A generalization of the Jordan–Schwinger map: classical version and its q–deformation.

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Abstract

For all three–dimensional Lie algebras the construction of generators in terms of functions on 4-dimensional real phase space is given with a realization of the Lie product in terms of Poisson brackets. This is the classical Jordan–Schwinger map which is also given for the deformed algebras $\mathcal{S}\mathcal{L}_q(2,\mathbb{R})$, $\mathcal{E}_q(2)$ and $\mathcal{H}_q(1)$. The $\mathcal{U}_q(n)$ algebra is discussed in the same context.

1 Introduction

The Jordan-Schwinger map [1],[2], is well known to physicists in connection with the realization of the $\mathcal{S}\mathcal{U}(2)$ algebra in terms of polynomials of creation and annihilation operators, $a^\dagger, a$, for the two–dimensional harmonic oscillator. This map allows to construct all the irreducible representations of the group $SU(2)$ using the states of two–dimensional harmonic oscillator and the technique of creation and annihilation operators: in fact through this map the angular momentum operators may be expressed as quadratic forms in $a^\dagger, a$. This map is a particular case of the second quantization procedure due to which generators of any Lie algebra may be realized as quadratic forms in creation and annihilation operators, of bosonic or fermionic kind, if a matrix representation of this Lie algebra is given.

This map has also been used ([3], [4]) in connection with realizations of $\mathcal{S}\mathcal{L}_q(2)$, where generators may be realized as quadratic forms of q-oscillators. Here and in the following

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we shall mean with $SL(2)$ the complex Lie algebra $A_1$ in the Cartan classification, whose real forms are $SU(2)$, $SL(2, \mathbb{R})$ and $SU(1, 1)$. When needed we will choose one of them.

An abstract Lie algebra, $G$, may be realized not only in terms of operators but also in terms of functions on Poisson manifolds ([5]), the Lie bracket now being the Poisson bracket. Realizations of this kind exist, in fact, naturally when Poisson manifolds are submanifolds of $G^*$, the dual of $G$ ([6],[7],[8]). It is then natural to ask whether Poisson maps exist between generic manifolds and $G^*$. In section 2 we will be interested in finding these Poisson maps between $\mathbb{C}^2$, equipped with the standard symplectic structure, and $SL(2)^*$. This is what we may call a classical Jordan–Schwinger map and it is already available in the literature ([9], [10]). As we already mentioned, it is through the Jordan–Schwinger map that a realization of $SL_q(2)$ was given in terms of 2-dimensional q–oscillators; analogous objects can be introduced in the classical setting. These classical hamiltonian systems may be of interest per se, showing a particular non linearity in the energy dependence of the frequency ([11]). Another reason for such a study is that the two kinds of realizations are connected by the Dirac map ([12]), that is by canonical quantization procedure. This would allow hopes to transfer results obtained in the classical context to the quantum one, a part from ordering problems.

In section 3 all the three–dimensional Lie algebras are considered with respect to their realization as Poisson algebras and the relevant maps, which we will call generalized classical Jordan–Schwinger maps, are given.

The subject of section 4 deals with effects of deformation on the Jordan–Schwinger map, while in sections 5 and 6 we comment on the possibility to extend the above contructions to higher dimensional Lie algebras, taking into particular consideration $SU_q(n)$.

2 The Jordan–Schwinger map and its classical analogue

We review very briefly the Jordan–Schwinger map as it was introduced by Schwinger to deal with angular momentum in terms of creation and annihilation operators.

A realization of the Lie algebra $SL(2)$ in terms of creation and annihilation operators, $a_i, a_i^\dagger, \ i \in \{1, 2\}$, satisfying standard commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}$$

is provided by

$$X_+ = a_1^\dagger a_2 , \quad X_- = a_2^\dagger a_1 , \quad X_3 = \frac{1}{2}(\hat{n}_1 - \hat{n}_2)$$

