On the Nature of Nonperturbative Effects in Stabilized 2D Quantum Gravity

Oscar Diego* and José González †

Instituto de Estructura de la Materia
Serrano 123, 28006 Madrid
Spain

Abstract

We remark that the weak coupling regime of the stochastic stabilization of 2D quantum gravity has a unique perturbative vacuum, which does not support instanton configurations. By means of Monte Carlo simulations we show that the nonperturbative vacuum is also confined in one potential well. Nonperturbative effects can be assessed in the loop equation. This can be derived from the Ward identities of the stabilized model and is shown to be modified by nonperturbative terms.

* e-mail: intod67@cc.csic.es
† e-mail: intjg64@cc.csic.es
1. Introduction

It is well known that discretized 2-D quantum gravity coupled to conformal matter with $c < 1$ can be represented by a zero-dimensional matrix model [1, 2, 3] and that the expansion in powers of $1/N$ of the matrix model, where $N$ is the dimension of the hermitian matrix field, defines the topological expansion of the original discretized 2-D quantum gravity. The partition function of zero-dimensional matrix model has the form

$$Z = \int d\Phi \exp\left\{-NTrW(\Phi)\right\}$$

where $W$ is the potential and $\Phi$ is an hermitian matrix field. The continuum limit of the discretized 2-D quantum gravity is achieved by taking the limit of the coupling constants of the potential $W$ to some critical set of coupling constants for which the partition function (1) has non-analytic behaviour. For pure gravity the argument of the integral (1) is an increasing function for large values of $\Phi$ and the integral does not exist for finite $N$, but in the large $N$ limit the integral exists up to some value of the coupling constant. This is the critical value of the coupling constant, and the non-analytic behaviour developed makes the partition function above it not well-defined. The matrix model is defined only in the large $N$ limit and its expansion in powers of $1/N$ represents the topological expansion of the quantum gravity, but the matrix model does not define uniquely the non perturbative 2-D quantum gravity [4].

Stochastic stabilization [5, 6] provides, on the other hand, a way of mapping the model (1) into one which is defined for all values of $N$ and the coupling constant, while reproducing the perturbative expansion in powers of $1/N$. There has been much hope that this stabilization of the original matrix
model could lead to an unambiguous nonperturbative definition of quantum gravity. If these expectations have not been met, it has been in part because of the poor understanding of nonperturbative effects in the stabilized theory. As we review afterwards, this is defined in one more dimension than the original matrix model, setting therefore the problem into the framework of quantum mechanics. In fact, in the zero-dimensional model (1) the source of nonperturbative effects lies on the transmission of eigenvalues through the walls of the confining potential well [7], so that one could expect the same kind of phenomenon in the stabilized theory. The main point of this article, however, is that the ground state of the one-dimensional model does not bear tunnelling between different wells of the potential. We are able to prove this, for the simplest matrix model, in perturbation theory as well as in the non-perturbative regime (using a Monte Carlo approach in this latter case). This leaves open the proper interpretation of the nonperturbative effects. Their significance can be assessed otherwise, since, as also shown in the paper, they modify the form of the loop equation in the stabilized theory.

2. Perturbative approach

The stochastic stabilization of the matrix model introduces a positive definite hamiltonian

$$H = \frac{1}{2} \text{Tr} \left\{ -\frac{1}{N^2} \frac{\partial^2}{\partial \Phi^2} + \frac{1}{4} \left( \frac{\partial W}{\partial \Phi} \right)^2 - \frac{1}{2N} \frac{\partial^2 W}{\partial \Phi^2} \right\}$$  \hspace{1cm} (2)$$

This hamiltonian is well defined for all values of $N$ and the coupling constants. The zero mode of the stabilized theory is given by

$$\Psi(\Phi) \sim \exp \left\{ -N \frac{W(\Phi)}{2} \right\}$$  \hspace{1cm} (3)$$
its norm being the partition function of the original matrix model. Hence, the original matrix model is well defined if and only if this zero energy state is a normalizable state. When this is the case, corresponding observables in both theories coincide

$$\langle Q \rangle_{\text{slab}} = \int Q | \Psi |^2 d\Phi = \frac{1}{Z} \int Q \exp\{-NW(\Phi)\} d\Phi = \langle Q \rangle_{\text{matrix}} \quad (4)$$

We can use the stabilized theory to calculate the observables of the original matrix model, with the ground state energy playing the role of an order parameter which gives us information about the range of definition of the integral (1). The non-analytic behaviour of the original matrix model becomes now non-analytic behaviour in the ground state energy of the hamiltonian (2). For pure gravity in the large $N$ limit, below the critical point the zero energy state is a normalizable state and the ground state energy is zero, while above the critical point the zero-dimensional matrix model does not exist and the ground state energy of the stabilized hamiltonian is greater than zero. The stabilized theory exists anyway for all values of $N$ and the coupling constant, even if the original matrix model is ill defined.

