STAR PRODUCTS AND DEFORMED OSCILLATORS$^1$

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Abstract

A star-product formalism describing deformations of the standard quantum mechanical harmonic oscillator is introduced. A number of existing generalized oscillators occur as particular choices of star-products between the elements of the ordinary oscillator algebra. Star dynamics and coherent states are introduced and studied.

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The recent activity in the theory of quantum groups [1,2,3] has invigorated the interest in the central issue of quantum theory, namely quantization. Instrumental in the studies of quantum groups is the notion of deformation. Quantization of a group is understood as deformation of the algebra of functions of the group manifold and as deformation of the respective enveloping algebra of generators, both using a deformation parameter $q$, to which in principle should be attributed a physical meaning. On the other hand, quantization of classical mechanics also can be understood, outside the conventional Hilbert space - with operators framework, as a deformation of the commuting algebra of functions defined on the classical phase space, with deformation parameter the Planck constant. This deformation quantization scheme is usually called star-product quantization [4,5] and its roots are dated back to the early days of quantum mechanics when it was known as Weyl-Wigner quantization [6,7,8].

In this paper we will attempt to describe the algebra quantization along the lines of the star-quantization method, using the $q$-deformed oscillator as a paradigm. Our "classical" algebra of functions will be the noncommutative enveloping algebra of creation and annihilation operators $a^\pm$, of the standard oscillator algebra of quantum mechanics $h_4$. Its deformation will proceed with the introduction of two different star-products defined between elements of the oscillator algebra. These products are operator valued and depend by construction only upon the number operator $N$, and a deformation parameter $q$. Utilizing these star-products we deform the oscillator algebra in a manner which affects only the commutation relation between the $a^+$ and $a^-$ and keeps unchanged all the rest, this algebra we call $\ast - h_4$. Then we proceed by showing the connection between the present star-product method of deforming the harmonic oscillator and the customary prescription of the so called $q$-deformed oscillator. The connection is established by means of the quantizing mappings relating deformed with undeformed generators. As is explained later in the text the introduction of two different star-products is necessary in order to obtain the $q$-oscillator after a particular choice of products. It is also shown how other generalized oscillators are reached by making other choises of star-products. Finally, dynamics and coherent states pertinent to $\ast - h_4$ are introduced and studied to some extend.

The star-quantization program constitutes an autonomous alternative to quantization [4,5]. The basic idea is to start with the commutative algebra of functions in the oscillator phase space with pointwise multiplication which is also a Lie algebra under the Poisson bracket. This algebra is now deformed with deformation parameter the $\hbar$. Let $(q,p) \equiv (z_1,z_2) \equiv (z)$ then for $f(z)$ and $g(z)$ functions of the phase space the Poisson bracket is ($\partial_z \equiv \frac{\partial}{\partial z}$)

$$\left\{f(z),g(z)\right\}_{PB} = f(z)\Delta g(z) = f(z)\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{c} \partial_{z_1} \\ \partial_{z_2} \end{array}\right) \left\{g(z)\right\}. \quad (1)$$

If for example we restrict ourselves to functions which are up to quadratic polynomials we obtain the $sl_2$ algebra or if we allow for polynomials of infinite-degree
(series) we obtain a infinite dimensional Lie algebra etc. We note also that the Poisson bracket is skew-symmetric, \( \{f, g\} = -\{g, f\} \), satisfies the Jacobi identity \( \{f, \{g, h\}\} + \{g, \{f, h\}\} + \{h, \{f, g\}\} = 0 \) and obeys the Leibnitz rule, \( \{f, gh\} = \{f, g\} + g\{f, h\} \). Now the deformation of such an algebra is obtained by first imposing a non-commutative \( * \)-product between the functions

\[
f(z) * g(z) := f(z)e^{\frac{i\hbar}{2}g(z)},
\]

due to course

\[
f * g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2).
\]

Then the so-called Moyal bracket is defined by

\[
f(z) * g(z) - g(z) * f(z) = i\hbar\{f(z), g(z)\} + \mathcal{O}(\hbar^2) := \{f(z), g(z)\}_*
\]

and is equivalent to the commutator of quantum mechanics, ordered such that the position operator is placed to the left of the momentum operator. It must be evident now that the philosophy of the \( * \)-quantization is entirely opposite to the quantum-mechanics-with-Hilbert-space method of ordinary quantization. The emphasis here has been shifted from the objects (operators) to the rule of composition (\( * \)-product) between the objects (classical functions). This approach to quantization is in fact similar to the Weyl-Wigner formalism of quantum mechanics (see e.g. Ref. 6,7,8), and to the Berezin type of quantization through coherent states by the so-called symbols of operators [9].