(1)

with $\hat{n}_i = a_i^\dagger a_i, \ i = 1, 2$. We have in fact

$$[X_\pm, X_3] = \mp X_\pm \quad [X_+, X_-] = 2X_3.$$  

(2)
To consider a classical analogue of this realization, we apply a "dequantization procedure", i.e., we replace operators with functions on phase space and commutators with Poisson brackets. To be definite we will consider the Lie algebra of $SL(2, \mathbb{R})$. Usually Poisson brackets are defined in real phase space for real-valued functions. As in our "dequantization procedure" we replace creation and annihilation operators by complex-valued functions on real phase space, we have to extend the usual definition of Poisson brackets to the case of complex-valued functions. We recall that this extension can be defined by linearity. For two complex-valued functions $f = f_1 + if_2$ and $g = g_1 + ig_2$ for which real and imaginary terms are real functions on the phase space, we have

$$\{f, g\} = \{f_1, g_1\} - \{f_2, g_2\} + i\{f_1, g_2\} + \{f_2, g_1\}. \quad (3)$$

Here $\{f_i, g_i\}$ are given by the standard Poisson brackets for real functions on the phase space with coordinates $p_i, q_i$

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0. \quad (4)$$

Then for complex variables $z_i, z_i^*, \ z_i = \frac{1}{\sqrt{2}}(p_i + iq_i)$ the symplectic Poisson brackets are given by

$$\{z_i^*, z_j\} = i\delta_{ij}. \quad (5)$$

A realization of $SL(2, \mathbb{R})$ in terms of complex-valued functions is obtained by setting

$$\mathcal{X}_+ = i\sqrt{i} z_2, \quad \mathcal{X}_- = i\sqrt{i} z_1, \quad \mathcal{X}_3 = \frac{i}{2}(z_1 z_2 - z_2 z_1) \quad (6)$$

and we get

$$\{\mathcal{X}_\pm, \mathcal{X}_3\} = \mp\mathcal{X}_\pm, \quad \{\mathcal{X}_+, \mathcal{X}_-\} = 2\mathcal{X}_3. \quad (7)$$

This is what we mean by a classical realization of the Lie algebra $\mathcal{S}\mathcal{L}(2, \mathbb{R})$, or the classical Jordan–Schwinger map for such an algebra.

To construct the angular momentum algebra in terms of Poisson brackets, we have used complex valued functions (6). The use of complex valued functions is convenient but not mandatory. In fact the same algebra can be realized using real functions of $q_i, p_i, i = 1, 2$.

The real analogue of Jordan–Schwinger map relations (6) are

$$\mathcal{Y}_+ = p_2 q_1, \quad \mathcal{Y}_- = p_1 q_2, \quad \mathcal{Y}_3 = \frac{1}{2}(p_1 q_1 - p_2 q_2). \quad (8)$$

In fact, using the brackets (4), it can be checked that these functions reproduce relations (7), i.e., they close on the algebra of $SL(2, \mathbb{R})$.

We have shown therefore that it is possible to have Poisson brackets realization of the Lie algebra commutation relations of $SL(2, \mathbb{R})$ in two ways. One way uses the functions on $\mathbb{R}^3$ with values in $\mathbb{R}$ and the standard Poisson brackets. The other one uses the extended definition of Poisson brackets for complex valued functions on $\mathbb{R}^4$.
Let us now investigate the geometrical and algebraic meaning of what has been done. We started with the symplectic Poisson algebra $\mathcal{F}(\mathbb{R}^4; \mathbb{R})$ with the standard Poisson bracket $\{p_a, q_b\} = \delta_{a b}$ and considered a Poisson subalgebra generated by $\mathcal{Y}_+, \mathcal{Y}_-, \mathcal{Y}_3$. These functions close on the Lie algebra of $SL(2, \mathbb{R})$, say $\mathcal{G}$. They are functions on $\mathbb{R}^4$ and linear functions on $\mathcal{G}^*$, the dual of $\mathcal{G}$, thus they may be taken as coordinates for $\mathcal{G}^*$. To put it more explicitly, we have constructed a map from $\mathbb{R}^4$ to $\mathbb{R}^3 \equiv \mathcal{G}^*$ given by

$$\pi : (p_1, q_1, p_2, q_2) \rightarrow \left( p_2 q_1, p_1 q_2, \frac{1}{2}(p_1 q_1 - p_2 q_2) \right)$$

