Graphical Representation of Invariants and Covariants in General Relativity *

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Abstract

We present a graphical way to describe invariants and covariants in the (4 dim) general relativity. This makes us free from the complexity of suffixes. Two new off-shell relations between (mass)**6 invariants are obtained. These are important for 2-loop off-shell calculation in the perturbative quantum gravity. We list up all independent invariants with dimensions of (mass)**4 and (mass)**6. Furthermore the explicit form of 6 dim Gauss-Bonnet identity is obtained.

§1. Introduction

In relation to the development of the unified theory and the investigation of the initial stage of the universe, the physical importance of the quantum gravity grows more and more. These several years the statistical aspect of the (euclidean) quantum gravity (critical dimension, etc.) has been vigorously investigated and has been clarified

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using lower-dimensional models. The renormalizability problem, however, does not go beyond the pioneering work by 'tHooft and Veltman [TV]. The problem seems crucial for approaching the quantum gravity perturbatively [II, I2]. In such research we must treat invariants with higher mass-dimensions: \((\text{mass})^{2+2n}\) for \(n\)-loop perturbation-order (in the ordinary gauge). In this circumstance it seems important to develop a systematic and convenient way to deal with those invariants. We present a graphical way, which enables us visually discriminate among various invariants. It makes us free from complexity of suffix-contraction and reduces the algebraic computation considerably. Mathematically this is a representation of invariants (covariants) in terms of graphs.

As is well known, the symmetry of Riemann curvature tensor is not so simple (see Sect.2). So far there has been no systematic and practically useful method to construct independent invariants [P,G,VW]. In particular the relations between invariants at the off-shell level (that is, with no use of a field equation) have not been so much investigated. It is known, however, that those relations are so important to investigate the quantum gravity [VW,II, I2]. We present a new method using a graph and derive all off-shell relations among the \((\text{mass})^{4}\) invariants and among the \((\text{mass})^{6}\) invariants.

In Sect.2 we list all well-known symmetries and relations of invariants and covariants in the general relativity. A graphical representation of Riemann tensor and the covariant derivative is introduced in Sect.3. Three rules, which are basic relations in the graphical calculation, are introduced in Sect.3. Three rules, which are basic relations in the graphical calculation, are presented. In Sect.4, all invariants with dimension of \((\text{mass})^{4}\) and with dimension of \((\text{mass})^{6}\) are expressed graphically. All listed graphs are classified in Sect.5. We reduce, in Sect.6, the number of \((\text{mass})^{6}\) invariants to 8 by using the three rules above and two new off-shell relations. In Sect.7 we comment on the independence of finally listed invariants in Sect.6. Furthermore the explicit form of Gauss-Bonett identity in 6 dim space-time is obtained.
§2. Preliminaries

Before the main text, we summarize the present notation and list up all well-known symmetry properties of covariants.

1. Notation

\[ R_{\nu\alpha\beta}^\mu = \partial_\alpha \Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\alpha\gamma} \Gamma^\gamma_{\nu\beta} - (\alpha \leftrightarrow \beta), \quad \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}) , \]

\[ R_{\mu\nu} = R_{\mu\nu}^\alpha , \quad R = R_{\mu}^\mu . \] (2.1)

From these definitions, we know the mass dimension of the Riemann curvature tensor, \([R_{\nu\alpha\beta}^\mu] = (\text{mass})^{*2} , \) under the definition : \([g^{\alpha\beta}] = (\text{mass})^{*0} , [\partial_\alpha] = (\text{mass})^{*1} .

2. Symmetry

\[ R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} = - R_{\nu\mu\lambda\sigma} = - R_{\mu\nu\sigma\lambda} , \] (2.2a)

\[ R_{\mu\nu\lambda\sigma} + R_{\nu\lambda\mu\sigma} + R_{\lambda\mu\sigma\nu} = 0 , \] (2.2b)

\[ R_{\mu\nu} = R_{\nu\mu} , \] (2.2c)

\[ \nabla_\delta R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\delta} + \nabla_\nu R_{\alpha\beta\delta\mu} = 0 \ (\text{Bianchi identity}) , \] (2.2d)

\[ \nabla^\alpha R_{\mu\nu\lambda\sigma} = \nabla_\mu R_{\nu\lambda\sigma} - \nabla_\nu R_{\mu\lambda\sigma} , \] (2.2e)

\[ \nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 , \] (2.2f)

In addition to the above symmetries which are derived by the manipulation of local quantites, there exist one relation ,between (mass)*4 invariants, which is related to a global (topological) quantity.