The Moyal bracket constitutes a non-trivial deformation of the Poisson bracket and is essential that it involves derivatives of infinite order, since according to Kirillov [10] any deformation depending on a finite number of jets of functions is trivial at least locally (Darboux theorem). Moreover the Moyal bracket is the unique, up to isomorphism, non-trivial deformation of the Poisson bracket [10, 11, 12]. (These and subsequent statements concerning the \( * \)-products and Moyal brackets are valid for any number of degrees of freedom, however here we state them for only one degree of freedom).

Two important tools of this formalism are first the \( * \)-exponential of a function,

\[
e^{\frac{i\hbar}{\hbar}f} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\hbar} \right)^n (f^*)^n
\]

where \((f^*)^n := f \cdots f\) \( (n\text{-times}) \). If \( H \) is the classical Hamiltonian then

\[
e^{\frac{i\hbar}{\hbar}H} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{i\hbar} \right)^n (H^*)^n
\]

and the time development of any function of the phase space is be given by,

\[
f(t) = e^{-\frac{i\hbar}{\hbar}H} * f(0) * e^{\frac{i\hbar}{\hbar}H},
\]
then \( f(t) \) obeys
\[
\frac{\dot{f}}{\hbar} = f \ast H - H \ast f = \{ f, H \}
\]
which is the full quantum equation of motion of this formalism.

The second tool is the star Backer-Champbel-Hausdorff, \( \ast \)-BCH, formula; the symbols \( a(z), b(z) \) and \( c(z) \) of three operators \( A, B \) and \( C \) for which the BCH formula is valid, namely
\[
e^A e^B = e^C
\]
with
\[
C = A + B + L_2 + L_3 + \cdots + L_n + \cdots
\]
and
\[
L_2 = \frac{1}{2} [A, B]
\]
\[
L_3 = \frac{1}{12} [A, [A, B]] + \frac{1}{12} [[A, B], B] \quad \text{etc.}
\]
satisfy the \( \ast \)-BCH formula, namely:
\[
e^a \ast e^b = e^c
\]
where
\[
c = a + b + \ell_2 + \ell_3 + \cdots + \ell_n + \cdots
\]
with
\[
\ell_2 = \frac{1}{2} \{ a, b \}
\]
\[
\ell_3 = \frac{1}{12} \{ a, \{ a, b \} \} + \frac{1}{12} \{ \{ a, b \}, b \} \quad \text{etc.}
\]

Let us finally note that if \( \alpha = \frac{1}{\sqrt{2}} (q + ip) \) and \( \bar{\alpha} = \frac{1}{\sqrt{2}} (q - ip) \) then the fundamental Poisson bracket of the classical oscillator
\[
\{ \alpha, \bar{\alpha} \}_p = 1,
\]
is in the \( \ast \)-product formalism replaced by the Moyal bracket,
\[
\{ \alpha, \bar{\alpha} \} = 1,
\]
which now describes the quantum-mechanical oscillator.
Let us now come to the \(q\)-deformed oscillator. The \(q\)-deformation of the quantum-mechanical harmonic oscillator as it is known in the literature [13, 14, 15] is along the lines of the quantization of the classical oscillator by means of the ordinary method of operators and Hilbert spaces. Since we expect that conceptual as well as technical profit can be made by recasting the \(q\)-deformation into some alternative formalism of a \(q\)-star-product type, we will proceed as the following diagram indicates:
Classical Mechanics  
e.g. Oscillator

Quantum Mechanics  
Operator-Hilbert-space quantization

\hbar-\text{-Star-product quantization}

Quantum Mechanics  
e.g. Oscillator

Quantum Groups  
Operator-Hilbert-space deformation

\textit{q}-\text{-Star-product deformation e.g. \textit{q}-oscillator

The present work
To this end let us consider the harmonic oscillator algebra denoted by $h_4$ [16]:

$$[a^-, a^+] = 1$$  \hspace{1cm} (15)

$$[N, a^\pm] = \pm a^\pm$$  \hspace{1cm} (16)

$$[1, \text{ everything}] = 0.$$  \hspace{1cm} (17)