With respect to the natural Poisson bracket on $\mathcal{G}^*$ this map is a Poisson map, that is, identifying $(p_2 q_1, p_1 q_2, \frac{1}{2}(p_1 q_1 + p_2 q_2))$ with $(x_+, x_-, x_3) \in \mathcal{G}^*$ and

$$\pi^* x^+ = \mathcal{Y}^+ \in \mathcal{F}(\mathbb{R}^4), \quad \pi^* x^- = \mathcal{Y}^- \in \mathcal{F}(\mathbb{R}^4), \quad \pi^* x^3 = \mathcal{Y}^3 \in \mathcal{F}(\mathbb{R}^4),$$

we have

$$\{\pi^*(x_\pm), \pi^*(x_3)\}_{\mathbb{R}^4} = \pi^*(\{x_\pm, x_3\}_{\mathcal{G}^*})$$

(9)

and similarly for the other bracket. We could use this observation to generalize the notion of Jordan–Schwinger map. We consider a symplectic Poisson algebra $\mathcal{F}$ on a symplectic manifold $M$ and a generic $n$-dimensional Lie algebra, so that our definition of (classical) Jordan–Schwinger map would be:

**Definition** A classical generalized Jordan–Schwinger map for a $n$-dimensional Lie algebra $\mathcal{G}$ with structure constants $c_{ij}^k$, is a realization of $\mathcal{G}$ as a $n$-dimensional Lie subalgebra $\tilde{\mathcal{G}}$ of $\mathcal{F}(M)$, where $\mathcal{F}(M)$ is equipped with the standard Poisson bracket.

If $\tilde{\mathcal{G}}$ is realized by $f_1, ... f_n$ with Poisson brackets

$$\{f_i, f_j\} = c_{ij}^k f_k$$

(10)

we may consider the Poisson algebra on $\mathcal{F}(\tilde{\mathcal{G}}^*)$ given by $\{x_i, x_j\} = c_{ij}^k x_k$. We have an associated map $\pi : M \rightarrow \mathcal{G}^*$, characterized by $\pi^* x_i = f_i$. We notice that this map needs not to be a projection, we will give indeed some examples where $\dim M < \dim \mathcal{G}^*$.

Let us consider $\mathbb{R}^2 \equiv T^* \mathbb{R}$ with standard coordinates and Poisson brackets

$$\{p, q\} = 1.$$

1) $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$(p, q) \rightarrow (pq, \frac{p^2}{2}, -\frac{q^2}{2}) \equiv (\mathcal{H}, \mathcal{X}_+, \mathcal{X}_-)$$

and find

$$\{\mathcal{H}, \mathcal{X}_{2}\} = \pm 2 \mathcal{X}_+ \quad \{\mathcal{X}_+, \mathcal{X}_-\} = \mathcal{H}$$

that is, we get a classical realization of the $SL(2, \mathbb{R})$ algebra as a subalgebra of $\mathcal{F}(\mathbb{R}^2)$. 
2) \[\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3\]
\[(p, q) \mapsto (e^p, e^{-q}, q) \equiv (X, Y, Z)\]
and find
\[\{X, Z\} = X, \quad \{Y, Z\} = -Y, \quad \{X, Y\} = 0\]
which is the Poincaré algebra in three dimensions.

3) \[\pi_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^3\]
\[(p, q) \mapsto (\sin q, \cos q, p) \equiv (A, B, C)\]
and find
\[\{A, C\} = -B, \quad \{B, C\} = A, \quad \{A, B\} = 0\]
which is the algebra of \(E(2)\), the euclidean group in three dimensions.

Going back to the general situation, in order to give a basis independent version of
(10) let us consider a \(\mathbb{R}\)-linear map on the second argument
\[F : M \times \mathcal{G} \rightarrow \mathbb{R}\]  \hspace{1cm} (11)
then (10) may be written in the following way
\[\{F(u), F(v)\} = F([u, v])\]  \hspace{1cm} (12)
where \(u, v \in \mathcal{G}\) and \(F(u) : M \rightarrow \mathbb{R}\). If \(u\) and \(v\) are taken to be the generators of the Lie algebra, say \(e_i, e_j\), then
\[\{F(e_i), F(e_j)\} = F([e_i, e_j]) = e_{ij}^k F(e_k)\]  \hspace{1cm} (13)
which is equivalent to (10).

Let us define
\[\hat{F} \equiv F_{\mathcal{G}} : M \rightarrow \mathcal{G}^* (\equiv \mathcal{L}) \setminus (\mathcal{G}, \mathbb{R})\].
The map \(\hat{F}\) could be identified with our previous definition of the classical generalized Jordan–Schwinger map for a generic Lie algebra.