We spend a few time showing that, despite the strange critical exponent for the leading contribution [8], the topological expansion of the ground state energy can be organized in terms of the scaling variable of the zero-dimensional matrix model. Dealing with the $1/N$ expansion we will also show our main conclusion in the weak coupling regime, i.e. that the eigenvalues of $\Phi$ in the ground state are all confined in the same well of the potential in (2). Let us consider the cubic matrix model given by

$$W(\Phi) = Tr\Phi^2 - \frac{2g}{3} Tr\Phi^3, \quad (5)$$
where $\Phi$ is the $N$ dimensional hermitian matrix. We consider throughout this section the large $N$ limit of the model, for which (1) is still defined within a certain range at $g > 0$. It is well-known that the one-dimensional matrix model is mapped into a gas of free fermions [9], which are the eigenvalues $\{\lambda_n\}$ of the $\Phi$ matrix variable$^1$. The Fokker-Planck hamiltonian of the stabilized model (2) becomes the sum of $N$ one-particle hamiltonians [5, 6, 12, 13, 14]

$$ h_n = -\frac{1}{2N^2} \frac{\partial^2}{\partial \lambda_n^2} + \frac{1}{2} \{ g^2 \lambda_n^4 - 2g \lambda_n^3 + \lambda_n^2 + 2g \lambda_n - 1 \} $$

(6)

The one-particle hamiltonian (6) has a main well and a local minimum below the critical coupling constant $g_c = \sqrt{1/(6\sqrt{3})}$, and only one well above it (see figure 1).

The ground state energy of the Fokker-Planck hamiltonian is

$$ E = \sum_{n=0}^{N-1} e_n $$

(7)

where $\{e_n\}_{0 \leq n \leq N-1}$ are the first $N$ eigenvalues of the one particle hamiltonian (6). In the large $N$ limit we can write down for it an integral representation, taking into account $1/N$ effects. From the Euler-Maclaurin sum-formula [15] the total energy is

$$ E = \sum_{n=0}^{N-1} e_n = \int_{E_0^*}^{E_F^*} dE E \rho(E) - \frac{1}{2} E_F^* + \frac{1}{2} E_0^* + \frac{1}{12} \rho(E_F^*) - \frac{1}{12} \rho(E_0^*) + \cdots $$

(8)

where $E_0^*$ is the first eigenvalue $e_0$, $E_F^*$ is the value of the analytic continuation of $e_n$ from $n \leq N - 1$ to $n = N$, and $\rho(E)$ is the density of states.

$^1$In other matrix models, the hamiltonian of the stabilized theory has interaction terms, but one can perform a Hartree-Fock approach to obtain the exact ground state energy in terms of suitable one-particle states [8, 10, 11].
From (6) we see that $1/N$ play the role of $\hbar$, and the large $N$ limit is a semiclassical expansion. The density of states is given by

$$\rho(E) = \frac{d}{dE} N(E) \quad (9)$$

The distribution function of states is given by [16]

$$N(E) = \frac{N}{2\pi} \int_a^b d\lambda \sqrt{2E - V} - \frac{1}{24\pi N} \int_a^b d\lambda \frac{d^2V}{d\lambda^2} \frac{1}{\sqrt{2(E - V)}} + O\left(\frac{1}{N^3}\right) \quad (10)$$

$a$ and $b$ being the classical turning points. $E_F^*$ and $E_0^*$ obey the quantization conditions

$$N(E_0^*) = \frac{1}{2} \quad (11)$$

and

$$\tilde{N}(E_F^*) = N + \frac{1}{2} \quad (12)$$

where $\tilde{N}(E)$ is the distribution function $N(E)$ restricted to the main well of the stabilized potential. In general, one has to keep track of higher order terms in the sum formula (8) altogether with the perturbative expansion (10), in order to obtain correctly the $1/N^2$ expansion of the total energy.