Let us endowed this algebra with two $*$-products: the right-star-product $*_R$,

$$g_1 *_R g_2 \equiv g_1 \cdot R(N) \cdot g_2,$$  \hspace{1cm} (18)

and the left-star-product $*_L$,

$$g_1 *_L g_2 = g_1 \cdot L(N) \cdot g_2$$  \hspace{1cm} (19)

for any $g_1, g_2 \epsilon h_4$. The $R(N)$ and $L(N)$ are smooth functions of the number operator $N$, which depend also on the deformation parameter $q = e^\gamma$, $\gamma \in \mathbb{R}$ or $\mathbb{C}$ in such a way that when $q \to 1$ then $R(N) \to 1$ and $L(N) \to 1$ (simultaneously). Each of these $*$-products are non-commutative and for any $g_1, g_2, g_3 \epsilon h_4$ which satisfy the Jacobi identity is valid that

$$g_1 *_R g_2 *_R g_3 + \text{cyclic permutation} = 0$$  \hspace{1cm} (20)

and

$$g_1 *_L g_2 *_L g_3 + \text{cyclic permutation} = 0.$$  \hspace{1cm} (21)

Let us now define the corresponding Moyal bracket of our case, in terms of $*_R$ and $*_L$. It is:

$$[g_1, g_2]_{*_{RL}} \equiv g_1 *_R g_2 - g_2 *_L g_1$$  \hspace{1cm} (22)

$$= g_1 Rg_2 - g_2 Lg_1.$$  \hspace{1cm} (23)

We will also need the opposite bracket, namely:

$$[g_1, g_2]_{*_{LR}} \equiv g_1 *_L g_2 - g_2 *_R g_1 = g_1 Lg_2 - g_2 Rg_1.$$  \hspace{1cm} (24)

Then the antisymmetry can be stated as

$$[g_1, g_2]_{*_{RL}} = -[g_2, g_1]_{*_{LR}}.$$  \hspace{1cm} (25)

We now proceed by defining the $q$-deformed oscillator in the present $*$-product approach. The $q$-deform harmonic oscillator algebra $*-h_4$ is defined to be:

$$[a^-, a^+]_{*_{RL}} = 1$$  \hspace{1cm} (26)

$$[N, a^\pm] = \pm a^\pm$$  \hspace{1cm} (27)

$$[1, \text{ everything}] = 0.$$  \hspace{1cm} (28)
The generalized $q$-Jacobi identity valid for that algebra reads [17]

$$\left[ a^-,[a^+, N]\right]_{*_{RL}} + \left[ [a^+, [N, a^-]]_{*_{LR}} + [N, [a^-, a^+]_{*_{RL}}] = 0 \right. \quad (29)$$

or

$$\left[ a^-,[a^+, N]\right]_{*_{LR}} + \left[ [a^+, [N, a^-]]_{*_{RL}} + [N, [a^-, a^+]_{*_{RL}}] = 0 \right. \quad (30)$$

Any choice of $*_{R}$ and $*_{L}$ products will induce a choice of the Moyal bracket and this will determine in turn a different $q$-oscillator. To come in contact with quantum groups (this is abuse of language, $q$-oscillator have no satisfactory co-product), we take the $R(N)$ and $L(N)$ functions to be as follows,

$$R(N) \equiv \frac{S(N)}{N} \cdot \frac{1}{S(N) - S(N - 1)} \quad (31)$$

and

$$L(N) \equiv \frac{S(N + 1)}{N + 1} \cdot \frac{1}{S(N + 2) - S(N + 1)} \quad (32)$$

where $S(N)$ parametrizes the products and is a function of $N$ and of the deformation parameter to be specified later. Let us now see how the $q$-deformed oscillator algebra of eqs. (26-28) includes different known deformed oscillators, by merely choosing different parametrizing functions $S(N)$ which in turn choose $*_{R}$, $*_{L}$ and Moyal brackets. We first elaborate on (26) using the identity $f(N)a^\pm = a^\pm f(N \pm 1)$;

$$a^- *_{R} a^+ - a^+ *_{L} a^- = 1 \quad (33)$$

means

$$a^- R(N)a^+ - a^+ L(N)a^- = 1 \quad (34)$$

or by virtue of (31-32),

$$a^- \left\{ \frac{S(N)}{N} \cdot \frac{1}{S(N) - S(N - 1)} \right\} a^+ - a^+ \left\{ \frac{S(N + 1)}{N + 1} \cdot \frac{1}{S(N + 2) - S(N + 1)} \right\} a^- = 1 \quad (35)$$