A third interpretation of the classical generalized Jordan–Schwinger map is suggested by the last map \(\hat{F}\). Indeed let us consider a symplectic manifold \((M, \omega)\) together with a Lie algebra \(\mathcal{G}\) realized in terms of vector fields on \(M\), say \(X_1, \ldots, X_n\),
\[[X_i, X_j] = c_{ij}^k X_k.\]  \hspace{1cm} (14)
Consider then the hamiltonian functions, \(f_1, \ldots, f_n\), associated with these vector fields, namely
\[i_{X_k} \omega = -df_k\]
we have: **Proposition** A classical generalized Jordan–Schwinger map is equivalent to a strongly symplectic action of $G$ on $M$.

We recall that an action is said to be strongly symplectic if (14) implies that

$$\{f_i, f_j\} = c_{ij}^k f_k$$

We would like to notice that, to make our definition meaningful, the fields $X_i$ need not to be complete, i.e. it is not necessary that they integrate to a group.

Previous examples show that when we consider a generalized Jordan–Schwinger map we should not expect that the subalgebra will be realized in terms of quadratic maps like for $SL(2, \mathbb{R})$.

We can now ask if for any Lie algebra $G$ there exists a realization in terms of a Poisson subalgebra of a symplectic Poisson algebra $\mathcal{F}(M)$, for some symplectic manifold $M$. The answer is positive, indeed if $G$ is the Lie algebra of a Lie group $G$, we can consider $M = T^*G$ and the (left or right) momentum map $J : T^*G \rightarrow G^*$ provides us with the required realization. Moreover the momentum map is what we called $\tilde{F}$.

It is possible to consider a symplectic manifold of smaller dimensions. Indeed we can consider the Lie group $G$ and any maximal closed subgroup $K$. Here there is an action of $G$ on $G/K$, say the left action, if the quotient was taken with respect to the right action of $K$. We set $M = T^*(G/K)$ and the lifted action of $G$ to $T^*(G/K)$ is a strongly symplectic map providing as with the required realization. In general, it is enough to consider any manifold carrying a symplectic action of $G$ if $H^2(G, \mathbb{R}) = \tau$, $H^1(G, \mathbb{R}) = \tau$. Many examples are also constructed by starting with a representation of $G$, i.e. a linear action on some vector space $V$ and doubling the space by going to $T^*V$ or $V \mathbb{C}$. We shall consider this situation in Section 6.

We notice at this point that our “dequantization” and “quantization” procedures associated with the Jordan-Schwinger map rely on the correspondence $a_i^+ \leftrightarrow z_i^*$, $a_i \leftrightarrow z_i$. Therefore, to be able to save this correspondence we shall try to find a symplectic realization of our Lie algebras in terms of a standard structure on $\mathbb{R}^{2n}$ or $\mathbb{C}^n$. We shall then say that we have a classical Jordan-Schwinger map (i.e. we drop the word ‘generalized’) if the Lie algebra $G$ can be realized as a Poisson sub-algebra of the real valued or complex valued functions on $\mathbb{R}^{2n}$. To put it differently, we are requiring that the symplectic manifold $(M, \omega)$ should be restricted to be $(\mathbb{R}^{2n}, dp_i \wedge dq^i)$. We have decided to introduce the notion of generalized Jordan–Schwinger map to make contact with the constructions associated with geometric quantization theory ([6], [7], [8]. In the coming section we show that for any three-dimensional Lie algebra we have a classical Jordan-Schwinger map on $T^*\mathbb{R}^2$. We notice that, by using $T^*(G/K)$, with $K$ any closed one-dimensional subgroup, it would be possible to give a generalized Jordan-Schwinger map on a four-dimensional symplectic manifold. Our result will show that it is possible to take our symplectic manifold to be $T^*\mathbb{R}^2$. 