3. Gauss-Bonnet identity(in 4 dim)

\[ \int d^4x \sqrt{g} R_{\mu\nu\alpha\beta} R_{\lambda\sigma\gamma\delta} \varepsilon^{\mu\nu\lambda\sigma} \varepsilon^{\alpha\beta\gamma\delta} = \]

\[ 4 \int d^4x \sqrt{g} ( R^2 + R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} ) = \text{topological invariant} , \] (2.3)

where \( \varepsilon^{\mu\nu\lambda\sigma} \) is the totally antisymmetric tensors.

4. covariant ingredients
In Table 1, we list up all covariant ingredients that make up invariants.

<table>
<thead>
<tr>
<th>Dim \ Suffix #</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(mass)**0</td>
<td></td>
<td></td>
<td>$g_{\mu\nu}$</td>
<td>$\varepsilon_{\mu\nu\lambda\sigma}$</td>
</tr>
<tr>
<td>(mass)**1</td>
<td>$\nabla_\mu$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(mass)**2</td>
<td>$R$</td>
<td>$R_{\mu\nu}$</td>
<td>$R_{\mu\nu\lambda\sigma}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1

§3. A Graphical Way to Represent Covariants and Invariants

We introduce a graphical representation for Riemann tensor, $R_{\mu\nu\lambda\sigma}$, as in Fig.1.

Note the following items.

**A1.** The graph is expressed by two kinds of lines (dotted lines and solid lines) and two *vertex points* which connect two different kinds of lines. We call the dotted line the *suffix line* because it represents the suffix flow. The arrow indicates the direction of the suffix-flow. The suffix which flows in (out) is defined to be the left (right) suffix of each doublet ($\mu\nu$) or ($\lambda\sigma$) which appear in $R_{\mu\nu\lambda\sigma}$. The solid line represents the Riemann tensor itself.

**A2.** We need not distinguish between upper and lower suffixes because we are interested only in invariants. ($R^\nu_{\mu\lambda\sigma}$, $R^\nu_{\mu\lambda\sigma}$, $R^{\mu\nu\lambda\sigma}$, $\cdots$ are also represented by
A3. We may write the graph in any way we like unless we change the graph topologically and change the suffix flow (Fig.2a).

A4. We may freely move the vertex point unless it jumps another vertex point (Fig.2b).

A5. Different lines may freely cross each other (Fig.2c, Fig.2d).
The graph satisfies the symmetries (2.2a) which Riemann tensor $R_{\mu\nu\lambda\sigma}$ has (Fig.3).

We represent the contraction of two suffixes simply by connecting them by a suffix line. Examples are given in Fig.4a,b and c which express $R^\mu_{\mu,\lambda\sigma} = 0$, $R^\mu_{\nu,\lambda\mu} = R_{\nu,\lambda\sigma}$, $R^\rho_\nu = R$ respectively.

We note the following facts.

**B1.** From the definition of the arrow, the graph changes its sign under the change
of the suffix flow (Fig.5a).

**B2.** Generally all suffix-lines are closed for invariants (Fig.4c). Fig.4c, which represents \( R \), is the simplest case of the graphical representation of invariants: the \( (\text{mass})^2 \) invariant.

**B3.** The arrow within a suffix line (closed or not-closed) may be omitted if even number of solid lines are connected to the line because there exists no ambiguity in the direction of arrow for such a case (Fig.4b, Fig.5b).