or

$$\left\{ \frac{1}{S(N + 1) - S(N)} \right\} \cdot \left\{ a^- \frac{S(N)}{N} a^+ - a^+ \frac{S(N + 1)}{N + 1} a^- \right\} = 1 \quad (36)$$

which becomes,

$$a^- \sqrt{\frac{S(N)}{N}} \sqrt{\frac{S(N)}{N}} a^+ - a^+ \sqrt{\frac{S(N + 1)}{N + 1}} \sqrt{\frac{S(N + 1)}{N + 1}} a^- = S(N + 1) - (N) \quad (37)$$
After defining $a_q^\pm$, by means of the deforming mappings

$$a_q^- \equiv a^- \sqrt{S(N) \over N} = \sqrt{S(N+1) \over N+1} a^-$$

and

$$a_q^+ \equiv a^+ \sqrt{S(N+1) \over N+1} = \sqrt{S(N) \over N} a^+ ,$$

eq. (37) can be rewritten in the form

$$a_q^- a_q^+ - a_q^+ a_q^- = S(N+1) - S(N) .$$

Next we elaborate on (27);

$$N a^+ - a^+ N = a^+ ,$$

which can be written as

$$N \sqrt{N \over S(N)} \sqrt{S(N) \over N} a^+ - a^+ N \sqrt{N+1 \over S(N+1)} \sqrt{S(N+1) \over N+1} N = a^+ ,$$

or

$$\sqrt{N \over S(N)} a^+ - a^+ \sqrt{N+1 \over S(N+1)} \sqrt{S(N+1) \over N+1} N = \sqrt{S(N) \over N} a^+ ,$$

or with the definitions issued by (38-39)

$$N a_q^+ - a_q^+ N = a_q^+ ,$$

and similarly from $N a^- - a^- N = -a^-$ we obtain

$$N a_q^- - a_q^- N = -a_q^-$$

by virtue of the identifications in (38-39). Finally from (28) we get easily that the unit operator commutes with all the $N, a_q^+$ and $a_q^-$. Summarizing the results derived for the deformed generators we have

$$[a_q^-, a_q^+] = S(N+1) - S(N)$$

$$[N, a_q^\pm] = \pm a_q^\pm$$

$$[1, \text{everything}] = 0 .$$

One should now contrast eqs. (26-28) with eqs. (46-48); they describe the same algebra in two fundamentally different ways. The former is the *-product road to
deformation, the latter is the deformed-operators type of description of the algebra deformation concept. The bridge between them is of course the deforming maps of (38-39); they trade the deformed product (26-28) for deformed generators or the opposite.

If we now select the parametrizing function\[18\] to be:

\[ S(N) = [N] = \frac{q^N - q^{-N}}{q - q^{-1}} = \frac{\sinh \gamma N}{\sinh \gamma}, \] (49)

we obtain from (46),

\[ a^{-}_q a^{+}_q - a^{+}_q a^{-}_q = [N + 1] - [N] \] (50)

and from (38,39)\[19\]

\[ a^{+}_q = a^{+} \sqrt{\frac{[N+1]}{N+1}} = \sqrt{\frac{[N]}{N}} a^{+}, \] (51)

\[ a^{-}_q = a^{-} \sqrt{\frac{[N]}{N+1}} = \sqrt{\frac{[N+1]}{N+1}} a^{-}. \] (52)

Equations (50-52) give evidently the $q$-deformed oscillator \[13, 14, 15\], or in its original commutator form

\[ a^{-}_q a^{+}_q - q a^{+}_q a^{-}_q = q^{-N}. \] (53)

For the choice (49) the $*_R$ and $*_L$ products are of course in view of (31-32) given by

\[ *_R = \frac{[N]}{N} \cdot \frac{1}{[N] - [N - 1]} \] (54)

and

\[ *_L = \frac{[N+1]}{N+1} \cdot \frac{1}{[N+2] - [N+1]}. \] (55)

We notice however that this scheme includes also other deformed oscillators with different star-products induced by other choises of the parametrizing function $S(N)$. The following list indicates some alternatives:
$S(N)$ parametrizing $*_R,L$  

$N$  

$\frac{q^N - q^{-N}}{q - q^{-1}}$  

$q$-deformed oscillator [13, 14, 15]  

$\frac{q^N - 1}{q - 1}$  

asymmetric deformed oscillator [20]  