From (8) and (10) the total energy is

$$NE = N^2 \left( E_0 + E_2 + O\left(\frac{1}{N}\right) \right) \quad (13)$$

where

$$E_0 = E_F^{(0)} + \frac{1}{2} \int_a^b d\lambda \left[ 2(E_F^{(0)} - V) \right]^2 \quad (14)$$

$$E_2 = \frac{1}{24\pi} \int_a^b d\lambda \frac{d^2V}{d\lambda^2} \sqrt{2(E_F^{(0)} - V)} - \frac{1}{24} \left[ \frac{1}{\pi} \int_a^b d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} \right]^{-1} \quad (15)$$
Here $E^{(0)}_F$ is the Fermi energy in the planar limit and is given by the condition

$$\frac{1}{\pi} \int_a^b d\lambda \sqrt{2(E^{(0)}_F - V)} = 1 \quad (16)$$

We expect the total energy to be zero for $g < g_c$ to all perturbative orders [5], which we have actually checked to second order in $1/N$ expansion. The leading order of the total energy (14) is zero up to the critical coupling $g_c = \sqrt{1/(6\sqrt{3})}$, and greater than zero above it. The critical behaviour of this quantity agrees with the exponent found in the quartic model [8]

$$E_0 \sim (g - g_c)^{11/4} \quad g > g_c \quad (17)$$

Regarding the subleading contribution (15) we find the critical behaviour

$$E_2 = 0 \quad g < g_c \quad (18)$$

$$E_2 \sim (g - g_c)^{1/4} \quad g > g_c \quad (19)$$

The appropriate double scaling limit $g \to g_c$ and $N \to \infty$ is such that $z = N(g_c - g)^{1/2}$ remains finite [3]. We see, in fact, that the topological expansion of the total energy organizes above the critical point as a power series in the scaling variable $z$ of the original matrix model

$$N E = N^2 (g - g_c)^{1/2} \left( B_1 + B_2 \frac{1}{N^2 (g - g_c)^{1/2}} + \cdots \right) \quad (20)$$

The striking result, however, concerns the position of the Fermi level $E_F$. To leading order of the $1/N$ expansion, $E^{(0)}_F$ is characterized by the condition (16), which places it precisely at the level of the local minimum of the potential, all the way up to $g_c$ [17]. The first order correction $E^{(1)}_F$ can be computed from the quantization condition

$$\frac{N}{\pi} \int_a^b d\lambda \sqrt{2(\epsilon_n - V)} + O\left(\frac{1}{N}\right) = n + \frac{1}{2} \quad (21)$$
bearing in mind that we have to fill the first $N$ levels from $e_0$ to $e_{N-1}$ as in
(7). We find then

$$E_F^{(1)} = -\frac{1}{2} \left[ \frac{1}{\pi} \int_a^b d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} \right]^{-1}$$

(22)

which is a negative quantity. This means that, at least in the weak coupling
regime, the ground state appears to be confined to the central well of the
potential, and that there is no issue about tunnelling to the region around
the local minimum. One may think that this only points at the unfeasibility
of discussing nonperturbative effects in the very framework of the $1/N$
expansion. Actually, this approximation implies taking the limit $N \to \infty$ from
the start, so that it would be conceivable that a more sophisticated way of
tuning the double scaling could lead to a different physical picture. In the
last section we will return to the question of the localization of the ground
state of the model at finite $N$.

3. Observables and the loop equation

Below the critical point, the observables of the stabilized theory and the
matrix model have to be the same, and may be calculated as follows [12, 11].

We add to the potential $V$ an auxiliary term $\beta \lambda^n$ and, using the Hellmann-
Feynman theorem, write the following formula for observables of the stabi-
lized theory

$$\frac{1}{N} \langle tr \Phi^n \rangle = \left( \frac{\partial E(\beta)}{\partial \beta} \right)_{\beta=0}. \quad (23)$$

Taking into account (13), (23) and results from appendix A, we perform
the calculation of the observables of the theory up to second order in $\frac{1}{N}$. To
first order all observables are given by

$$\langle tr \Phi^n \rangle = N^2 \int_a^b d\lambda \lambda^n \rho(\lambda) \quad (24)$$
and

$$\rho(\lambda) = \frac{1}{\pi} \sqrt{2(E_F - V)}$$

(25)

is the fermionic density and is equal to the eigenvalue density of the original matrix model in the planar limit for \(g < g_c\) [10]. Now we perform the limit \(g \longrightarrow g_c\), which represents the continuum limit of the discretized 2-D quantum gravity, and calculate numerically the observables. After computing the most singular part of the observables turns out to be given by

$$\langle tr \Phi^n \rangle = N^2 (g_c - g)^{\frac{n}{2}} \left( A_1^{[n]} + A_2^{[n]} \frac{1}{N^2 (g_c - g)^{\frac{n}{2}}} + \cdots \right)$$

(26)

for \(g < g_c\), and

$$\langle tr \Phi^n \rangle = N^2 (g - g_c)^{\frac{n}{2}} \left( C_1^{[n]} + C_2^{[n]} \frac{1}{N^2 (g - g_c)^{\frac{n}{2}}} + \cdots \right)$$

(27)

for \(g > g_c\).