6
3 Three dimensional algebras in terms of Poisson brackets

As it is known ([13]), Poisson structures on $\mathbb{R}^3$ can be characterized by 1-forms which admit an integrating factor. The argument goes along the following lines. On $\mathbb{R}^3$ we consider coordinates $(x_1, x_2, x_3)$ along with a volume form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. Any Poisson bracket defines a bivector field

$$\Lambda = c_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$ 

By contracting $\Lambda$ with $\Omega$ we find

$$i_\Lambda \Omega = \epsilon^{ijk} c_{ij}(x) dx_k$$

i.e.

$$i_\Lambda \Omega = \alpha = A^k dx_k.$$ 

The Jacobi identity is equivalent to

$$d\alpha \wedge \alpha = 0$$

i.e.

$$\epsilon_{ijk}(\frac{\partial A^k}{\partial x^i} - \frac{\partial A^i}{\partial x^k})A^j = 0.$$ 

Locally it means that $\alpha = f d\phi$ and $c_{ij} = \epsilon_{ijk} f \frac{\partial \phi}{\partial x^k}$. Symplectic leaves for $\Lambda$, are characterized to be level sets of $\phi$, say $\Sigma_\phi$, or by the fact that the pull-back of $\alpha$ to each leaf vanishes. The 1-form $\alpha$ will be called a Casimir 1-form for the given brackets.

Now we are able to demonstrate

**Theorem** Any 3–dimensional Lie algebra acts in a strongly symplectic way on $(\mathbb{R}^4, \omega_0)$.

The statement of the theorem is equivalent to say that every three dimensional Lie algebra may be realized as a Poisson subalgebra of $\mathcal{F}(\mathbb{R}^4)$ with the standard Poisson structure, or which is the same, for any three–dimensional Lie algebra it is possible to exhibit a classical Jordan–Schwinger map.

Let us demonstrate this explicitly. The following Poisson brackets provide a realization of any real three dimensional Lie algebra when the real parameters $a$, $b$, $c$, $n$, are selected accordingly ([13]):

$$\{x, y\} = cw + ny, \quad \{y, w\} = ax, \quad \{w, x\} = by + nw.$$ \hspace{1cm} (15) 

These Poisson brackets satisfy the Jacobi identity if and only if

$$na = 0$$ \hspace{1cm} (16)
The Casimir 1–form is

\[ \alpha = (cw + ny)dw + axdx + (by - n)dy \]  \hspace{1cm} (17)

or in more convenient form

\[ \alpha = n(ydw - wdy) + d \frac{1}{2}(ax^2 + by^2 + cw^2) \]  \hspace{1cm} (18)

with

\[ d\alpha \wedge \alpha = n a x \; dy \wedge dw \wedge dx \]  \hspace{1cm} (19)

We will show by explicit construction that all the three–dimensional algebras, which are classified by equation (15), may be realized as subalgebras of \( \mathcal{F}(\mathbb{R}^3) \). Namely we will consider all inequivalent cases described by the choice of the parameters.

Let us consider first the case when the only non zero parameter is \( n \). We have then

\[ \{x, y\} = ny, \; \{y, w\} = 0, \; \{w, x\} = -nw. \]  \hspace{1cm} (20)

This is the Lie algebra of \( SB(2, \mathbb{C}) \) for any \( n \neq 0 \). We can define our map \( \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) to be given by

\[ x = p_1 + p_2, \; y = e^{ny_1}, \; w = e^{ny_2} \]  \hspace{1cm} (21)

or, in terms of complex valued variables,

\[ x = i(z_1 + z_2), \; y = ie^{nz_1^2}, \; w = ie^{nz_2^2}. \]  \hspace{1cm} (22)

Both the subalgebras (21), (22), reproduce relations (20).

A different case corresponds to \( n=0 \) and all other parameters are positive numbers. In this case we obtain the \( SU(2) \) algebra. A real realization of this algebra is given by

\[ x = \frac{1}{2}(q_1 q_2 + p_1 p_2)\sqrt{bc}, \; y = \frac{1}{2}(p_1 q_2 - q_1 p_2)\sqrt{ac}, \; w = \frac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2)\sqrt{ab} \]  \hspace{1cm} (23)

while a complex one is

\[ x = \frac{1}{2}(z_1^* z_2^* - z_1 z_2) + \frac{i}{4}(z_1^2 - z_2^2 - z_1^* z_2^* + z_1^* z_2), \; y = \frac{1}{2}(-z_1^* z_2^* + z_1 z_2) + \frac{i}{4}(z_1^2 - z_2^2 - z_1^* z_2^* + z_1^* z_2) \]

\[ w = \frac{1}{2}(z_1^* z_2^* - z_2^* z_1). \]

If each one of the three parameters \( a, b, c \) is negative, a realization of the same algebra will be obtained by changing the sign of all the three generators.

