Intersecting Braids and Intersecting Knot Theory

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Abstract
An extension of the Artin Braid Group with new operators that generate double and triple intersections is considered. The extended Alexander theorem, relating intersecting closed braids and intersecting knots is proved for double and triple intersections, and a counter example is given for the case of quadruple intersections. Intersecting knot invariants are constructed via Markov traces defined on intersecting braid algebra representations, and the extended Turaev representation is discussed as an example.
Possible applications of the formalism to quantum gravity are discussed.

1. Introduction

During the last years a great deal of interest has been paid to the construction and study of intersecting link invariants. This motivation is due to the important role that intersecting links seem to play in several fields. Beginning with the construction by V. Vassiliev [16] of the so called 'Vassiliev invariants' of knots via singularity theory. Vassiliev considered not the knot itself, but the map from $S^1$ to $S^3$ that generates the knot. Within this approach, different knots correspond to the disconnected components of the space of isotopy classes of these maps (those which are embeddings). The 'walls' among the components are then knots with intersections and other types of singularities. The construction by Vassiliev of a recursive (sklein-type) procedure to compute the invariants involves an intermediate step a going trough these walls, and thus the Vassiliev invariants take values on intersecting links, which appear then as naturally as non intersecting links. A second example, the Witten's formula [15] relating Chern-Simons field theory and knot invariants was originally derived from conformal field theory. In alternative derivations via the so called variational approach, singular invariants are involved, since [13, 6, 14, 7] the derivation of the skein relation is done by the breaking of a double intersection. Furthermore some of these works [6, 7]
support the viewpoint that Witten’s formula also holds for double and triple intersections, and that there are no natural distinction between the intersecting and the non intersecting case. 

As a final example, in the loop representation of Quantum Gravity [17] the physical states are functionals of the knot classes, ie. knot invariants. It has also been proved that for nondegenerate solutions (those for which the spatial volume does not vanish) the wave function take non-zero values on knots with at least triple intersections [18, 6].

Our motivation in this paper comes from our work in Quantum Gravity, where physical states must be invariant under diffeomorphisms. This leads us to consider a slight shift of the standard conceptual frame of knot theory. We will consider diffeomorphisms connected to the identity on the manifold (\(R^3\) or \(S^3\) in this work) instead of homeomorphisms, as usual. This makes no difference for nonintersecting knot theory, and it is implicitly assumed in previous works about double intersections. In the tangent space at any point the diffeomorphisms induces an invertible linear transformation, that preserves the linear independence (dependence) between tangent vectors. Furthermore a diffeomorphism connected to the identity preserves the sign of the volume spanned by three arbitrary vectors in the tangent space.

This paper is organized as follows: in section 2 we consider an extension of the braid algebra for double and triple intersections. In section 3 matrix representations of the extended algebra are considered. Finally in section 4 we show how intersecting link invariants can be obtained, using these representations or using arbitrary non intersecting invariants, no matter how they were derived. We would like to point out that the rigor level is that of a theoretical physicist.

2. New Reidemeister moves, extended braid algebra and the Alexander theorem

In this section we will extend the usual Reidemeister moves planar knots (links) diagrams to include double and triple intersections.

Later we will prove that these generalized knots (links) may be drawn as closed braids (generalized Alexander theorem). Finally we enlarge the braid algebra by including new elements that generate double and triple intersections.

The Reidemeister moves for double intersections, that we will call \(D\), are sketched in figure 1. This kind of intersections and their associated moves were considered first in [1] and later in many works [6, 2, 4, 3, 5]. For the case of triple intersections[7] we must note that the sign of the volume element spanned by the tangents at the triple point is an invariant under a diffeomorphism connected with the identity. Hence there are three classes of triple points, that we will call \(T^+, T^-\) and \(T^0\). These intersections and their Reidemeister moves are sketched in figures 2 and 3.
Note that for triple intersections we may assume (after an isotopy) that our incoming and outgoing lines in the diagram are arranged so there are no incoming (outcoming) lines in between two outcoming (incoming) lines. The only exception is the flat intersection drawn in figure 4, that we will ignore in what follows. Due to this fact one can show that except for the above mentioned case, the Alexander theorem [8] still holds (for double intersections this was shown by J. Birman [3]).

**Theorem 1.** Given an arbitrary representative $K$ of a intersecting knot or link with double and triple points and an axis $A$ in $\mathbb{R}^3 - K$, $K$ may be deformed to a closed intersecting braid with axis $A$.