**B4.** Fig.4a says 'tadpole' graphs are prohibited.

We can express the relation (2.2b) by a graphical rule: Rule 1 (Fig.6).
If we make use of the graphical representation, various relations between products of Riemann tensors are easily obtained. We demonstrate here a simple example: the relation of $R_{\gamma\beta\tau\omega} R_{\alpha}^{\tau\omega \beta \gamma} = \frac{1}{4} R_{\gamma\beta\tau\omega} R_{\alpha}^{\omega\tau \alpha\beta}$ (Fig.7) and its proof using Rule 1(Fig.8). We call the graphical relation of Fig.7 Rule 2 since it will be used as one of basic relations in the following.
Similarly we can obtain another useful graphical rule: Rule 3 (Fig.9a), using Rule 1.
By contracting the suffixes $\mu$ and $\lambda$ in Fig.9a, we obtain Fig.9b. Furthermore Fig.9c is obtained by contracting $\nu$ and $\sigma$ in Fig.9b. We add here some notes.

**C1.** Rule 2 is useful when we want to increase or decrease the number of suffix-loops.

**C2.** Rule 3 is useful when we want to change the flow of a suffix line.

In addition to the graphical representation for the curvature tensors, we introduce that for the covariant derivative as in Fig.10a($\nabla_\alpha R_{\mu\nu\lambda\sigma}$) and Fig.10b($\nabla_\beta \nabla_\alpha R_{\mu\nu\lambda\sigma}$).
In Fig.10a and b, the symbol $\nabla$ represents the covariant derivative. We must depict it noting the following points.

**D1.** The symbol $\nabla$ must be written near enough a solid line to show clearly which Riemann tensor the derivative acts on.

**D2.** The order of the covariant derivatives is defined by; from the right to the left.

We can represent the Bianchi relations (2.2d),(2.2e) and (2.2f) as in Fig.11a, Fig.11b
and Fig.11c respectively.

§4. Graphical Representation of Invariants

Now we list up all invariants with a fixed mass-dimension and represent them graphically.

§4.1 Invariants with Dimension (Mass)∗∗ 4

We can easily find the three invariants, with the mass dimension (mass)∗∗ 4, as shown in Fig.12a,b and c which represent $R^2$, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ respectively.
Note that we have the relation Fig.9c and we need not consider the total derivative 
\[ \nabla_\mu \nabla^\mu R. \]

It is well known, in 4 dim quantum gravity, that there exists one relation between three invariants above: Gauss-Bonnet identity (2.3). Here we note that the integrand of (2.3), 
\[ R_{\mu \nu \alpha \beta} R_{\lambda \sigma \gamma \delta} \epsilon_{\mu \nu} \epsilon_{\alpha \beta \gamma \delta}, \]
can be expressed graphically as in Fig.13.

One part (say, the upper part) of the contracted suffixes (the upper and lower suffixes) are explicitly written in Fig.13 in order to specify which suffixes are antisymmetrized. We will compare Fig.13 with its 6-dim counter-part in sect.6.

Therefore we can take two out of three invariants above, as (locally) independent invariants; say,

\[ \begin{align*} 
R^2 & , \\
R_{\mu \nu} R^{\mu \nu} 
\end{align*} \]

(4.1)

We will comment on their independence in sect.6.
§4.2 Invariants with Dimension (Mass)* *6

There exist two types of invariants with the mass dimension (mass)* *6: $\nabla R \times \nabla R$ and $R \times R \times R$.

(i) $\nabla R \times \nabla R$ (Fig.14a,b)

Fig.14a and b represent $O_1 \equiv \nabla^\mu R \cdot \nabla_\mu R$ and $O_2 \equiv \nabla^\mu R_{\alpha\beta} \cdot \nabla_\mu R^{\alpha\beta}$ respectively.

(ii) $R \times R \times R$

Products of three curvature tensors can be listed as in Fig.15a-f and Fig.16a-f. The number (0 or 2 or 4) indicates that of suffixes each curvature tensor has. (0,2 and 4 imply the scalar curvature $R$, Ricci curvature tensor $R_{\mu\nu}$, and Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$ respectively. We call here all these quantities curvature tensors.)
Fig.15a,b,c,d,e and f represent \( P_1 = R R R, P_2 = R R \rho_\nu R^{\rho_\nu}, P_3 = R R \rho_\mu\lambda_\sigma R^{\mu\nu\lambda\sigma}, P_4 = R_\rho_\nu R^{\rho_\lambda} R^{\lambda_\mu}, P_5 = R_\rho_\nu\lambda_\sigma R^{\rho_\lambda} R^{\nu\sigma}, P_6 = R_\rho_\nu\lambda_\sigma R^{\nu\lambda\sigma} R^{\mu\sigma}, \) respectively.