$\frac{q^N - p^{-N}}{q - p^{-1}}$  

$N(p + 1 - N)$  

two-parameter deformed oscillator [21, 22]  

para-fermionic oscillator [23]  

$\frac{\sin \hbar(\tau N) \sin \hbar((p+1)-N)}{\sin \hbar^{2}(\tau)}$  

deformed para-fermionic oscillator [24]  

$N^K$  

power deformed oscillator [25]  

$\frac{\sin(\tau N)}{\sin(\tau)}$  

elliptic deformed oscillator [25]  

Let us add that this list of alternatives is far from being exhausted. The $su(2)$ algebra, in accordance to the spirit of the present work, can also be derived by means of deformed star-products defined between the elements of the standard oscillator algebra, and similarly the relation among other algebras can also be casted in the deformation language as will be shown elsewhere.

We will now turn to the study of the generalized dynamics of the $q$-oscillator. We postulate, by analogy with the $*$-product case as described earlier, the following Heisenberg equation of motion[26, 27]

$$i\hbar \dot{A} = [A, H]_{*,RL}$$  

(56)

This is satisfied by the formal solution

$$A(t) = e^{i \hbar H L} A(0) e^{-i \hbar RH}$$  

(57)

as is easily verified.

The appearance of the $*_R,L$-commutator in (56) implies that the dynamics is not canonical i.e the Hamiltonian in general is not $*_R,L$-commuting with itself, therefore we have no conservation of energy. Moreover such dynamics is not an automorphism for the fundamental commutation relations of the $q$-oscillator algebra. These imply a non-canonical dynamics for the $q$-deformed oscillator which physically accounts for e.g. dissipation, constraints, or non-linear self interactions.

To extent our considerations relating the star-products with the algebra deformation, we now come to study generalized $*_R,L$-coherent states pertinent to the algebra of eq. (26). To this end we introduce a deformed exponential function in analogy to the $*$-product formalism of the classical case above.
Let

\[ e_{*RL}^x \equiv 1_{*RL} + \frac{1}{1!} x + \frac{1}{2!} x \Box x + \frac{1}{3!} x \Box x \Box x + \cdots \]

\[ \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (x \Box)^n \]  

(58)

where

\[ 1_{*RL} \equiv (\frac{R + L}{2})^{-1} \]

and \(\Box\) is the abbreviation (not a new product!), \(\Box \equiv \frac{1}{2}(*R + *L)\) i.e

\[ x \Box x = \frac{1}{2} x (*R + *L)x = \frac{1}{2} (xRx + xLx) \]  

(59)

Obviously \((x \Box)^2 \equiv x \Box x \xrightarrow[q \to 1]{\sim} x^2\) and also in the same zero deformation limit,

\[ R \to 1 \]
\[ L \to 1 \]

and

\[ e_{*RL}^x \xrightarrow[q \to 1]{\sim} e^x. \]

Two alternative forms of the \(*RL\)-exponential function given in terms of the ordinary exponential are the following (\(x\) any operator),

\[ e_{*RL}^x = (\frac{R + L}{2})^{-1} e^{(\frac{R+L}{2})x} \]  

(60)

and

\[ e_{*RL}^x = e^{x (\frac{R+L}{2})} (\frac{R + L}{2})^{-1}. \]

(61)

From above we deduce the following composition of the \(*RL\)-exponentials for any two commuting elements \(x, y\) (with \(N\) as well):

\[ e_{*RL}^x \Box e_{*RL}^y = e_{*RL}^{x + y}. \]  

(62)

Also for any two non-commuting elements \(x, y\) for which the ordinary BCH formula is valid, i.e \(e^x e^y = e^z\) etc. see eqs. (9-10), then the \(*RL\)-BCH formula is also satisfied i.e

\[ e_{*RL}^x \Box e_{*RL}^y = e_{*RL}^{x + y + z_2 + z_3 + \cdots} (\frac{R+L}{2}) \]  

(63)

where

\[ z_2 = \frac{1}{2} [x, y] \Box \]  

(64)

\[ z_2 = \frac{1}{12} [x, [x, y]] \Box + \frac{1}{12} [x, y] \Box [x, y] \Box \]  

etc.  