**Proof.** If we think of $A$ as the $z$ axis, by making an isotopy of $K$ in $\mathbb{R}^3 - K$ we may put it in the $(r, \theta)$ plane (except in a small neighborhood of a crossing or a $T^{+,-}$ intersection). By another isotopy we may arrange that the polar angle function restricted to a small neighborhood of each intersection be monotonically increasing. We now proceed as in the usual proof [8, 3] by considering the arcs of $K$ that have at most one crossing and have not intersecting points, and classifying these arcs as good (bad) if the polar angle function is an increasing (decreasing) function of the arc. Replacing the bad arcs (if there are) for good arcs by deforming $K$ we finally get a closed intersecting braid.

Note that the crucial step in the proof is the possibility of arranging the lines at an intersecting point so that between two incoming (outcoming) lines there are no outgoing (incoming) lines. This is not possible for one of the two classes of flat triple intersection and several classes of quadruple intersections. In fig.4 examples of this are shown. These intersections, if they appear in a knot, prevent it from being expressed as a closed braid.

We can conclude that a knot with $n$-uple intersections, $n \geq 3$, cannot be expressed generally as a closed braid, despite of the fact that any set of three tangents at the intersecting point are linearly independent.

An algebraic expression of the topological properties of braids is given by the Artin braid group $B_N$ that is generated by $N$ elements $g_i$, $i = 1, \ldots, N - 1$ satisfying the algebra

$$
\begin{align*}
  g_i g_i^{-1} &= g_i^{-1} g_i = 1, \\
  g_i g_j &= g_j g_i \quad \text{if } |i - j| > 1
\end{align*}
$$

The braid algebra $B_N$ can be extended to include double [1, 5, 4, 3] and triple intersections [7] by adding the generators $a_j$, $b_j^+$, $b_j^-$ and $b_j$, $j = 1, \ldots, N - 1$, $i = 2, \ldots, N - 1$ (see figure 5). In what follows we call $SB_N$ the extended braid algebra. These generators must satisfy algebraic relations, that follow from the generalized Reidemeister moves of fig.1 and fig.3. They are

$$
\begin{align*}
  a_j g_i &= g_i a_j, & g_i^{-1} a_i g_j &= g_j a_i g_i^{-1} \\
  g_i a_j &= a_j g_i & a_i a_j &= a_j a_i \quad \text{if } |i - j| > 1
\end{align*}
$$

(2)
and

\[ b_i^- = g_{i-1} b_i^+ g_i^{-1}, \quad b_i^- = g_{i-1} b_i^+ g_i^{-1}, \quad b_{i+1}^+ = g_{i-1} g_i b_i^+ g_i^{-1} g_i^{-1} g_i^{-1} g_i^{-1} \]

\[ b_i g_j = g_j b_i \quad \text{and} \quad b_i a_j = a_j b_i \quad \text{if} \quad j > i + 1 \quad \text{or} \quad j < i - 2 \]

\[ b_i b_j = b_j b_i \quad \text{if} \quad |i - j| > 2, \quad b_i^0 = g_i g_{i-1} b_i^0 g_i^{-1} g_i^{-1} g_i^{-1}. \] (3)

We would like to point out that the extended braid algebra is not a group, because the new generators have no inverse.

The fact that knots with n-uple intersections cannot, in general, be expressed as braids makes it less interesting further extensions of the braid algebra\(^1\). There exist other classes of double and triple intersections for which the incoming and outgoing tangents to any of the strands are not the same (there are kinks at the intersection). These cases will not be discussed here but are relevant in quantum gravity [19].

3. Representations of the extended braid algebra

We will consider representations of the braid algebra of the \(g_i\)’s by complex matrices \(G_i\) satisfying eq.(1). To construct a representation of the extended braid algebra we must add matrices \(A_i\) [6], \(B_i^\pm\) and \(B_i^0\) [7] satisfying eq.(2-3). The general procedure would be, given a representation of the standard braid algebra, to impose eq.(2-3) in its matrix form. We can see, however, that matrices of the form

\[ A_i = \alpha_1 G_i + \alpha_2 G_i^{-1} + \alpha_3 I_i \]

\[ B_i^+ = \beta_1 G_i G_{i-1} G_i + \beta_2 G_i G_{i-1} G_i + G_i^2 G_i + G_i G_i^{-1} \]

\[ B_i^0 = \beta_3 G_i G_{i-1} G_i + \beta_4 G_i G_{i-1} G_i + G_i G_i^{-1} G_i^2 + G_i^{-1} G_i^2 \]