Note that \{ \( O_i \) \} and \{ \( P_i \) \} are those invariants which vanish on shell, \( R_\rho_\nu = 0 \); the field equation of Einstein-Hilbert lagrangian \( \mathcal{L} = (1/\kappa) \sqrt{g} R \). The remaining kind are \( (4,4,4) \); product of three Riemann curvature tensors which does not vanish on shell. We can list them up as in Fig.16a-f.
Fig. 16a, b, c, d, e and f represent $A_1 = R_{\mu\nu\lambda\sigma} R^{\alpha\infty}_{\lambda\tau\omega} R^{\alpha\infty\nu\mu}$, $B_1 = R_{\mu\nu\tau\sigma} R^{\nu}_{\lambda\omega} R^{\lambda\mu\sigma\omega}$, $B_2 = R_{\mu\nu\omega\tau} R^{\lambda\infty}_{\lambda\tau\omega} R^{\infty\mu\nu\lambda}$, $B_3 = R_{\mu\nu\omega\tau} R^{\nu}_{\lambda\tau} R^{\lambda\mu\sigma\omega}$, $C_1 = R_{\mu\nu\sigma\tau} R^{\nu}_{\nu\lambda} R^{\nu\lambda\sigma\tau\omega}$, and $C_2 = R_{\mu\nu\sigma\tau} R^{\nu}_{\nu\lambda} R^{\nu\lambda\sigma\nu\mu}$, respectively.

We will derive all relations between \{ $O_i, P_i, A_i, B_i, C_i$ \} in Sect. 6. (Here we list the relation between invariants defined in other references and those defined in the present paper. As for [VW], $A_1^{VW} = A_1$, $A_2^{VW} = -B_1$, $A_3^{VW} = C_2$. As for [11], $O_1^I = P_1$, $O_2^I = P_2$, $O_3^I = P_4$, $O_4^I = P_5$, $O_5^I = P_3$, $O_6^I = P_6$, $O_7^I = -A_1$, $O_8^I = O_1$, $O_9^I = O_2$. )
§5. Classification of Graphs

Here we classify all graphs, which have appeared in previous sections, in order to confirm no missing graphs. Let us consider a graph with $N_R$ solid lines and $N_E$ external suffix lines. Its mass-dimension is $(\text{mass})^{2N_R}$. The number of vertices is $V = 2N_R$. The number of internal suffix lines, $N_I$, equals the number of suffix-contraction and is given by $N_I = 2N_R - \frac{1}{2}N_E$. The number of total (internal and external) suffix lines is given by $S = N_I + N_E = 2N_R + \frac{1}{2}N_E$. Particularly the relation $3V = 2(N_I + N_R) + N_E$ holds true. Some simple examples are listed in
Table 2.

<table>
<thead>
<tr>
<th>Graph</th>
<th>$N_R$</th>
<th>$N_E$</th>
<th>$V$</th>
<th>$N_I$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2

1) $N_E = 0$

In this case the graph in our consideration is an invariant. All suffix lines are internal and make up a number of closed-loops. Let us define the number of those closed loops which have $n$ vertices, $L_n$. There cannot exist 'self-energy' graphs of purely suffix lines. This means $L_0 = 0$. No tadpole graphs (B4 in Sect.3) means $L_1 = 0$. 

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Then the following relation holds true.

\[ N_I = 2L_2 + 3L_3 + 4L_4 + \cdots \quad , \quad L_n : \text{non-negative integer} \quad (5.1) \]

We can use this formula to classify all invariants. Three cases \( N_R = 1, 2, 3 \) are classified in Table 3 and corresponding graphs in the previous sections are given.

