of the $CS$ for any $SU(N)$ group was constructed and investigated in our work [11], using representations of the groups in the space of polynomials of a fixed power. One can also find there references devoted to the $CS$ of the $SU(2)$ group and related questions. In the present work we are going to extend that approach to construct the $CS$ for all $SU(l, 1)$ groups. One ought to say that $CS$ of $SU(1, 1)$ group from that family were first constructed in [[7, 12]] on the base of the well investigated structure of the $SU(1, 1)$ matrices in the fundamental representation. A quantization on the Lobachevsky plane, which is the coset space $SU(1, 1)/U(1)$, was considered by Berezin [[8, 13]], using these $CS$. It is difficult to use the method of the works [[7, 12]] or commutation relations for generators only to construct explicitly $CS$ for any group $SU(l, 1)$, since technical complications are growing with the number $l$. Nevertheless, a generalization of the method, used by us in [11], allows one to obtain the result, despite of the fact that $SU(l, 1)$ groups are noncompact and their unitary representations are infinite-dimensional (see Appendix).

We construct $CS$ of the $SU(l, 1)$ groups as orbits of highest or lowest weights factorized with respect to stationary subgroups, using representations in spaces of quasi-polynomials of a fixed integer negative power $P$. The $CS$ are parametrized by points of a coset space, which is, in that particular case, the open complex ball $CD^l$. As was already said before, this space together with the projective space $CP^l$, which parametrizes $CS$ of the $SU(N), N = l + 1$, group, exhaust all complex spaces of constant curvature. The $CS$ constructed form an overcompleted system in the representation space and, as quantum states, possess a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of symbols of operators. The role of the Planck
constant plays $h = |P|^{-1}$, where $P$ is the signature of the representation. The limit of the so called star commutator of operators symbols generates the Poisson’s bracket in $CD^I$, the latter plays the role of the phase space for the corresponding classical mechanics.

In Appendix we add some necessary information about representations of the noncompact groups we are working with.

## 2 Construction of $CS$ of $SU(l, 1)$ groups

Following to the general definition [[4, 7]] and the way we used in the case of $SU(N)$, we are going to construct $CS$ of the $SU(l, 1)$ groups as orbits in some irreducible representations (IR) of the groups, factorized with respect to stationary subgroups. First, we describe the corresponding representations.

Let $g$ be matrices $N \times N$, $N = l + 1$ of a fundamental representation of the group $SU(l, 1)$, $g \in SU(l, 1)$. They obey the relations

$$
\Lambda g^\dagger \Lambda = g^{-1}, \quad \det g = 1, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -I_l \end{pmatrix}, \quad \Lambda = \Lambda^* = \Lambda^{-1},
$$

where $I_l$ is the $l \times l$ unit matrix.

Define by $C^N$ the $N$-dimensional space of complex row vectors $z = (z_\mu)$, $\mu = (0, i), \ i = 1, \ldots, l$, with the scalar product $(z, z')_C = z_\mu \Lambda^{\mu \nu} z'_\nu$, and by $\hat{C}^N$ the dual space of complex columns $\hat{z} = (\hat{z}^\mu)$, with the scalar product $(\hat{z}, \hat{z}')_{\hat{C}} = \hat{z}^\mu \Lambda_{\mu \nu}^{-1} \hat{z}'^\nu$. The anti-isomorphism of the spaces $C^N$ and $\hat{C}^N$ is given by the relation

$$
z \leftrightarrow \hat{z} \leftrightarrow z_\mu = \Lambda_{\mu \nu} \hat{z}^\nu, \quad \tag{1}
$$

on account of eq. $(\hat{z}, \hat{z}')_{\hat{C}} = (z, z')_C$. It is convenient to define the mixed Dirac scalar product between elements of $C^N$ and $\hat{C}^N$ as

$$
< z', \hat{z} > = (z', \hat{z})_{\hat{C}} = (\hat{z}', z)_C = z'_{\mu} \hat{z}^\mu. \quad \tag{2}
$$

The group acts by its fundamental representations in the spaces $C^N$ and $\hat{C}^N$,

$$
z_g = zg, \quad \hat{z}_g = g^{-1} \hat{z}. \quad \tag{3}
$$
The form $< z', \tilde{z} >$ is invariant under the group action, $< z'_g, \tilde{z}_g >=< z', \tilde{z} >$. That means that whole domain of $z_\mu$ can be divided in three invariant subdomains, where $< z, \tilde{z} >$ is positive, negative or zero. We restrict ourselves to the subdomain where $< z, \tilde{z} >$ is positive, choosing the normalization condition

$$< z, \tilde{z} > = |z_0|^2 - \sum_{i=1}^l |z_i|^2 = 1 ,$$

what is sufficient for our purpose to construct $CS$ connected with the quantization on the coset space $CD^l$.

Consider spaces $\Pi_P$ and $\tilde{\Pi}_P$ of quasi-polynomials $\Psi_P(z)$ and $\Psi_P(\tilde{z})$ in $z$ and $\tilde{z}$,

$$\Psi_P(z) = \sum_{\{n\}} K_{\{n\}} \prod_{\mu} (z_\mu)^{n_\mu}, \quad \Psi_P(z) \in \Pi_P ,$$

$$\Psi_P(\tilde{z}) = \sum_{\{n\}} K_{\{n\}} \prod_{\mu} (\tilde{z}_\mu)^{n_\mu}, \quad \Psi_P(\tilde{z}) \in \tilde{\Pi}_P ,$$

$$\{n\} = \{n_0, n_1, \ldots, n_i | \sum_{\mu} n_\mu = P \} ,$$

where $P$ are integer and negative, $P < -l$; all $n_\mu$ are also integer and $n_0 \leq P$, $n_i \geq 0$, $i = 1, \ldots, l$.