If one of the parameters is negative, say \( b \), and the others are positive, we have a classical realization of the \( SL(2, \mathbb{R}) \) algebra and a possible choice for the classical Jordan–Schwinger map in terms of real functions is given by

\[ \]
\[ x = \frac{1}{2}(p_1 q_2 + p_2 q_1) \sqrt{-bc}, \quad y = \frac{1}{2}(p_1 q_2 - p_2 q_1) \sqrt{ac}, \quad w = \frac{1}{2}(p_1 q_1 - p_2 q_2) \sqrt{-ab} \] (24)

while in terms of complex functions it is

\[ x = \frac{i}{2}(z_1 z_2^* + z_2 z_1^*) \sqrt{-bc}, \quad y = \frac{i}{2}(z_1 z_2^* - z_2 z_1^*) \sqrt{ac}, \quad w = \frac{i}{2}(z_1 z_1^* - z_2 z_2^*) \sqrt{-ab}. \] (25)

We will discuss now the cases when the parameter \(a\) is equal to zero and \(b, c\) are different from zero. If both \(b\) and \(c\) are positive we have the three-dimensional \(E_2\) algebra. Then, a realization in terms of real phase space variables is

\[ x = (p_1 + p_2) \sqrt{bc}, \quad y = \sin \left( \frac{q_1 + q_2}{2} \right), \quad w = -\sqrt{\frac{b}{c}} \cos \left( \frac{q_1 + q_2}{2} \right) \] (26)

and a complex realization is

\[ x = i(z_1 + z_2) \sqrt{bc}, \quad y = i \sin \left( \frac{z_1^* + z_2^*}{2} \right), \quad w = -i \sqrt{\frac{b}{c}} \cos \left( \frac{z_1^* + z_2^*}{2} \right) \] (27)

If one coefficient is positive, say \(c\), and the other one, say \(b\), is negative, we have the Poincaré algebra in three dimensions. A possible real realization is

\[ x = (p_1 + p_2) \sqrt{-bc}, \quad y = \sinh \left( \frac{q_1 + q_2}{2} \right), \quad w = \sqrt{-\frac{b}{c}} \cosh \left( \frac{q_1 + q_2}{2} \right) \] (28)

and a complex one is

\[ x = i(z_1 + z_2) \sqrt{-bc}, \quad y = i \sinh \left( \frac{z_1^* + z_2^*}{2} \right), \quad w = i \sqrt{-\frac{b}{c}} \cosh \left( \frac{z_1^* + z_2^*}{2} \right) \] (29)

The last unexploited case, namely when only one coefficient is different from zero, describes the Heisenberg-Weyl group, \(H(1)\). By choosing \(c\) to be the non-zero coefficient, we may consider, for example, the following real realization for the algebra of this group

\[ x = c q_1, \quad y = p_1 q_2, \quad w = -q_2 \] (30)

and the complex one

\[ x = iz_1^*, \quad y = iz_1 z_2^*, \quad w = -iz_2^*. \] (31)

Of course, there are many other realizations in terms of functions on \( \mathbb{R}^4 \), for the Heisenberg-Weyl case.

The final case of an abelian algebra, which is obtained by taking all the coefficients in (15) equal to zero, may also be realized by many functions. We give for completeness one of the possible realizations

\[ x = q_1 p_1, \quad y = q_2 p_2, \quad w = I \] (32)
Now we will discuss the case when \( a \) is equal to zero and the other three parameters \( b, c, n \) are not equal to zero. This case may be reduced to the considered ones by linear transformation of generators \( y \) and \( w \) which is equivalent to diagonalizing the 2-dimensional matrix

\[
\begin{pmatrix}
  n & c \\
  -b & n
\end{pmatrix}
\]

the eigenvalues will be

\[
\lambda_1 = n + \sqrt{-bc} \quad \lambda_2 = n - \sqrt{-bc}
\]

while the new basis of functions is

\[
X = x \\
Y = \frac{1}{2}y + \sqrt{-\frac{c}{b}}w \\
W = \frac{1}{2}y - \sqrt{-\frac{c}{b}}w
\]

and satisfies the Poisson brackets

\[
\{X, Y\} = \lambda_1 Y \quad \{X, W\} = \lambda_2 W \quad \{Y, W\} = 0
\]

Then, depending on the values of the two parameters \( \lambda_1, \lambda_2 \), we may identify one of the previous algebras.

