COSMOLOGICAL SOLUTIONS IN 2D POINCARÉ GRAVITY

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ABSTRACT
The 2D model of gravity with zweibeins $e^a$ and the Lorentz connection one-form $\omega^a_b$ as independent gravitational variables is considered. The solutions of classical equations of motion which can be interpreted as cosmological ones are studied.

Numerous recent attempts to formulate the theory of gravity in the framework of a consistent gauge approach resulted in constructing the gauge gravity models for the de Sitter and Poincaré groups. The independent variables are zweibeins $e^a = e^a_\mu dx^\mu$ and Lorentz connection one-form $\omega^a_b = \omega^a_{b\mu} dx^\mu$. The application of these methods in two dimensions was justified by attempts to give an alternative description of two-dimensional dynamical gravity in terms of variables $(e^a, \omega^a_b)$. It was argued also that investigation of simple two-dimensional model leads to a better understanding of four-dimensional gravity and its quantization$^{1,2,3}$. The classical equations of motion were analyzed in conformal gauge in Ref.$^1$ and in light cone gauge in Ref.$^2$ and their exact integrability was demonstrated. The canonical quantization of the model was recently studied in Ref.$^3$. In Ref.$^4$ was shown that classical equations of motion are exactly integrated in coordinate system determined by components of 2D torsion. Here we study in more details the solutions found in Ref.$^4$ which can be interpreted as the cosmological ones.

In two dimensions the gauge gravity is described in terms of zweibeins $e^a = e^a_\mu dx^\mu, a = 0, 1$ (the 2D metric on the surface $M^2$ has the form $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$) and Lorentz connection one-form $\omega^a_b = \omega e^a_b, \omega = \omega_\mu dx^\mu$ ($\varepsilon_{ab} = -\varepsilon_{ba}, \varepsilon_{01} = 1$). The curvature and torsion two-forms are: $R = d\omega, T^a = de^a + \varepsilon^a_{\beta\gamma} \omega \wedge e^\beta \wedge e^\gamma$.

The dynamics of gravitational variables $(e^a, \omega)$ is determined by the action:

$$ S = \frac{\alpha}{2} \int_M * T^a \wedge T^a + \frac{1}{2} * R \wedge R - \frac{\lambda}{4} \varepsilon_{ab} e^a e^b $$

(1)

where $*$ is the Hodge dualization and $\alpha, \lambda$ are arbitrary constants and we fixed constant in front of the second term.

Let us consider variables $\rho = * R$ and $q^a = * T^a$. Variation of action (1) with
respect to zweibeins $e^a$ and Lorentz connection $\omega$ leads to the following equations of motion:

$$d\rho = -aq^a_{\phi b} e^b$$

$$\nabla q^a = -\frac{1}{2\alpha}[\rho^2 + aq^2 - \alpha^2 - \Lambda\alpha]e^a_{\phi b},$$

where $\nabla q^a \equiv dq^a + \omega e^a_{\phi b} q^b$; here $q^2 = q^a q^b \eta_{ab} (\eta_{ab} = diag(+1, -1))$. In (2), (3) the following notation was introduced: $\Lambda = \frac{\Lambda}{\alpha} - \alpha$.

One particular solution of (2)-(3) is evident. Assuming $q^2 = \text{constant}$ one gets from (2)-(3), provided $e^a$ are linearly independent everywhere on $M^2$:

$$\rho = \pm \alpha, \quad q^a = 0$$

in all points of the two-dimensional manifold. That is, torsion is zero and $M^2$ is the de Sitter space.

In Ref. 4 was shown that if $q^2$ be non zero identically everywhere in $M^2$ then the general solution of (2), (3) is easily found in coordinate system determined by components of 2D torsion. More exactly, assuming (for definiteness) that $q^2 = (q^0)^2 - (q^1)^2 > 0$, one can write the torsion components in the form: $q^0 = q \cosh \phi$, $q^1 = q \sinh \phi$. It follows from (2), (3) that $q^2$ is function of curvature $\rho$:

$$q^2(\rho) \equiv -\frac{1}{\alpha}(\rho + \alpha)^2 + \Lambda + \epsilon \epsilon \frac{\hat{\phi}}{\hat{\phi}},$$

where $\epsilon$ is integrating constant, which is proportional to ADM mass. So we may consider $\rho, \phi$ as the new local coordinates on $M^2$. Then as shown in Ref. 4 the metric has the form:

$$ds^2 = q^2(\rho) \exp \left(-\frac{2\rho}{\alpha}(d\phi)^2 - \frac{1}{\alpha^2q^2(\rho)}(d\rho)^2 \right)$$

where $q^2(\rho)$ is known function (5). This metric can be rewritten in Schwarzschild like form. Indeed, let change variable: $r = e^{-\frac{\phi}{\epsilon}}$. Then the metric (6) takes the form:

$$ds^2 = g(r) dt^2 - \frac{1}{g(r)} dr^2$$

$$g(r) = c r + \Lambda r^2 - \alpha r^2 (1 - \ln r)^2.$$  

Generally (7) describes the black hole of charged type with mass $M = \frac{\Phi}{2\pi}$ embedded in the de Sitter space-time4. If $g(r) \leq 0$ everywhere then metric (7) is the nonstationary but homogeneous one ($\rho$ is the time coordinate in this case) and describes 2D cosmology. One may identify in this case points $\phi = 0$ and $\phi = \phi_0$. The metric (7) describes the space-time of topology $S^1 \oplus R$. The solutions with such a topology are standard subject for canonical quantization3. Therefore it is of interest to see what they describe on the classical level. Since $r$ is time coordinate we may denote: $r \rightarrow t$. Then metric (7) can be written as follows:

$$ds^2 = \frac{1}{f(t)} dt^2 - f(t) d\phi^2$$
where $f(t) = -g(t)$. This metric can be written in standard "cosmological" form:

$$\sqrt{f(t')} \frac{dt'}{d\tau} = d\tau^2 - a(\tau) d\phi^2,$$

where the cosmological time $\tau$ and cosmic scale factor $a(\tau)$ are defined as follows:

$$\tau = \int^t \frac{dt'}{\sqrt{f(t')}} ; \quad a(\tau) = f(t(\tau))$$

The metric (9) describes the expanding of closed one-dimensional Universe. The eq.(5) and equation $\rho = -\alpha \ln t(\tau)$ show the evolution of torsion and curvature with cosmological time $\tau$.

We begin with consideration of 2D de Sitter solution (4). Let us choose coordinates $(t, \phi)$ in which the metric takes the form (8). Then condition (4) leads to equation: $f''(t) = \pm 2\alpha$. We will consider here the case $\rho = -\alpha > 0$. Then $f(t)$ takes the form:

$$f(t) = \alpha(t - t_1)(t - t_2)$$

where $t_1 \leq t_2$. The coordinate $\tau$ can be determined only in the region $t \geq t_2$. From (10) we obtain:

$$\tau = \frac{1}{\sqrt{\alpha}} \ln \left( t - \frac{t_1 + t_2}{2} + \sqrt{(t - t_1)(t - t_2)} \right),$$

$$a(\tau) = \alpha(t(\tau) - t_1)(t(\tau) - t_2)$$

We see that if $t_1 \neq t_2$ then the Universe starts the expanding from zero radius ($a = 0$) at $\tau = \frac{1}{\sqrt{\alpha}} \ln(\frac{t_1 + t_2}{2})$ and expands exponentially such that for large $\tau$:

$$a(\tau) \approx \frac{\alpha}{4} \exp(2\sqrt{\alpha} \tau).$$

The eqs.(12) takes simplest form if $t_2 = t_1$:

$$\tau = \frac{1}{\sqrt{\alpha}} \ln(2(t - t_1)) ; \quad a(\tau) = \frac{\alpha}{4} \exp(2\sqrt{\alpha} \tau)$$

The Universe expands now exponentially from infinite past ($\tau = -\infty$, $a = 0$) to infinite future ($\tau = +\infty$). Thus the constant curvature solution (4) describes the ordinary inflation.

Let us consider now the general solution with nontrivial torsion (7). We will restrict ourselves to the case $\Lambda = 0$. Let now $\epsilon = 0, \alpha > 0$. Then (7) takes the form (8) with $f(t) = \alpha t^2 (1 - \ln t)^2$. The view of functions $\tau(t)$, $a(\tau)$ depends in what interval variable $t$ changes. Let $t \geq \epsilon$ ($\epsilon$ is Euler number) then we obtain:

$$\tau = \frac{1}{\sqrt{\alpha}} \ln(\ln t - 1) ; \quad a(\tau) = \alpha \exp[2(1 + \epsilon \sqrt{\alpha} + \sqrt{\alpha} \tau)]$$

The Universe starts evolution from constant curvature ($\rho = -\alpha$) configuration of zero radius ($a = 0$) at infinite past ($\tau = -\infty$). Then it expands with law $\exp(\exp(\sqrt{\alpha} \tau))$ and such a stage can be called "super-inflation".
If $0 < t \leq \epsilon$ then we get:

$$\tau = -\frac{1}{\sqrt{a}} \ln(1 - \ln t) \ ; \ a(\tau) = a \exp[2(1 - \epsilon^{-\sqrt{\alpha \tau}} - \sqrt{\alpha \tau})]$$

(15)

The metric (9) with (15) describes then expanding of the Universe from singular state ($\rho = +\infty$) of zero radius in infinite past ($\tau = -\infty$). It has maximal radius ($a = \alpha$) at $\tau = 0$ and then collapses to constant curvature ($\rho = -\alpha$) configuration with $a = 0$ in infinite future ($\tau = +\infty$).

Let now $\epsilon < 0$ then

$$f(t) = |\epsilon| t + \alpha t^2 (1 - \ln t)^2$$

(16)

For small $t \sim 0$ we obtain from (10) that evolution of the Universe is determined by $\epsilon$-term: $\tau \approx 2 \sqrt{\frac{\epsilon}{\alpha}}$; $a \approx \frac{\epsilon^2}{4} \tau^2$, while for large $t$ (and consequently for large $\tau$) the second term in (16) dominates and we obtain the super-inflationary stage as before (14). The corresponding evolution of curvature is following: $\rho = -\alpha \ln(\frac{\epsilon^2}{4\tau^2})$ for small $\tau$ and $\rho = -\alpha \exp(\sqrt{\alpha \tau})$ for large cosmic time $\tau$. The $\epsilon$-term in (7), (16) is due to the presence of non-gravitating matter with energy-momentum tensor $T_{00} \sim \epsilon \delta(\tau)$ in the initial state of the Universe. So we may interpret this result as that the matter determines the evolution of the Universe at the beginning of the expanding while for later cosmological times the self-gravitating forces dominate.

Concluding we considered the solutions of cosmological type in 2D Poincare gravity. The different possible regimes are analyzed. The most interesting one is stage of super-inflation when the scale factor $a(\tau)$ changes as $\exp(\exp\sqrt{\alpha \tau})$ with cosmological time $\tau$.

References