The fundamental irreducible IR of the group induce unitary IR in the spaces $\Pi_P$ and $\tilde{\Pi}_P$,

$$T(g) \Psi_P(z) = \Psi_P(z_g), \quad z_g = z g, \quad \Psi_P(z) \in \Pi_P ,$$

$$\tilde{T}(g) \Psi_P(\tilde{z}) = \Psi_P(\tilde{z}_g), \quad \tilde{z}_g = g^{-1} \tilde{z}, \quad \Psi_P(\tilde{z}) \in \tilde{\Pi}_P .$$

We will further call $P$ the signature of the IR. Such representations and their place among other ones of $SU(l, 1)$ groups are described in the Appendix to this paper.

Define a scalar product of two polynomials from $\Pi_P$,

$$< \Psi_P | \Psi'_P > = \int \overline{\Psi_P(z)} \Psi'_P(z) d\mu_P(\tilde{z}, z) ,$$

5
\[
d\mu_P(\bar{z}, z) = \frac{(-P - 1)!}{(2\pi)^{i+1} (-P - l - 3)!} \delta(\bar{z}_0^2 - \sum_{i=1}^{l} |\bar{z}_i|^2 - 1) \prod_{\nu=0}^{l} d\bar{z}_\nu dz_\nu, \\
d\bar{z} dz = d(|z|^2) d(\arg z),
\]

which can also be interpreted as a mixed Dirac scalar product between elements $|\Psi_P\rangle = \Psi_P(z)$ from $\Pi_P$ and $\langle \Psi_P | = \Psi_P(z)$ from $\hat{\Pi}_P$, because of the anti-isomorphism (1).

Note that the restriction to $P$ integer is a way to avoid representations in spaces of multivalued functions; the additional restriction $P < -l$ ensures the existence of the scalar product (7).

The monomials

\[
\Psi_{P_{\{n\}}}(z) = \sqrt{(-1)^{P-n_0} \Gamma^{-n_0} z_0^{n_1} \cdots z_l^{n_l} }, \tag{8}
\]

\[
\{n\} = \{n_0, n_1, \ldots, n_l \mid \sum_{\mu} n_\mu = P \},
\]

form a discrete basis in $\Pi_P$, whereas the monomials $\Psi_{P_{\{\bar{n}\}}}(\bar{z}) = \Psi_{P_{\{n\}}}(z)$ form a basis in $\hat{\Pi}_P$.

Using the integral

\[
\int_1^\infty d\rho_0 \int_0^\infty d\rho_1 \cdots \int_0^\infty d\rho_l \delta(\rho_0 - \sum_{i=1}^l \rho_i - 1) \prod_{\nu=0}^l \rho_\nu^\omega \\
= \prod_{k=1}^l n_k! (-\sum_{\mu=0}^l n_\mu - l - 3)! \\
\frac{(-n_0 - 1)!}{(-n_0 - 1)!},
\]

it is easy to verify that orthonormality and completeness relations hold,

\[
\langle \Psi_{P_{\{n\}}} | \Psi_{P_{\{n'\}}} \rangle = \langle P, n | P, n' \rangle = \delta_{\{n\}, \{n'\}}, \tag{9}
\]

\[
\sum_{\{n\}} |P, n \rangle \langle P, n| = I_P.
\]

where $I_P$ is the identity operator in the space of representation of signature $P$. The monomials (8) obey the remarkable relation
\[
\sum_{\{n\}} \Psi_{P;\{n\}}(z') \Psi_{P;\{n\}}(z) = \sum_{\{n\}} \Psi_{P;\{n\}}(z') \Psi_{P;\{n\}}(\tilde{z}) = \langle z', \tilde{z} \rangle^P,
\]

which is group invariant on account of the invariance of the scalar product (2) under the group transformation, \( \langle z'_g, \tilde{z}_g \rangle = \langle z', \tilde{z} \rangle \). The validity of (10) can be checked by means of the formula

\[
(a \pm b)^{-m} = \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!n!} a^{-m+n}(\mp b)^n, \quad m > 0, \ |b| < |a|,
\]
together with the binomial formula.

The generators \( A_\mu^\nu \) of the groups \( U(l,1) = SU(l,1) \otimes U(1) \) obey the relations (see Appendix)

\[
(A_\mu^\nu)^+ = (-1)^{\delta_\mu_0 + \delta_\nu_0} A_\mu^\nu,
\]

where the hermitian conjugation is defined with respect to the scalar product (7). Their explicit form in the space \( \Pi_P \) is \( A_\mu^\nu = z_\mu \frac{\partial}{\partial z_\nu} \), and in the space \( \tilde{\Pi}_P \) is \( A_\mu^\nu = \frac{\partial}{\partial \tilde{z}_\mu} \tilde{z}_\nu \) (action on the left).

Independent generators \( \hat{\Gamma}_a, \ a = 1, N^2 - 1, \) of \( SU(l,1) \) can be written through \( A_\mu^\nu \),

\[
\hat{\Gamma}_a = (\Gamma_a)_\mu^\nu A_\mu^\nu, \quad [\hat{\Gamma}_a, \hat{\Gamma}_b] = i f_{abc} \hat{\Gamma}_c,
\]

where \( \Gamma_a \) are generators in a fundamental representation, \( [\Gamma_a, \Gamma_b] = i f_{abc} \Gamma_c \). However, in contrast with the case of \( SU(N) \) group, where \( \Gamma_a^+ = \Gamma_a \), the \( \Gamma_a \) can be either hermitian or anti-hermitian in case of \( SU(l,1) \) group. Namely, \( N \) matrices \( \Gamma_a \), with zero diagonal elements and \( (\Gamma_a)_\mu^\rho = -(\Gamma_a)_\rho^\mu \), differ from the corresponding matrices \( SU(N) \) by a factor \( i \) only. To be sure, we take those to be the first \( \Gamma_a, \ a = 1, \ldots, N \). In particular, for \( SU(2) \) and \( SU(1,1) \) we have

\[
SU(2) : \quad \Gamma_k = \sigma_k, \ k = 1, 2, 3,
\]

\[
SU(1,1) : \quad \Gamma_\lambda = i \sigma_\lambda, \ \lambda = 1, 2, \Gamma_3 = \sigma_3,
\]

where \( \sigma_k \) are the Pauli matrices.
It is easy to verify that the condition (11) and the above convention provide the hermicity of the generators $\hat{\Gamma}_a$.