<table>
<thead>
<tr>
<th>( N_R )</th>
<th>( N_I = 2N_R )</th>
<th>( (L_n \not= 0) )</th>
<th>Corresponding Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>( L_2 = 1 )</td>
<td>Fig.4c</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>( L_2 = 2 )</td>
<td>Fig.12a (disconnected), Fig.12c</td>
</tr>
<tr>
<td>( L_4 = 1 )</td>
<td>Fig.12b</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>( L_2 = 3 )</td>
<td>Fig.15a (disconnected), Fig.15c (disconnected), Fig.16a</td>
</tr>
<tr>
<td>( L_3 = 2 )</td>
<td>Fig.16b, Fig.16d</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_2 = 1, L_4 = 1 )</td>
<td>Fig.15b (disconnected), Fig.15f, Fig.16c</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_6 = 1 )</td>
<td>Fig.15d, Fig.15e, Fig.16e, Fig.16f</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3

2) \( N_E \not= 0 \)

The graph in our consideration is a covariant (or contravariant) with \( N_E \) suffixes. Among internal suffix lines, some of them make up closed-loops. The other ones do not make up loops. They are linked, by vertices, directly to external suffix lines or to adjacent internal suffix lines which are linked finally, through successive linkage, to external suffix lines. Let us define the number of such non-loop internal suffix lines, \( K \). Then we have

\[ N_I = 2L_2 + 3L_3 + 4L_4 + \cdots + K \quad , \quad L_n : \text{non-negative integer} \quad (5.2) \]

We define the number of those sets of non-loop internal suffix lines which are linked adjacently by \( n \) vertices, \( K_{n-1} \), for \( n \geq 2 \). Then we have

\[ K = K_1 + 2K_2 + 3K_3 + \cdots \quad , \quad K_n : \text{non-negative integer} \quad (5.3) \]
Some simple examples are listed in Table 4. (5.2) and (5.3) are substituted for (5.1) in the present case of \( N_E \neq 0 \).

<table>
<thead>
<tr>
<th>Graph</th>
<th>( N_R )</th>
<th>( N_E )</th>
<th>( L_n \neq 0 )</th>
<th>( K_n \neq 0 )</th>
<th>( N_I )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>( L_2 = 1 )</td>
<td>( K_1 = 1 )</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2</td>
<td>( L_3 = 1 )</td>
<td>( K_2 = 1 )</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2</td>
<td>( L_4 = 1 )</td>
<td>( K_3 = 1 )</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4

As an example, let us classify the case \( N_R = 2, N_E = 4 \). (5.2) and (5.3) are written as

\[
2 = 2L_2 + 3L_3 + 4L_4 + \cdots + K ,
\]

\[
K = K_1 + 2K_2 + 3K_3 + \cdots , \quad (5.4)
\]
All possible cases for the choice of \((L_n)\) and \((K_n)\) are listed up in Table 5.

<table>
<thead>
<tr>
<th>(K)</th>
<th>((L_n) \neq 0)</th>
<th>((K_n) \neq 0)</th>
<th>(N'_E)</th>
<th>Connected Graphs</th>
<th>Disconnected Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(L_2 = 1)</td>
<td></td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(K_2 = 1)</td>
<td>2</td>
<td></td>
<td>No Graph</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>(K_1 = 2)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5

In Table 5, \(N'_E\) is the number of those external suffix lines which are not linked to internal suffix lines. \(N'_E\) is given by \(N'_E = N_E - 2(1 + 2 + \cdots)\). The con-
nected graphs with \( N_L^E = 0 \) correspond to 'one-particle irreducible' graphs. The all connected graphs in Table 5 appear in Fig.7 (Rule 2) and in Fig.9a (Rule 3).