In the double scaling limit (26) defines the puncture operator of the 2D quantum gravity. As long as the critical exponents in expressions (26) and (27) are the same, we can define an analogous double scaling limit above the critical point given by \(z = N(g - g_c)^{\frac{1}{2}}\). This limit does not exist in the zero-dimensional matrix model, hence, it defines new effects beyond the original formulation of matrix model. In reference [13] this limit is used to show that the stabilized model does not satisfy the nonperturbative KdV flow equations.

The loops equations arise from the stabilized model as follows: We add to the potential \(W\) a perturbation

$$W = Tr \Phi^2 - \frac{2g}{3} Tr \Phi^3 - \frac{2\beta}{n} Tr \Phi^n.$$  

(28)
The potential of the stabilized theory becomes now

\[
V(\beta_n) = \frac{1}{2} \left\{ Tr \Phi^2 + g^2 Tr \Phi^4 - 2g Tr \Phi^3 + 2g Tr \Phi - N \right. \\
+ \beta_n^2 Tr \Phi^{2n-2} - 2\beta_n Tr \Phi^n + 2g \beta_n Tr \Phi^{n+1} \\
+ \left. \frac{\beta_n^{n-2}}{N} \sum_{k=0}^{n-2} Tr \Phi^k Tr \Phi^{n-2-k} \right\}
\]  

(29)

now from the Hellmann-Feynmann theorem

\[
\left( \frac{\partial E}{\partial \beta_n} \right)_{\beta_n=0} = -\langle Tr \Phi^n \rangle + g \langle Tr \Phi^{n+3} \rangle + \frac{1}{2N} \sum_{k=0}^{n-2} \langle Tr \Phi^k Tr \Phi^{n-2-k} \rangle.
\]

(30)

The loop length \( L \) is given by

\[
W(L) = \frac{1}{N} Tr e^{LL}
\]

(31)

and is not difficult to prove that

\[
\dot{V} \left( \frac{\partial}{\partial L} \right) \langle W(L) \rangle - \int_0^L dJ \langle W(J) W(L - J) \rangle = -\frac{2}{N} \sum_{n=0}^{\infty} \frac{L^n}{n!} \left( \frac{\partial E}{\partial \beta_{n+1}} \right)_{\beta_{n+1}=0}
\]

(32)

then, if the energy of the ground state is zero for all values of \( \beta_n \), the Hellmann-Feynmann theorem gives the set of Ward identities

\[
\left( \frac{\partial E}{\partial \beta_n} \right)_{\beta_n=0} = 0 \quad n = 0, \ldots, \infty
\]

(33)

which are equivalent to the first loop equation

\[
\dot{V} \left( \frac{\partial}{\partial L} \right) \langle W(L) \rangle = \int_0^L dJ \langle W(J) W(L - J) \rangle.
\]

(34)

We expect that the other loop equation can be found when we add to the potential \( W \) perturbation like:

\[
W = Tr \Phi^2 - \frac{2g}{3} Tr \Phi^3 - \frac{2\beta_{n1n2\ldots}}{n1n2\ldots} \prod_i Tr \Phi^n_i.
\]

(35)
The stabilized hamiltonian (2) is the supersymmetric hamiltonian of reference [17] restricted to the bosonic sector. The Ward identities of the supersymmetric matrix model becomes the Schwinger Dyson equation of the zero-dimensional matrix model.

In the case of pure gravity the supersymmetry is broken non perturbatively, then the Ward identities or the loop equation are modified by non perturbative corrections.

4. Nonperturbative approach

We come back to the question of the localization of the ground state in the stabilized theory, working near the critical coupling at finite values of $N$. In order to get information about the eigenvalue distribution of the ground state, we apply the Monte Carlo method in the computation of the path integral for the quantum system. In reference [6] a Monte Carlo method was also applied to simulate observables of the matrix variable $\Phi$. We consider here an alternative approach which optimizes the Monte Carlo calculation reducing the number of variables to the eigenvalues of $\Phi$.