The quadratic Casimir operator

$$C_2 = \sum_{a=1}^{N^2-1} \epsilon_a \hat{\Gamma}_a^2, \quad \epsilon_a = \begin{cases} -1, & a = 1, \ldots, N \\ +1, & a = N + 1, \ldots, N^2 - 1 \end{cases},$$

(14)
can be written through the $A_\mu^\nu$ and evaluated explicitly,

$$C_2 = \frac{1}{2} \hat{A}_\mu^\nu \hat{A}_\nu^\mu = \frac{P(N + P)(N - 1)}{2N}, \quad \hat{A}_\mu^\nu = A_\mu^\nu - \frac{\delta_\mu^\nu}{N} \sum_\lambda A_\lambda^\lambda,$$

(15)

if one uses the formula

$$\sum_{a=1}^{N^2-1} \epsilon_a (\Gamma_a)_\mu^\nu (\Gamma_a)_\nu^\lambda = \frac{1}{2} \delta_\lambda^\lambda \delta_\mu^\mu - \frac{1}{2N} \delta_\mu^\nu \delta_\nu^\lambda,$$

which is a generalization to the case of SU(l, 1) group of the well known formula for matrices of SU(N) group.

Let us construct orbits of a lowest ($D^+(P0)$) or a highest ($D^-(0P)$) weights (of vectors of the basis (8) with the minimal length $\sqrt{\sum_{\mu=0}^{l} n_\mu^2} = |P|$, namely $n_0 = P$, $n_i = 0$. For $D^+(P0)$ the lowest weight is the state $\Psi_{P,\{P0\ldots0\}}(z) = (z_0)^P$. Then we get, in accordance with (6),

$$T(g)\Psi_{P,\{P0\ldots0\}}(z) = [z_\mu g_0^\mu]^P = <z, \hat{u}^P, \hat{u}^\mu = g_0^\mu,$$

(16)

where the vector $\hat{u} \in \tilde{C}^N$ is the zero column of the SU(l, 1) matrix in the fundamental representation.

One can notice, that the transformation arg $\hat{u}^\mu \rightarrow$ arg $\hat{u}^\mu + \lambda$ changes all the states (16) by the constant phase exp(iP$\lambda$). To select only physical different quantum states (CS) from all the states of the orbit, one has to impose a gauge condition on $\hat{u}$, which fixes the total phase of the orbit (16). Such a condition may be chosen in the form $\sum_\mu \text{arg} \hat{u}^\mu = 0$. Taken into account that the quantities $\hat{u}$ obey the condition $|\hat{u}^0|^2 - \sum_{i=1}^l |\hat{u}^i|^2 = 1$, by definition, as elements of the first column of the SU(l, 1) matrix, we get the explicit form of the CS of the SU(l, 1) group in the space $\Pi_P$:

$$\Psi_{P,\hat{u}}(z) = <z, \hat{u}^P,$$

(17)

8
\[ |\tilde{u}^0|^2 - \sum_{i=1}^{l} |\tilde{u}^i|^2 = 1, \sum_{\mu} \arg \tilde{u}^\mu = 0. \]  \hspace{1cm} (18)

In the same way we construct the orbit of the highest weight \( \Psi_{P,\{P_0,\ldots\}}(\tilde{z}) = (\tilde{z}^0)^P \) of \( D^-(0P) \) in the space \( \tilde{\Pi}_P \), the corresponding \( CS \) have the form:

\[ \Psi_{P,u}(\tilde{z}) = <u, \tilde{z}>^P, \]  \hspace{1cm} (19)

\[ |u_0|^2 - \sum_{i=1}^{l} |u_i|^2 = 1, \sum_{\mu} \arg u_\mu = 0. \]  \hspace{1cm} (20)

One can see that \( \Psi_{P,\tilde{u}}(z) = \overline{\Psi_{P,u}(\tilde{z})}, \ z \leftrightarrow \tilde{z}, u \leftrightarrow \tilde{u} \).

The quantities \( \tilde{u} \) and \( u \), which parametrize the \( CS \) (17) and (19), are elements of the coset space \( SU(l,1)/U(l) \), in accordance with the fact that the stationary subgroups of both the initial vectors from the spaces \( \Pi_P \) and \( \tilde{\Pi}_P \) are \( U(l) \). At the same time, the coset space is the \( l \) dimensional open complex ball \( CD^l \) of unit radius. The eq.(18) or (20), are just possible conditions which define the space. The coordinates \( u \) or \( \tilde{u} \) are called homogeneous in the \( CD^l \). One can also introduce local independent coordinates \( \alpha_i, \ i = 1, \ldots, l, \sum_{i=1}^{l} |\alpha_i|^2 < 1 \) on \( CD^l \). For instance, in the domain where \( u_0 \neq 0 \), the local coordinates are

\[ \alpha_i = u_i/u_0, \]  \hspace{1cm} (21)

\[ u_i = \alpha_i u_0, \ u_0 = \frac{\exp(-i \sum_{k=1}^{l} \arg \alpha_k)}{\sqrt{1 - \sum_{k=1}^{l} |\alpha_k|^2}}. \]