(4)

are solution of eq.(2-3). Thus, given a representation of the braid group it is straightforward to construct a representation of the extended braid algebra. A wide class of \(B_N\) representations are the so called R-matrix representations, where a finite dimensional vector space \(V_i\) is associated to the \(i\)-th strand and the total representation space is \(V(N) = V_1 \otimes \ldots \otimes V_N\). The \(G_i\)’s are given by

\[ G_i = I \otimes \ldots \otimes \mathbb{R} \otimes \ldots \otimes I \] (5)

where \(R\) acts on \(V_i \otimes V_{i+1}\). To have a representation of \(B_N\) we require that \(R\) be invertible and

\[ R_{12} R_{23} R_{12} = R_{23} R_{12} R_{12} \]

(6)

where

\[ R_{12} = R \otimes I, \quad R_{23} = I \otimes R \]

(7)

\(^1\)Perhaps the way to do this is to enlarge the Temperley-Lieb algebra
The eq.(6) is called the quantum Yang-Baxter equation.
To enlarge the $B_N$ representation to a representation of the extended braid algebra $SB_N$ we must add the new generators given by
\[
A_i = I \otimes \ldots \otimes A \otimes \ldots \otimes I \\
B_i^{\pm,0} = I \otimes \ldots \otimes B^{\pm,0} \otimes \ldots \otimes I
\]
where $A$ acts on $V_i \otimes V_{i+1}$ and $B^{\pm,0}$ acts on $V_{i-1} \otimes V_i \otimes V_{i+1}$ and to impose the relations given by Eq (2-3).
\[
AR = RA, \quad A_{23}R_{12}R_{23} = R_{12}R_{23}A_{12}
\]
with
\[
A_{12} = A \otimes I, \quad A_{23} = I \otimes A
\]
and
\[
B^+ = R_{12}B^+ R_{23}^{-1}, \quad B^- = R_{23}^{-1}B^+ R_{12}, \quad B^0 = R_{23}R_{12}B^0 R_{23}^{-1}R_{12}^{-1}R_{23}^{-1}
\]
\[
I \otimes B^{\pm,0} = (R_{12} \otimes I)(I \otimes R_{12})(I \otimes R_{23})(B^{\pm,0} \otimes I) \times
\]
\[\times(R_{12} \otimes I)^{-1}(I \otimes R_{12})^{-1}(I \otimes R_{23})^{-1}
\]
We explicitly worked out the computations involved in eq.(9-11) for the Turaev representation of $SB_N$. In the Turaev representation the space $V_i$ is the two dimensional complex vector space $C^2$ with basis vectors $\{u_i, v_i\}$, and the R-matrix is
\[
R = q^\frac{\alpha}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - q^{-1} & q^{-\frac{1}{2}} & 0 \\
0 & q^{-\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
in the basis $\{u_i \otimes u_{i+1}, u_i \otimes v_{i+1}, v_i \otimes u_{i+1}, v_i \otimes v_{i+1}\}$ of $V_i \otimes V_{i+1}$. The complex parameter $q$ is arbitrary and the generators in this representation satisfy the relation
\[
q^\frac{\alpha}{2}G_i - q^{-\frac{1}{2}}G_i^{-1} = (q^\frac{\alpha}{2} - q^{-\frac{1}{2}})I
\]
that leads to the skein relations.
Our direct calculation for $A$ showed that the general form of this matrix is given by eq.(4), that is
\[
A = \alpha_1 R + \alpha_2 R^{-1}
\]
where the $\alpha$’s are arbitrary complex parameters. In eq.(14) $\alpha_3 I$ does not appear because from eq.(13) we know that $I$ is a linear combination of $G_i$ and $G_i^{-1}$.
The explicit computation of \( B^+, B^- \) and \( B^0 \) is again in accordance with eq.(4), the \( \beta \)'s not being all independent. The general result in this representation is

\[
\begin{align*}
B^+ &= \beta_1 M_1 + \beta_3 M_3 + \beta_4 M_4 \\
B^- &= \beta_1 M_1 + \beta_3 M_3^{-1} + \beta_4 M_3^{-1} \\
B^0 &= \beta_5 [B^+ + B^-] + \beta_6 I
\end{align*}
\]  

(15)

with

\[
M_1 = (I \otimes R)(R \otimes I)(I \otimes R) \\
M_3 = (I \otimes R) \\
M_4 = (R^{-1} \otimes I)
\]  

(16)

or explicitly

\[
M_1 = \begin{pmatrix}
q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} + \frac{1}{2} & -q^{-\frac{1}{2}} + q^{\frac{1}{2}} & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 \\
0 & -q^{-\frac{1}{2}} + q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{\sqrt{q}} & 0 & -q^{-\frac{1}{2}} + q^{\frac{1}{2}} & q^{-\frac{1}{2}} & 0 \\
0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -q^{-\frac{1}{2}} + q^{\frac{1}{2}} & 0 & q^{-\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}}
\end{pmatrix}
\]  