The basic object to look for is the probability amplitude between two states $\Psi_i$ and $\Psi_f$ at times 0 and $T$, respectively. This admits the path integral representation [18]

$$\langle \Psi_f(T) | \Psi_i(0) \rangle = \int \prod_{n=1}^{N} D\lambda_n(t) \Delta(\lambda(0)) \Delta(\lambda(T)) \bar{\Psi}_f(\lambda(T)) \Psi_i(\lambda(0)) \exp \left\{-N \int_0^T dt \sum_{n=1}^{N} \left( \frac{1}{2} \lambda_n(t) + \frac{1}{2} V_{FP}(\lambda_n(t)) \right) \right\} \quad (36)$$

where $\Delta(\lambda)$ is the Van der Monde determinant and $V_{FP}$ the stabilized potential appearing in (6). If both $\Psi_i$ and $\Psi_f$ have nonvanishing projection over the ground state, this is the state which dominates at large $T$. The
amplitude behaves, in terms of the ground state energy $E$,

$$\langle \Psi_f(T) | \Psi_i(0) \rangle \sim e^{-ET} \quad (37)$$

We have performed, in practice, a discretization of the time $T$, such that $t_j = \epsilon j$, where $j = 0, \ldots, N_T$ and $t_{N_T} = \epsilon N_T = T$. The measure of integration in (36) becomes

$$e^{-S} = \exp \left\{ -N \sum_{i=1}^{N} \sum_{j=0}^{N_T-1} \left( \frac{1}{2} \frac{(\lambda_i(t_{j+1}) - \lambda_i(t_j))^2}{\epsilon} + \epsilon V_{FP}(\lambda_i(t_j)) \right) 
+ \sum_{i<j} \log (\lambda_i(t_0) - \lambda_j(t_0)) 
+ \sum_{i<j} \log (\lambda_i(t_{N_T}) - \lambda_j(t_{N_T})) \right\} \quad (38)$$

This can be simulated by the Monte Carlo method, as if we were dealing with a two-dimensional statistical system of size $N \times N_T$. The variables are the eigenvalues $\lambda_i(t_j)$ where $i = 1, \ldots, N$, $j = 0, \ldots, N_T$. The kinetic term defines a nearest-neighbor interaction along the time axis and the determinants in (36) —the only vestige of Fermi statistics in the path integral— define a long-range interaction at the boundaries $t = 0$ and $t = T$. There are no interactions between bulk variables along the $N$ axis.

The crucial point in the simulation is to reach a large enough value of the time $T$, in order to measure with confidence the ground state properties. We have taken steps with $\epsilon = 0.1$, and found that above $T = 8$ observables measured with (38) do not show significant contribution from the first excited states. We have imposed boundary conditions in the form of constant wave functions at both ends of the time interval $\Psi_i(\lambda) = \Psi_f(\lambda) = const.$, which have certainly nonvanishing overlap with the ground state.
We have implemented the Monte Carlo simulation in a VAX 9000 machine with Metropolis algorithm. We have made a random choice of the point $t_i$ each time, updating then the $N$ variables at that site. The number of iterations between measures has been 2000 MC sweeps and we have left a thermalization period equivalent to $5 \times 10^4$ MC sweeps of all the variables in the lattice. We have simulated systems with $N = 5$ and $N = 10$, starting with different initial conditions for the set of eigenvalues. In some of the simulations part of the eigenvalues were initially confined in the well around the local minimum and in the others all of them were in the central well. Irrespective of these different choices, after the thermalization period we ended up with a distribution of eigenvalues equal to zero in the well around the local minimum. This happened even for values of $g$ very close to $g_c$.

Figure 2 shows a typical distribution for $N = 5$ and $g - g_c = 10^{-4}$ after $8 \times 10^6$ MC sweeps of all the lattice. This supports strong evidence that at finite $N$ the Fermi level is placed below the level of the local minimum of the potential, all the way up to $g_c$.

5. Conclusions

We have shown that in the framework of the $1/N$ approximation there is only one perturbative vacuum. The Fermi level in the weak coupling regime is below the local minimum of the potential and, hence, instanton configurations which start or end at the local minimum do not exist. In the nonperturbative approach we have also made plausible that the ground state does not bear tunnelling between different wells of the potential. The explanation of this picture should be the following. The stochastic stabilization may be viewed as the usual stochastic quantization with asymptotic final
and initial states fixed and given by well defined configurations of the original model [19]. Therefore, in the stabilized matrix model all configurations have to start and end at the main well of the stabilized potential. The Fermi level has to be placed below the local minimum and an instanton which starts or ends at the local minimum cannot exist.