To decompose the \( CS \) in the discrete basis one can use the relation (10), since the right side of eq.(10) can be treated as \( CS \) (17) or (19),

\[ \Psi_{P,\tilde{z}}(z) = \sum_{\{n\}} \Psi_{P,\{n\}}(\tilde{u}) \Psi_{P,\{n\}}(z). \]  \hspace{1cm} (22)

Using Dirac’s notations, we get

\[ <P, u|P, n> = \Psi_{P,\{n\}}(u), \ \ <P, n|P, u> = \Psi_{P,\{n\}}(\tilde{u}). \]  \hspace{1cm} (23)

Thus, the discrete bases in the spaces \( \Pi_P \) and \( \tilde{\Pi}_P \) are the ones in the \( CS \) representation.
The completeness relation can be derived similarly to the case of the $SU(N)$ groups [11],
\[ \int |P, u > < P, u| d\mu_P(\bar{u}, u) = I_P . \] (24)

3 Uncertainty relation and $CS$ overlap

The elements of the orbit of each vector of the discrete basis $|P, n >$ and, particularly, the $CS$ constructed, are eigenstates for a nonlinear operator $C'_2$, which is defined by its action on an arbitrary vector $|\Psi >$ as

\[ C'_2 |\Psi > = \sum_a \epsilon_a < \Psi | \hat{\Gamma}_a |\Psi > \hat{\Gamma}_a |\Psi > . \]

with $\epsilon_a$ from (14). The proof of this fact is fully analogous to the one for the $SU(N)$ group [11]. Direct calculations result in

\[ C'_2 |P, n > = \lambda(P, n) |P, n >, \] (25)
\[ \lambda(P, n) = \frac{1}{2} \left( \sum_{\mu} n_{\mu}^2 - P^2 / N \right) = \frac{1}{2} \sum_{\mu} (n_{\mu} - P / N)^2 . \]

The eigenvalue $\lambda(P, n)$ attains its minimum for the lowest weight $(D^+(P0))$, for which $\sum_{\mu} n_{\mu}^2 = P^2 = \text{min}$. The $CS$ $|P, u >$ belong to the orbit of the lowest weight $\{ n \} = \{ P0 \ldots 0 \}$. Thus, we get:

\[ C'_2 |P, u > = \frac{P^2(N-1)}{2N} |P, u > . \] (26)

Define a dispersion of the square of the "hyperbolic length" of the isospin vector,

\[ \Delta C_2 = < \Psi | \sum_a \epsilon_a \hat{\Gamma}_a^2 |\Psi > - \sum_a \epsilon_a < \Psi | \hat{\Gamma}_a |\Psi >^2 = < \Psi | C_2 - C'_2 |\Psi > . \]

where $C_2$ is quadratic Casimir operator (14). The dispersion serves as a measure of the uncertainty of the state $|\Psi >$. Due to the properties of the operators $C_2$ and $C'_2$, it is group invariant and its modulus attains its lowest value $P(N-1)/2$ for the orbits of lowest $(D^+(P0))$ or highest $(D^-(0P))$
weights, particularly for the CS constructed, compared to all the orbits of the discrete basis (8). The relative dispersion of the square of the "hyperbolic length" of the isospin vector has the value in the CS

\[ \Delta C_2 / C_2 = \frac{N}{N + P}, \quad P < -N - 1, \tag{27} \]

and tends to zero with \( h \to 0, \ h = \frac{1}{|P|} \). Note, that the relative dispersion obeys here the relation \(-\infty < \Delta C_2 / C_2 < 0\), in contrast with the case of compact groups \( SU(N) \), where \( 0 < \Delta C_2 / C_2 \leq 1 \).

Proceeding to the consideration of the CS overlap, one has to say that many of its properties in general were investigating in [[7, 15, 16, 17]]. Using the completeness relation (9) and formulas (23), (10) and (17), we get for the overlap of the CS in question

\[
\langle P, u | P, v \rangle = \sum_{\{n\}} \langle P, u | P, n \rangle \langle P, n | P, v \rangle = \sum_{\{n\}} \Psi_{P, \{n\}}(u) \Psi_{P, \{n\}}(\tilde{v}) \\
= \langle u, \tilde{v} \rangle^P = \Psi_{P, \tilde{v}}(u). \tag{28}
\]

As in case of the Heisenberg-Weyl and \( SU(N) \) groups, the CS overlap plays here the role of the \( \delta \)-function (so called reproducing kernel). Namely, if \( \Psi_P(u) \) is a vector \( |\Psi\rangle \) in the CS representation, \( \Psi_P(u) = \langle P, u | \Psi \rangle \), then

\[
\Psi_P(u) = \int \langle P, u | P, v \rangle \Psi_P(v) d\mu_P(\tilde{v}, v). \]

The modulus of the CS overlap (28) has the following properties:

\[
| \langle P, u | P, v \rangle | \leq 1, \quad \lim_{P \to \infty} | \langle P, u | P, v \rangle | = 0, \quad \text{if } u \neq v, \\
| \langle P, u | P, v \rangle | = 1, \quad \text{only, if } u = v, \tag{29}
\]

which allow to introduce a symmetric\(^1\) \( s(u, v) \) in \( CD^I \),

\[
s^2(u, v) = -\ln |\langle P, u | P, v \rangle|^2 = -P \ln |(u, \tilde{v})|^2. \tag{30}
\]