(17)

\[
M_3 = \begin{pmatrix}
q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} + \frac{1}{2} & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{\sqrt{q}} & q^{-\frac{1}{2}} & 0 \\
0 & 0 & 0 & 0 & q^{-\frac{1}{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0
\end{pmatrix}
\]  

(18)
\[ M_4 = \begin{pmatrix}
q^{-\frac{\alpha}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{\frac{\alpha}{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{\frac{\alpha}{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{\frac{\alpha}{2}} & 0 & 0 & 0 \\
0 & 0 & q^{\frac{\alpha}{2}} & 0 & 1 - \frac{q^\alpha}{q^{\frac{\alpha}{2}}} & 0 & 0 \\
0 & 0 & 0 & 0 & q^{\frac{\alpha}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-\frac{\alpha}{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-\frac{\alpha}{2}}
\end{pmatrix} \] (19)

Note that we omit the \( \beta_2 \) term because it may be expressed as linear combination of the others. It could happen that the most general B's are of the form given by eq.(4) plus other terms irreducible to this form. If this were the case, we would not have simple skein relations for our intersecting knot invariants, as will be shown later.

4. From intersecting braids to intersecting knot invariants

In this section we will show how to construct intersecting link invariants from intersecting braids.

As we already saw, any oriented intersecting link of the classes that we considered can be expressed as a closed intersecting braid. Conversely given any braid diagram we can associate a link diagram by joining the endpoints of the strands in an order preserving way and without adding new crossings. We call the resulting link the closure of this braid element \( b \in \mathbb{B}_N \) and we denote it as \( \hat{b} \). It is straightforward to see that the extended braid algebra relations account for the type 2 and 3 Reidemeister as well as for the new moves. But we also have that braids related by the following moves (Markov moves) correspond to ambient isotopic links [12]:

M1. \( b_1, b_2 \in \mathbb{B}_N \), \( \to b_1 b_2 \equiv b_2 b_1 \)
M2. \( b_1 \in \mathbb{B}_N \), \( b_2 = b_1 q_{N+1}^{-1} \in \mathbb{B}_{N+1} \), \( \to b_1 \equiv b_2 \)

If we are working with regular isotopy only the first type of Markov moves holds.

To construct intersecting links invariants from an \( R \)-matrix representation of the extended braid algebra we will follow the standard procedure of Markov traces.

It has been proved [11] that if there is a matrix \( \mu \) acting in \( \mathcal{V} \) such that

\[
R \mu_1 \otimes \mu_2 = \mu_1 \otimes \mu_2 R
\]

\[
\text{Tr} |_{\mathcal{V}_2} (R^{\pm 1} \mu_2) = \alpha^{\pm 1} \cdot 1 |_{\mathcal{V}_i}
\] (20)

then, if \( L \) is the link obtained from the closure \( \hat{b} \) of the braid \( b \in \mathbb{B}_N \) represented by the matrix \( B \), we have that

\[
\chi^R(L) = \text{Tr} |_{\mathcal{V}(N)} (B \mu_{\otimes N})
\] (21)
is a regular isotopy invariant and that
\[
\chi^{A}(L) = \alpha^{-w(L)} Tr_{\chi^{N}} \left( B_{\mu} \otimes N \right)
\]  
(22)

where \( w(L) \) is the writhe of the diagram, is an ambient isotopy invariant. Since the new Reidemeister moves are accounted for in the extended algebra, it is straightforward to see that invariants for intersecting links can be defined in the same way. This allows us to extend known link invariants (for regular or ambient isotopy) to the intersecting case. In general we will have, in addition of the usual defining relations for the non intersecting case (usually skein relations and the value of the unknotted), the generalized skein relations

\[
\chi(D) = \alpha_{1} \chi(L_{+}) + \alpha_{2} \chi(L_{-}) + \alpha_{3} \chi(L_{0})
\]

\[
\chi(T^{+}) = \beta_{1} \chi(L_{24}L_{14}L_{24}) + \beta_{2} \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24})
\]

\[
\chi(T^{-}) = \beta_{1} \chi(L_{24}L_{14}L_{24}) + \beta_{2} \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24}) + \chi(L_{24}L_{14}L_{24})
\]

\[
\chi(T^{0}) = \beta_{5} \chi(T^{+}) + \chi(T^{-}) + \beta_{3} \chi(L_{10}L_{20})
\]

(23)

where the meaning of the notation is explained in fig.6.