§6. Application of Graphical Representation

We can prove the following relations among \( \{ A_1, B_i, C_i \} \) in the way similar to the proof of Rule 2 (Fig.8) in Sect.3. We list the relations with the names of rules which are necessary for their proofs.

\[
B_2 = \frac{1}{2} A_1 \quad \text{(Rule 2)}, \quad B_3 = C_2 - C_1 \quad \text{(Rule 3)} = -B_1 \quad \text{(relation below)}, \quad C_1 = \frac{1}{4} A_1 \quad \text{(Rule 2, Rule 2)},
\]

\[
C_2 = C_1 - B_1 \quad \text{(Rule 1)} = \frac{1}{4} A_1 - B_1 \quad \text{(previous relation)}.
\]

Therefore we see all invariants of the kind (4,4,4) are described only by \( A_1 \) and \( B_1 \) [VW].

There exist a different kind of relations which come not from the rules in sect.3 but from the dimension of the space-time: 4.

Off-Shell Relation 1

Let us consider the identity of Fig.17. This idea was explicitly noticed in [GS].

Note that the identity Fig.17 holds true because each Greek suffix runs from 0 to 3 (or from 1 to 4 for the Euclidean gravity) in 4 dim space-time. This identity can be
written as (by use of a computer)

\[-P_2 + \frac{1}{2}P_3 + 2P_4 - 4P_5 - 5P_6 + A_1 - 2B_1 = 0\]  \quad (6.2)

The on-shell case of (6.2), \(A_1 = +2B_1\), was obtained in [VW] by use of the spinor formalism.

Off-Shell Relation 2

Similarly we can consider the identity of Fig.18.

This identity can be written as (by use of a computer)

\[I \equiv 8\left(-P_1 + 12P_2 - 3P_3 - 16P_4 + 24P_5 + 24P_6 - 4A_1 + 8B_1\right) = 0\]  \quad (6.3)

The on-shell case of (6.3) again gives \(A_1 = +2B_1\). At the off-shell level, however, (6.2) and (6.3) are independent relations.

Off-Shell Relation 3

From the relations (6.2) and (6.3), we obtain the relation between \(\{P_i\}\) as follows.

\[-P_1 + 8P_2 - P_3 - 8P_4 + 8P_5 + 4P_6 = 0\]  \quad (6.4)

This relation can also be directly derived by the identity of Fig.19.

The independent relations are two out of the above three relations (6.2),(6.3) and
(6.4). We have checked any other choice of anti-symmetrized suffixes and a starting graph, in the above procedure, does not lead to another independent relation.

Therefore we can list up $8 = 2(O_i) + 6(P_i) + 2(A_1, B_1) - 2$ (off-shell relations) independent (mass)** 6 invariants, say, as follows. \(^1\)

\[
O_1 , \; O_2 , \; P_1 , \; P_2 , \; P_3 , \; P_4 , \; P_5 , \; A_1 . \quad (6.5)
\]

As for their independence we make comment in Sect.7. The importance of $A_1$, which is the unique, in (6.5), non-vanishing term on shell, was first pointed out in [K].

§7. Independence of Invariants, 6 Dim Gauss-Bonnet Identity and Conclusion

The final proof of the independence of the listed invariants (4.1) and (6.5) can be done by the weak-gravity expansion approach. Let us introduce the 'linear field' $h_{\mu \nu}$ instead of the metric $g_{\mu \nu}$ by considering the case of weak gravity:

\[
g_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu} \quad |h_{\mu \nu}| \ll 1 \quad , \quad (7.1)
\]

where $\delta_{\mu \nu}$ is the Minkowski metric (the flat space-time). Each invariants can be expanded around the flat metric and can be expressed by the infinite series of powers of $h_{\mu \nu}$ (with derivatives). We have checked the independence of the listed invariants (4.1) and (6.5) by looking at the first few orders of $h_{\mu \nu}$ by use of a computer.