\(^1\)We remember that a real and positive symmetric obeys only two axioms of a distance \( s(u, v) = s(v, u) \) and \( s(u, v) = 0, \) if and only if \( u = v \), except the triangle axiom.
The symmetric $s(u, v)$ generates the metric tensor in the space $CD^l$. To demonstrate that, it is convenient to go over to the local independent coordinates (21). In the local coordinates the symmetric takes the form

$$s^2(\alpha, \beta) = -P \ln \frac{\lambda(\alpha, \beta) \lambda(\beta, \vec{\alpha})}{\lambda(\alpha, \vec{\alpha}) \lambda(\beta, \beta)},$$

with $\lambda(\alpha, \beta) = 1 - \sum_i \alpha_i \beta_i$. Calculating the square of the ”distance” between two infinitesimally close points $\alpha$ and $\alpha + d\alpha$, one finds

$$ds^2 = g_{ik} d\alpha_i d\bar{\alpha}_k, \quad g_{ik} = -P \lambda^{-2}(\alpha, \bar{\alpha}) \left[ \lambda(\alpha, \bar{\alpha}) \delta_{ik} + \bar{\alpha}_i \alpha_k \right],$$

$$F = P \ln \lambda(\alpha, \bar{\alpha}),$$

$$\det \| g_{ik} \| = P^l \lambda^{-N}(\alpha, \bar{\alpha}), \quad g^{\bar{\alpha}_i} = -\frac{1}{P} \lambda(\alpha, \bar{\alpha})(\delta_{ki} - \bar{\alpha}_k \alpha_i).$$

The quantity $g_{ik}$ is the metric on the open complex ball $CD^l$ with constant holomorphic sectional curvature $C = 2/P < 0$, [9], whereas $g^{\bar{\alpha}_i}$ defines the corresponding Poisson bracket on this Kahlerian manifold

$$\{ f, g \} = ig^{\bar{\alpha}_i} \left( \frac{\partial f}{\partial \alpha_i} \frac{\partial g}{\partial \bar{\alpha}_k} - \frac{\partial f}{\partial \bar{\alpha}_k} \frac{\partial g}{\partial \alpha_i} \right).$$

As we have just said, the logarithm of the modulus of $CS$ overlap defines a symmetric on the coset space. The expression for the symmetric through $CS$ has one and the same form for any group; its existence follows directly from properties of $CS$. As for the real distance $\rho$ on the coset space, its expression through $CS$ depends on the group. For example, in case of the $CP^l$ ($SU(l + 1)$ group), $\cos(\rho/P) = | < u, \tilde{v} > |$, so that for $l = 1$, $\rho$ is the distance on the sphere with the radius $P/2$. For our case of $CD^l$ ($SU(l, 1)$ group) the distance $\rho$ shows up in the relation $\cosh(\rho/P) = | < u, \tilde{v} > |$. Thus, for both cases (see [11] as well) we have the following relations between $CS$ overlaps and the distances

$$CP^l: \quad | < P, u|P, v' > | = [\cos(\rho/P)]^P,$$

$$CD^l: \quad | < P, u|P, v' > | = [\cosh(\rho/P)]^P.$$
4 Operators symbols and classical limit

We are going to investigate the classical limit on the language of operators symbols, constructed by means of the CS. Remember that the covariant symbol $Q_A(u, \bar{u})$ and the contravariant one $P_A(u, \bar{u})$ of an operator $\hat{A}$ are defined as [[13, 14]]

$$Q_A(u, \bar{u}) = \langle P, u | \hat{A}| P, u \rangle, \quad \hat{A} = \int P_A(u, \bar{u})|P, u \rangle < P, u | d\mu_P(\bar{u}, u) ,$$

$$Q_A(u, \bar{u}) = \int P_A(u, \bar{u})| < P, u | P, v > |^2 d\mu_P(\bar{u}, u) . \quad (35)$$

One can calculate the $P$ and $Q$ symbols of operators explicitly, if one generalizes formally creation and annihilation operators method to the case under investigation. Consider for example IR $D^+(P0)$ and introduce, as in case of $SU(N)$, operators $a^\dagger_{\mu}$ and $a^\nu$, which act on basis vectors and CS by the formulas

$$a^\dagger_{\mu}| P, n \rangle = \sqrt{\frac{n_{\mu} + 1}{P + 1}}| P + 1, \ldots, n_{\mu} + 1, \ldots > = z_{\mu} \Psi_{P,\{n\}}(z) ,$$

$$a^\nu| P, n \rangle = \sqrt{P n_{\mu}| P - 1, \ldots, n_{\mu} - 1, \ldots > \frac{\partial}{\partial z_{\mu}} \Psi_{P,\{n\}}(z) ,$$

$$< P, n | a^\dagger_{\mu} = \sqrt{\frac{n_{\mu}}{P}} < P - 1, \ldots, n_{\mu} - 1, \ldots | = \frac{1}{P} \frac{\partial}{\partial z_{\mu}} \Psi_{P,\{n\}}(z) ,$$

$$< P, n | a^\nu = \sqrt{(P + 1)(n_{\mu} + 1)} < P + 1, \ldots, n_{\mu} + 1, \ldots >$$

$$= (P + 1) \tilde{z}_{\mu} \Psi_{P,\{n\}}(\tilde{z}) ,$$

$$a^\mu| P, u \rangle = P \tilde{u}^\mu| P - 1, u \rangle = \frac{\partial}{\partial z_{\mu}} \Psi_{P,\tilde{u}}(z) ,$$

$$< P, u | a^\dagger_{\mu} = u_{\mu} < P - 1, u \rangle = \frac{1}{P} \frac{\partial}{\partial z_{\mu}} \Psi_{P,u}(\tilde{z}) ,$$

$$[a^\mu, a^\nu] = \delta^\mu_{\nu}, \quad [a^\mu, a^\nu] = [a^\dagger_{\mu}, a^\dagger_{\nu}] = 0 . \quad (36)$$

(Note that the sign $\dagger$ does not mean the hermitian conjugation with respect to the scalar product (7)). In contrast with the case of $SU(N)$ group where $P$ and $n_{\mu}$ are always positive, $P$ and $n_0$ are negative for the $SU(l,1)$ group,
so that complex factors can appear when the operators \( a_\mu^\dagger \) and \( a^\mu \) act on states. Because of negative \( n_0 \), the space of states can not be treated as Fock space.