(65)
with the symbol (not a new *-commutator!),
\[
[x, y] □ \equiv x □ y - y □ x \equiv \frac{1}{2} x (\star_R + \star_L) y - \frac{1}{2} y (\star_R + \star_L) x
\]
\[
= x \left( \frac{R + L}{2} \right) y - y \left( \frac{R + L}{2} \right) x .
\] (66)

We notice that if \([x, y] □ = 0\), which means
\[
[x, y]_{\star_{RL}} = 0 = [x, y]_{\star_{LR}},
\]
then
\[
\epsilon^{\Re} \otimes \epsilon^{\Im} = \epsilon^{\Re + \Im}
\] (67)

We finally introduce the \(\star_{RL}\)-coherent states,
\[
|\alpha\rangle := \epsilon^{\alpha \Re + \alpha \Im} |0\rangle = \left( \frac{R + L}{2} \right)^{-1} \epsilon^{(\frac{R + L}{2}) \alpha \Re} |0\rangle
\]
\[
= \epsilon^{\alpha \Re \left( \frac{R + L}{2} \right)} \left( \frac{R + L}{2} \right)^{-1} |0\rangle
\] (68)

where \(|0\rangle\) stands for the vacuum Fock state. Calling \(\frac{R(N) + L(N)}{2} = f(N)\), and utilizing the number state \(|n\rangle\), on which \(f(N)|n\rangle = f(n)|n\rangle\) we obtain the expansion of the coherent state in the Fock states,

\[
|\alpha\rangle = f^{-1}(0) \left\{ |0\rangle + \alpha \sqrt{2!} f'(0) |1\rangle + \frac{\alpha^2}{\sqrt{3!}} f''(1) f(0) |2\rangle + \frac{\alpha^3}{\sqrt{4!}} f'''(2) f(1) f(0) |3\rangle + \cdots \right\}
\] (69)

The normalized \(\star_{RL}\)-CS is taken to be
\[
|\alpha\rangle := \frac{1}{\sqrt{(|\alpha\rangle |\alpha\rangle)}} |\alpha\rangle
\] (70)

where the overlap which provides the normalization factor reads,

\[
(\alpha |\alpha\rangle) = f^{-2}(0) + |\alpha|^2 \frac{1}{1!} f(0) |1\rangle + \frac{|\alpha|^2}{2!} f(1) |2\rangle + \frac{|\alpha|^3}{3!} f(2) f(1) |3\rangle + \cdots .
\] (71)

It is straightforward to prove that this coherent state is eigenstate of the annihilation operator in the sense that,

\[
f^{-1}(N) a f(N) \ |\alpha\rangle = \alpha |\alpha\rangle
\] (72)

or that

\[
a \cdot \left( \frac{R(N) + L(N)}{2} \right) \ |\alpha\rangle = \alpha \left( \frac{R(N) + L(N)}{2} \right) |\alpha\rangle
\] (73)

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which with the abbreviation issued by eq. (59) becomes,

\[ a \Box |\alpha > = a \Box |\alpha > . \]  \hspace{1cm} (74)

This eigenvalue problem is satisfied for any choice of the *_{RL}-products i.e. for any \(q\)-oscillator. Before closing some remarks are in order. Despite the demonstrated generality of the *-product approach to the deformed oscillator we should note that all the particular choises above have the *_{R} and *_{L} products unequal, namely \( R(N) \neq L(N) \) and they are both depended on \( N \). Open are left a large class of deformed products (new \(q\)-oscillator) where *_{R} and *_{L} are symmetric (i.e. \( R(N) = L(N) \)). Also cases of asymmetric products such as \( R(N) = 1 \) and \( L(N) \neq 1 \) are challenging, or cases where one (or both) of the *-products are c-numbers (e.g. \( R(N) = f(q) \) and \( L(N) = f^{-1}(q) \)) etc. are also interesting to explore. The selection of the *-product is dictated only from the physical relevance of the ensuing \(q\)-deformed oscillator.

Finally, concerning future developments of the formalism put forward here it would be interested to look at the classical level and search for a classical version of the *_{RL} -deformation of the commutator. This will require in addition to the \( \hbar \) -deformation of the Poisson bracket, expressed by the Moyal bracket, a new \(q\) -deformation so that a two-parameter deformation scheme should be put forward [28]. We expect to be able to formulate that problem using the r-matrix theory of the Lie-Poisson algebras[29] and we aim to return to these matters elsewhere.
References


Chaichian M Ellinas D and Popowicz Z (1990) Phys. Lett. 248B 95, in these papers generalized two and three-parameter deformed Jacobi identities where introduced.


[27] This deformation of the dynamical equation is similar to that suggested by ref.22, it would be interested to investigate further the relationships between the present formalism with that put forward in ref.22.