Eq.(23) follows from eq.(4), eq.(21-22) and the properties of the trace. For example, for the generalized Turaev representation and with the matrix \( \mu \)

\[
\mu = \left( \begin{array}{cc}
q^{-\frac{1}{2}} & 0 \\
0 & q^{\frac{1}{2}}
\end{array} \right)
\]

(24)

we obtain a generalized Jones polynomial[9] in the variables \( q, \alpha_i, \beta_j \).

The approach via extended braid algebra and Markov traces makes crystal clear our construction of intersecting links invariant, and allows to detect redundancy in the chosen parameters. However a direct (and more general) construction is possible.

**Theorem 2.** Given any ambient (regular) isotopic invariant for non-intersecting links \( \chi(L) \) its generalization to the intersecting case \( \chi^{E}(L) \) given by eq.(23) is an intersecting link invariant

*Proof.* From the defining relations of eq.(23), making a sequence of transformations on planar diagrams entirely analogous to the algebraic calculations needed to check that the matrices of eq.(4) satisfy the algebra (2-3), and using that \( \chi(L) \) is a non-intersecting link invariant i.e. invariant under type 2 and 3 Reidemeister moves, the invariance of \( \chi^{E}(L) \) under the new moves can be verified, that concludes the proof \( \Box \).

Note that the defining relations eq.(23) imply that the extended invariants inherit many properties from the non-intersecting original ones. For example the polynomials defined above from R-matrix representations and Markov traces
don’t detect the orientation of the links, despite of its appearance, because their original ones do not do it either.

5. Concluding remarks and open problems

In the study of Vassiliev invariants only double intersections have been considered (these are unavoidable). The skein-type relation used is of the form of eq (23) with \( \alpha_1 = -\alpha_2 \) and \( \alpha_2 = 0 \) (see for example [3, 4]). It is interesting to ask what happens for triple (or n-uple) intersections and to study if we also have relations of the form given by eq (23).

For Chern-Simons theory the work in the intersecting case is quite heuristic; it would be desirable to have a rigorous proof of the validity of Witten’s formula in the intersecting case.

In loop quantum gravity, some physical wave functions are given in integral form, an example being the Gauss linking number

\[
\chi(L_1, L_2) = \frac{1}{4\pi} \int_{L_1} \int_{L_2} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}
\]

To have a nondegenerate solution we need it to take non-zero values on at least triple intersections, but the integral is ill defined in this case. The usual procedure is to consider a framing like \( x^\mu \rightarrow x^\mu + \delta^\mu(x) \) and an analogous one to \( y^\nu \) that splits the intersections. Theorem 2 tells us how the framing must be taken. Not all the framings work. We will always have an invariant of the framed links, but in general this will not be the same as having an invariant of the original intersecting loop.

On the other hand it has been proved that link invariants with non-zero values only on non intersecting links are degenerate physical states of loop quantum gravity [17], provided they do not detect the orientation of the link. In the intersecting case the so called hamiltonian constraint imposes new conditions. For example, if the cosmological constant \( \Lambda \) does not vanish, we have that the Kauffman Bracket \( S(L; q, \alpha_i, \beta_j) \) (see for example [14]), a polynomial invariant closely related to the Jones polynomial, is a physical state \( \psi^A[L] \) [6, 7], where the parameters \( q, \alpha_i \) and \( \beta_j \) are fixed functions of \( \Lambda \). That is:

\[
\psi^A[L] = S(L; q(\Lambda), \alpha_i(\Lambda), \beta_j(\Lambda)).
\]

One can think that the hamiltonian does select one 'point' of the 'parameter space' spanned by \( q, \alpha_i \) and \( \beta_j \).

In principle, each non intersecting invariant, that do not detect the orientation, may be extended to the intersecting case in such a way that the Mandelstam constraints of Quantum Gravity are satisfied. This raises the question [19] of what values of the \( \alpha_i \)'s and \( \beta_j \)'s are required in order to have a non degenerate physical state of quantum gravity for a given value of \( \Lambda \).
References

Figure 1. Reidemeister moves for a double intersection.

Figure 2. Classes of triple intersections.
Figure 3. Reidemeister moves for triple intersections.

Figure 4. Intersections in knots (links) that cannot be expressed as a closed braid.

\[ \cdots \]
Figure 5. Graphical representation of the generators of the extended braid algebra.

\[ L_{10} L_{20} \quad L_{2+} L_{1+} L_{2+} \]

\[ L_{1+} \quad L_{1-} \quad L_{2+} \quad L_{2-} \]

\[ L_{2+} L_{1+} \quad L_{1+} L_{2+} \quad L_{1+} L_{1+} \quad L_{2+} L_{2+} \]

Figure 6. Crossings that appear in the skein relations for intersections.