Quadratic combinations \( A_\mu^\nu = a_\mu^\dagger a^\nu = \gamma_\mu \frac{\partial}{\partial \gamma^\nu} \) obey the commutation relations (42) and are generators of the groups \( U(l, 1) = SU(l, 1) \otimes U(1) \). That is the reason why operators, which are polynomial in the generators, can be written through the \( a_\mu^\dagger \) and \( a^\nu \) and presented in the normal or anti-normal form,

\[
\hat{A} = \sum_K A_{\nu_1...\nu_K}^\mu_1...\mu_K a_{\mu_1}^\dagger ... a_{\mu_K}^\dagger a^\nu_1 ... a^\nu_K \\
= \sum_K \hat{A}_{\nu_1...\nu_K}^\mu_1...\mu_K a_{\mu_1}^\dagger ... a_{\mu_K}^\dagger a^\nu_1 ... a^\nu_K .
\]

(37)

Direct calculations give for the symbols of such operators:

\[
Q_A(u, \bar{u}) = \sum_K (-1)^{K-K_0} \frac{(-P + K - 1)!}{(-P - 1)!} A_{\nu_1...\nu_K}^\mu_1...\mu_K u_{\mu_1} ... u_{\mu_K} \bar{u}_{\nu_1} ... \bar{u}_{\nu_K}, \\
P_A(u, \bar{u}) = \sum_K (-1)^{K-K_0} \frac{(-P - N - K)!}{(-P - N)!} \hat{A}_{\nu_1...\nu_K}^\mu_1...\mu_K u_{\mu_1} ... u_{\mu_K} \bar{u}_{\nu_1} ... \bar{u}_{\nu_K}, \\
K_0 = \sum_{i=1}^i \delta_{\nu_i, 0} .
\]

(38)

In manipulations it is convenient to deal with nondiagonal symbols

\[
Q_A(u, \bar{v}) = \frac{\langle P, u | \hat{A} | P, v \rangle}{\langle P, u | P, v \rangle},
\]

which can be derived from the corresponding diagonal symbols (38) by the replacement \( \bar{u} \to \bar{v} \) and by multiplying of each term by the factor \( \langle u, \bar{v} \rangle \).

In the local independent variables (21) these symbols are analytical functions of both their arguments.

Consider for example covariant symbols \( \langle J_\alpha \rangle = \langle P, u | J_\alpha | P, u \rangle \) of generators \( J_\alpha = (\Gamma_\alpha)^\mu_\nu A_\nu^\mu \) for the \( SU(1, 1) \) group, so that \( \Gamma_\alpha \) are matrices (13)). In this case it is convenient to parameterize the \( CS \) by \( j, \theta, \varphi \); \( P/2 = j \), \( \tilde{u}^1 = \cosh \frac{\theta}{2} e^{-i\varphi} \), \( \tilde{u}^2 = \sinh \frac{\theta}{2} e^{-i\varphi} \),

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\[ <\hat{J}_1> = \mp j \sinh \theta \cos \varphi = j_1 \]
\[ <\hat{J}_2> = \mp j \sinh \theta \sin \varphi = j_2 \]
\[ <\hat{J}_3> = \mp j \cosh \theta = j_3 , -j_1^2 - j_2^2 + j_3^2 = j^2 , \]

where the upper sign belongs to \( D^+(P) \) and lower one to \( D^-(P) \).

Fig.1

The dots on the axis \( <\hat{J}_3> \) correspond to the states of discrete basis \( |j, m> \), \( \hat{J}_3|j, m>= m|j, m> \); the \( CS \) are placed on the upper \( (D^+(P0)) \) or lower \( (D^-(0P)) \) sheet of the two sheets hyperboloid on the Fig.1.

The classical limit can be considered as in [11]. So, one can get for the star product of two covariant symbols in the local coordinates (21) the following expression

\[
Q_{A_1} * Q_{A_2} = Q_{A_1 A_2}(\alpha, \tilde{\alpha}) = \int Q_{A_1}(\alpha, \tilde{\alpha})Q_{A_2}(\beta, \tilde{\alpha})e^{-\frac{\pi}{2}(\alpha, \beta)}d\mu_P(\tilde{\beta}, \beta)
\]

\[
= Q_{A_1}(\alpha, \tilde{\alpha})Q_{A_2}(\alpha, \tilde{\alpha}) + g^{i\mathbf{k}} \frac{\partial Q_{A_1}(\alpha, \tilde{\alpha})}{\partial \alpha_k} \frac{\partial Q_{A_2}(\alpha, \tilde{\alpha})}{\partial \alpha_i} + o(\hbar) ,
\]

\[
d\mu_P(\tilde{\beta}, \beta) = \frac{(-P - 1)!}{(-P - l - 3)!P!} \det \|g_{\gamma\kappa}(\beta, \tilde{\beta})\| \prod_{i=1}^{l} \frac{dRe\beta_i dIm\beta_i}{\pi} , \tag{40}
\]

where the matrix \( g^{i\mathbf{k}} \) was defined in (32) and is proportional to \( h = 1/\|P\| \).

Note, the decomposition of \( Q_{A_1}(\alpha, \tilde{\alpha})Q_{A_2}(\beta, \tilde{\alpha}) \) into a series with respect to \( \beta - \alpha \) is possible if symbols are nonsingular (differentiable) functions on \( \alpha, \tilde{\alpha} \) in the limit \( P \rightarrow \infty \). That is valid for polynomial operators, but not for the operators of finite transformations, which are singular in that limit.