The procedure presented in this paper is valid for any space-time dimension. All relations in this paper, except (2.3) (Fig.13), (6.2) (Fig.17), (6.3) (Fig.18) and (6.4) (Fig.19), are valid irrespective of the space-time dimension. Particularly in 6 dim space-time, we can obtain the explicit form of Gauss-Bonnet identity as follows.

\[
\int d^6 x \sqrt{|g|} R_{\mu \nu \alpha \beta} R_{\lambda \sigma \tau \xi} e^{\mu \nu \lambda \sigma \tau \xi} e^{\alpha \beta \gamma \delta \epsilon} = \int d^6 x \sqrt{|g|} (\text{Left Hand Side of Fig.18})
\]

\[
= \int d^6 x \sqrt{|g|} I = \text{topological invariant} \quad , \quad (7.2)
\]

\(^1\)In [GS], 9 terms are listed. They missed the relation (6.4).
where $I$ has appeared in (6.3) and the above formula is explicitly re-written as

$$
8 \times \int d^8 x \sqrt{|g|} \left( -R^3 + 12 R R_{\mu \nu} R^\mu R^\nu - 3 R R_{\mu \nu \lambda \sigma} R^{\mu \nu \lambda \sigma} - 16 R_{\mu \nu} R^\nu R^\lambda R^\mu \right)
$$

$$
+ 2 A R_{\mu \nu \lambda \sigma} R^{\mu \lambda} R^{\nu \sigma} + 2 A R_{\mu \nu \lambda \sigma} R^{\nu \lambda} R^{\mu \tau} - 4 A R_{\mu \nu \lambda \sigma} R^{\nu \tau} R^{\lambda \tau} R^{\nu \mu} + 8 A R_{\mu \nu \tau} R^{\nu \lambda \mu} R^{\lambda \nu \mu} \right)
$$

= topological invariant. \quad (7.3)

It is known, in 6 dim quantum gravity, 1-loop counterterms are given by \text{(mass)}**6 invariants in the ordinary gauge\cite{VW}. They are given by the linear combination of the \text{9} \left( =2(O_1)+6(P_i)+2(A_1, B_1) - 1 \right) \text{Gauss-Bonnet identity}) \text{ independent invariants, say, as follows.}

$$
O_1, \quad O_2, \quad P_1, \quad P_2, \quad P_3, \quad P_4, \quad P_5, \quad P_6, \quad A_1. \quad (7.4)
$$

Because $A_1$ does not vanish on shell, 1-loop counterterms, in 6 dim quantum gravity, do not trivially vanish on-shell. The importance of this fact was stressed in \cite{VW}. It must be compared with 4 dim case (4.1), where 1-loop counterterms trivially vanish.

The relation (7.3) and the left-hand-side of Fig.18 are the 6 dim counter-part of the relation (2.3) and Fig.13 in 4 dim.

Finally we comment on those invariants which have not been considered in the present paper.

1. The extension to invariants with higher mass-dimension (mass)**8, \ldots \text{in 4 space-time dimension, is straightforward. They are important in higher-order perturbative quantum gravity.}

2. Pseudoscalars: $\epsilon^{\mu \nu \lambda \sigma} R_{\mu \nu \alpha \beta} R_{\lambda \sigma}^{\alpha \beta}, \epsilon^{\mu \nu \lambda \sigma} R_{\mu \nu \alpha \beta} R_{\lambda \sigma \gamma \delta} R^{\alpha \beta \gamma \delta}, \ldots$. They does not appear in the quantum effect in Einstein-Hilbert theory. They might appear in the unified theory including the gravitational and weak interactions.

3. There are some non-local invariants which become local in a specific gauge.

The famous example is $R + 2 R$ in the 2 dim gravity ,which is local with the conformal gauge\cite{PP} It was exploited as the solvable model of 2 dim Euclidean quantum gravity. Similar non-local invariants are discussed in 4 dim \cite{AM}. At present, however, the role of those non-local invariants remain obscure in 4 dim space-time.

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We recall that Feynman diagrams (graphs), which represent various scattering processes of particles, have been playing a very important role in the practical and formal development of QED, QCD and other renormalizable theories. We hope the present approach will play an analogous role in the further progress of the quantum gravity.

The results (6.2) and (6.3) and the proof of the independence of terms in (4.1) and (6.5) by the weak-gravity expansion are obtained by the algebraic software FORM [V.VV].

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References


\(^2\)The symbolic manipulation program FORM was written by J.A.M. Vermaseren. Version 1.0 of the program and the manual are available via anonymous ftp from nikhef.nikhef.nl.