One possible way of understanding the nonperturbative effects may be envisaged as follows. There are two kind of static solutions of the classical equation in the large $N$ limit: (a) all the fermions are restricted to the main well of the potential, and (b) a fermion is placed at the local minimum. These two solutions are degenerate at first order in $1/N$. Hence we expect that the time dependent solution of the classical equation of motion connects the main well and the local minimum. The perturbative effects lift the degeneracy, and the solution with a fermion in the local minimum becomes an excited state. The nonperturbative effects have to arise from trajectories of one fermion which start and end at the main well. These closed trajectories are given by a succession of instantons and anti-instantons [20]. Hence, the instantonic action of one fermion is

$$S_{\text{inst}} = 2 \int_a^b \sqrt{2(V - E_F)}$$

which agrees with the instantonic action calculated in [10].

In this paper we have considered the double scaling limit in the order: first take the large $N$ limit and then $g \to g_c$. This is the only double scaling limit defined in the zero-dimensional matrix model. But in the stabilized model it is possible to perform the double scaling limit from finite values of $N$ and $g - g_c$. We have performed a preliminary numerical calculation. In order to understand how an alternative double scaling limit may be achieved
from finite values of \( N \), a more detailed investigation of the phases in the space of parameters \((N, g)\) should be carried out.
Appendix A

In this appendix we show how formulas like
\[
\frac{\partial}{\partial \beta} \int_a^b d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}}
\]
(40)

which have end points singular integrand, can be calculated.

We define two fixed arbitrary points \( \Lambda_1 \) and \( \Lambda_2 \) such that
\[
\int_a^b d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} = \int_{\Lambda_2}^{\Lambda_1} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} + \int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}}
\]
(41)

For \( g \neq g_c \), \( \frac{dV}{d\lambda} \) has only one zero in the interval \([a, b]\), then we choose \( \Lambda_1 \) and \( \Lambda_2 \) such that the zero of \( \frac{dV}{d\lambda} \) is placed between \( \Lambda_1 \) and \( \Lambda_2 \), see (fig.1, a).

The second integral can be derived directly
\[
\frac{\partial}{\partial \beta} \int_{\Lambda_2}^{\Lambda_1} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} = \int_{\Lambda_2}^{\Lambda_1} d\lambda \frac{\partial}{\partial \beta} \left\{ \frac{1}{\sqrt{2(E_F^{(0)} - V)}} \right\}
\]
(42)

and the result is well defined for all values of the coupling constant.

The other integrals are posibly singular and we perform them as follows
\[
\int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} = \int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} \frac{\dot{V}}{V} = -\int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{1}{V} \frac{d}{d\lambda} \left( \frac{1}{\sqrt{2(E_F^{(0)} - V)}} \right)
\]
(43)

where the dots are \( \lambda \) derivatives, and integrations by parts gives
\[
\int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{1}{\sqrt{2(E_F^{(0)} - V)}} = \text{Analytic terms} + \int_{\Lambda_1}^{\Lambda_2} d\lambda \frac{d}{d\lambda} \left( \frac{1}{V} \right) \sqrt{2(E_F^{(0)} - V)}
\]
(44)
the result is well defined for \( g \neq g_c \) and can be derived directly, but if \( g = g_c \) then \( \dot{V}(b) = 0 \) and the integral is possibly singular. This is the origin of the critical point \( g_c \), which is defined by the condition \( \dot{V}(b) = 0 \), see (fig.1, c).

To all orders in \( 1/N \), the energy and observables are some combinations of integrals which have the form

\[
\frac{d^j}{d\beta^j} \frac{d^k}{dE^k} \left( \int_\lambda d\lambda \frac{P(\lambda)}{\sqrt{2(E - V(\beta))}} \right)_{E=E^{(0)}, \beta=0} \tag{45}
\]

This integrals have the same critical point \( g_c \). The critical point \( g_c \) remains constant to all orders in \( 1/N \) expansion.
References


Figure Captions

Figure 1. Plot of the stabilized potential a) below the critical point, b) above the critical point and c) at the critical point

Figure 2. Plot of the normalized fermionic density: continuum line. For $N = 5$, $g - g_c = 10^{-4}$, 100 time intervals. We have performed 4200 measures. The planar fermionic density is given by the dashed line. The vertical dashed line is placed at the absolute minimum of the stabilized potential. The local maximum and the cut of the support of $\rho$ are very near of the local minimum.