Taking into account the expression (33) for the Poisson bracket in the space \( C\bar{D}^l \), and eq. (40) we get for the star multiplication of two symbols of polynomial operators

\[
\lim_{\hbar \to 0} Q_{A_1} * Q_{A_2} = Q_{A_1} Q_{A_2} , \tag{41}
\]

\[
Q_{A_1} * Q_{A_2} - Q_{A_2} * Q_{A_1} = i\{Q_{A_1}, Q_{A_2}\} + o(\hbar) .
\]

The equations (41) are just Berezin’s conditions of the classical limit in terms of operators symbols [7, 13], where the quantity \( h = 1/\|P\| \) plays the role
of the Planck constant. That property of \( h \) has been remarked already in Sect. 3, while investigating the uncertainty relation. From that consideration it is also easy to see that the length of the isospin vector is proportional to the signature \( P \) of a representation. Thus, the classical limit in this case is connected with large values of the isospin vector. In contrast with the ordinary case of the Heisenberg-Weyl group, where the Planck constant is fixed, as for \( SU(N) \), the Planck constant can really take different values, which are however quantized since the quantity \( P \) is discrete.

It is easy to demonstrate that the contravariant and covariant symbols coincide in the classical limit. For instance,

\[
Q_A(\alpha, \bar{\alpha}) = P_A(\alpha, \bar{\alpha}) + g^{ik} \frac{\partial P_A(\alpha, \bar{\alpha})}{\partial \alpha_k} \partial \dot{\alpha}_i + o(h) .
\]

For the operators of finite transformations one can derive

\[
Q_{T(g_2)T(g_1)}(u, \bar{u}) = < P, u | T(g_2)T(g_1) | P, u > = < u, g_2 g_1 \dot{v} | P > ,
\]

\[
Q_{T(g_2)} \ast Q_{T(g_1)} = Q_{T(g_2 g_1)} .
\]

We see that the law of multiplication of these symbols is similar to one of matrices of finite transformations and does not depend on \( P \). Thus, we have an example of operators, which do not obey to the eq. (39) in the classical. According to Yaffe’s terminology [15] these are so called nonclassical operators.

5  Appendix

We give here a brief description of discrete positive \( D^+ \) and negative \( D^- \) series of unitary IR of \( SU(l, m) \), in particularly, of \( SU(l, 1) \) ones, which are related to the \( CS \) in question.

Remember first, if \( r \) be a rank of a semi-simple algebra \( \text{Lie} \), which is our case, then there exist \( r \) fundamental IR \( D_1, \ldots, D_r \), having the highest weights \( M_1, \ldots, M_r \) correspondingly. Consider the tensor product of the representations

\[
D_1^{P_1} \otimes D_2^{P_2} \ldots D_r^{P_r} ,
\]
where \( P_i \) are nonnegative integers, and \( D_i^{P_i} \) means the \( P_i \) times direct product of the \( D_i \). Let \( D(P_1, \ldots, P_r) \) be the irreducible part of this product, containing the highest weight \( M(P) = \sum P_i M_i \), then all finite-dimensional IR (and therefore all unitary IR of compact groups) are exhausted by such representations. The set of numbers \( P_1, \ldots, P_r \) is called the signature of IR. Fundamental IR are characterized by one nonzero index of signature, which is unity. For unitary IR of noncompact groups one needs to consider, in general, complex \( P_i \), i.e. to generalize the tensor calculus and consider tensors of noninteger or complex ranks \([18, 18]\). In contrast with the case of compact groups, all linear unitary IR of noncompact groups are infinite-dimensional. In this case there are two different kinds of representation spaces, which correspond to discrete and to continuous series. The theory of the discrete series is mostly analogous to the finite-dimensional case. A classification of unitary IR of \( SU(l, m) \) one can find in \([18, 19, 20, 21, 22, 23, 24]\] and in ref. cited there. The case of \( SU(l, 1) \) is considered separately in \([25]\), besides, one can find the case of \( SU(2, 1) \) in \([26]\) and the case of \( SU(2, 2) \) in \([27]\).

The fundamental IR \( D(10\ldots0) \) and \( D(0\ldots0) \) of \( SU(l, m) \) groups are representations by \( N \otimes N, N = l + m, \) quasi-unimodular matrices \( g \) and \( g^{-1}, \) \( \Lambda g^t \Lambda = g^{-1}, \) \( \Lambda = \text{diag}_i, m(1, \ldots, 1, -1, \ldots, -1) \) in spaces of \( N \) dimensional rows \( z_{\mu} \) or columns \( \tilde{z}^\mu, \) see e.g. \([3]\). Others fundamental IR \( D(010\ldots0), D(001\ldots0), \ldots, D(0\ldots100), D(0\ldots010) \) are realized in spaces of antisymmetric elements \( z_{ik}, \) \( z_{ikm}, \ldots, \tilde{z}^{ikm}, \tilde{z}^{ik} \), \([18, 19, 26, 30]\).

As it is well known the commutation relations of \( U(l, m) \) generators have the form

\[
[A^\nu_{\mu}, A^k_{\lambda}] = \delta^\nu_{\lambda} A^k_{\mu} - \delta^k_{\mu} A^\nu_{\lambda},
\]

and furthermore, for unitary IR \([19, 21]\)

\[
(A^k_{j})^+ = c^i_k A^i_j, \quad c^i_k = \begin{cases} +1 & \text{if } k, j \leq l \text{ or } k, j > l \\ -1 & \text{if } k \leq l < j \text{ or } j \leq l < k. \end{cases}
\]

It is convenient to introduce a basis, consisting of eigenfunctions of the commuting operators \( A^i_1, \)

\[
A^i_1 |n_1 n_2 \ldots n_N > = n_i |n_1 n_2 \ldots n_N >, \quad N = l + m,
\]

where \( n_i \) are called occupation numbers. By means of the commutation relations \((42)\) we get for \( i \neq k \)
\[ A^i_k \mid n_i \ldots n_k \ldots = \sqrt{n_i(n_k + 1)} \mid n_i - 1 \ldots n_k + 1 \ldots >. \] (44)

The conditions

\[ n_k(n_j + 1) \geq 0, \ k, j \leq l \text{ or } k, j > l, \]
\[ n_k(n_j + 1) \leq 0, \ k \leq l < j \text{ or } j \leq l < k, \] (45)

must hold for unitary IR.

One can reach any weight of a given IR by means of operators \( A^i_k \), moving from any other weight of the representation; the weight diagram stops suddenly when one reaches a highest weight, the factor in (44) appears to be zero at this step. The occupation number space is \( N \) dimensional; weights, which correspond to a given IR fill in a area with \( \sum n_i = P \), where \( P \) is an eigenvalue of the operator \( \sum A^i_k \), commuting with all the operators \( A^i_k \).

Consider some particular cases. For the groups \( SU(2,1) \) and \( SU(3) \) the weights fill in the three dimensional space (Fig.2a); the weights which correspond to a one IR, fill in areas on the planes \( n_1 + n_2 + n_3 = P \) (such areas for integer \( n_i \) are represented on Fig.2b).

Fig.2

For unitary IR of \( SU(3) \) either \( n_i \geq 0 \) or \( n_i \leq -1 \) and are integers. Unitary IR are finite-dimensional; areas with \( P \geq 0 \) correspond to IR \( D^0(P0) \), ones with \( P \leq -3 \) correspond to \( D^0(0Q) \), \( Q = -P - 3 = \sum q_i \), \( q_i = -p_i - 1 \geq 0 \). The representations \( D^0(P0) \) and \( D^0(0P) \) are conjugated. One can find the following unitary IR for \( SU(2,1) \), using (44) and (45):

bounded below by the weight with \( Y_{\text{min}} \), \( Y = -P/3 - n_3 \),

\[ D^+(P0), \ P < 0, \ n_1, n_2 \geq 0 \text{ and integers, } n_3 \leq 0 \text{ and real}, \]
\[ D^+(0Q), \ Q < 0, \ q_1, q_2 \geq 0 \text{ and integers, } q_3 \leq 0 \text{ and real}, \]

and IR conjugated to the former, bounded above by weights with \( Y_{\text{max}} \), \( Y = -Q/3 + q_3 \),

\[ D^-(0Q), \ Q \geq 0, \ q_1, q_2 \geq 0 \text{ and integers, } q_3 \leq -1 \text{ and integer}, \]
\[ Q = -P - 3, \ q_i = -n_i - 1. \]
The replacement of the signature of IR $D(P0) \to D(0-P-3)$ is a particular case of the group of parameters transpositions of IR [19]. Such replacements leave eigenvalues of the Casimir operator unchanged.

Weights of IR for the $SU(4), SU(3, 1), SU(2, 2)$ groups fill in areas in the space $n_1 + n_2 + n_3 + n_4 = P$; such areas, for $n_i$ integers, are shown on Fig. 2c.

The unitary IR $D(P0 \ldots 0)$ and $D(0 \ldots 0P)$ of the $SU(N)$ are well known full symmetrical representations. They can be realized, for instance, in spaces of polynomials of a fixed power $P$.

Weights diagrams of the unitary IR, corresponding to the discrete series $D^+(P0 \ldots 0)$ and $D^-(0 \ldots 0P)$ of the $SU(l, 1)$ groups are presented on the Fig. 3.

Fig. 3
They fall in a sum of full symmetrical IR by the reduction on the compact subgroup $SU(l)$,

$$D^+(P0 \ldots 0)_{SU(l, 1)} = \sum_{\alpha=0}^{\infty} D(\alpha0 \ldots 0)_{SU(l)},$$

$$D^-(0 \ldots 0P)_{SU(l, 1)} = \sum_{\alpha=0}^{\infty} D(0 \ldots 0\alpha)_{SU(l)}.$$

That is well seen on the Fig. 3: each level of the weight diagram corresponds to a IR of a subgroup. Besides the eigenvalues of the $l-1$ commuting Cartan generators $H_i$ from the compact subgroup $SU(l)$ (these are linear combinations of $A^1_i$, $i \neq 0$), the weights of $SU(l, 1)$ are characterized by an additional number $Y = \mp(P/N - n_0)$; the upper sign for $D^+(P0 \ldots 0)$, the lower one for $D^-(0 \ldots 0P)$, and weight diagrams are bounded below and above correspondingly,

$$Y_{\min} = -P(N-1)/N, \quad Y_{\max} = P(N-1)/N. \quad (46)$$

The weight structure of these IR does not depend on $P$; only the position of the diagram in the weight space depends on $P$, according to the (46).
6 Conclusion

Thus, an explicit construction of the $CS$ for all the $SU(l, 1)$ groups appears to be possible as well as for all the $SU(N)$ ones due to the appropriate choice for the irreducible representations of the group in the space of polynomials and quasi-polynomials of a fixed power. Many formulas look very similar in the two cases, nevertheless, there are also many differences connected with principal difference between the compact $SU(N)$ and noncompact $SU(l, 1)$. Construction of the $CS$ of the two groups provide an explicit analysis of quantization problem on complex spaces of constant curvature in full agreement with the general theory [17] of quantization on Kahlerian manifolds.

References


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