Let us consider some realization of these relations. If the coefficients \( b \) and \( c \) have different signs a real valued realization may be chosen to be

\[
x = p_1 + p_2 \\
y = e^{\lambda_1 q_1} + e^{\lambda_2 q_2} = e^{(n+\sqrt{-bc})q_1} + e^{(n-\sqrt{-bc})q_2} \\
w = \sqrt{-\frac{b}{c}}(e^{\lambda_1 q_1} - e^{\lambda_2 q_2}) = \sqrt{-\frac{b}{c}}(e^{(n+\sqrt{-bc})q_1} - e^{(n-\sqrt{-bc})q_2})
\]

and a complex one is

\[
x = i(z_1 + z_2) \\
y = ie^{\lambda_1 z_1} + e^{\lambda_2 z_2} = ie^{(n+\sqrt{-bc})z_1} + e^{(n-\sqrt{-bc})z_2} \\
w = i\sqrt{-\frac{b}{c}}(e^{\lambda_1 z_1} - e^{\lambda_2 z_2}) = i\sqrt{-\frac{b}{c}}(e^{(n+\sqrt{-bc})z_1} - e^{(n-\sqrt{-bc})z_2})
\]

If the coefficients have equal signs we may choose

\[
x = p_1 + p_2 \\
y = e^{(n+\sqrt{-bc})q_1} + e^{(n-\sqrt{-bc})q_2} \\
w = -\sqrt{-\frac{b}{c}}(e^{(n+\sqrt{-bc})q_1} - e^{(n-\sqrt{-bc})q_2})
\]
in terms of real functions, and
\[ x = i(z_1 + z_2) \]
\[ y = i(e^{(n + \sqrt{c})z} - e^{(n - \sqrt{c})z^*}) \]
\[ w = -i \frac{b}{c} (e^{(n + \sqrt{c})z} - e^{(n - \sqrt{c})z^*}) \] (39)

If one of the coefficients is equal to zero, for example \( c = 0 \), equations (15) become relations
\[ \{x, y\} = ny, \quad \{y, w\} = 0, \quad \{x, w\} = nw - by \] (40)
and can be realized by
\[ x = p_1 + p_2 \quad y = e^{nq_2} \quad w = -bq_1 e^{nq_2} \] (41)
or by
\[ x = i(z_1 + z_2) \quad y = ie^{nz_2} \quad w = -bi z_1^* e^{nz_2^*} \] (42)

All the other cases may be reduced to the considered ones.

Thus we have shown by construction that all 3-dimensional Lie algebras can be realized through a classical Jordan–Schwinger map.

4 \ Deformed Jordan–Schwinger map

In the previous section we have realized all the 3-dimensional Lie algebras as algebras of functions on \( \mathbb{R}^3 \) and we have found a map, \( \pi \), from \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \) (what we call classical Jordan–Schwinger map) which gives a realization of 3-dimensional algebras in terms of functions on \( \mathbb{R}^4 \). By replacing \( z, z^* \), with creation and annihilation operators we obtain a realization of these algebras in terms of operators, modulo ordering problems, which have to be taken into account case by case when they occur.

Our next goal will be to consider deformed algebras, which are known in the literature as non commutative Hopf algebras or quantum groups, and we will look for a realization of the associated commutator algebras in terms of Poisson brackets. We will pose the problem in the following way:

Given a 3-dimensional Lie algebra, \( G \), together with its classical realizations in \( \mathcal{F}(\mathbb{R}^3) \) and in \( \mathcal{F}(\mathbb{R}^4) \), connected by the classical Jordan–Schwinger map
\[ \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \]
which we assume to be known now, we will look for the possibility to exhibit a Jordan–Schwinger map for the corresponding deformed algebra \( G_H \), that is to say a realization
of $G_{11}$ in terms of functions on $\mathbb{R}^4$ and Poisson brackets. We will approach the problem first by looking for a map

$$\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

which gives the deformed generators as functions of the undeformed ones, then, by using the classical Jordan–Schwinger map $\pi$ of the undeformed algebra, we will obtain the wanted map by composing $\phi$ and $\pi$

$$\pi' = \phi \circ \pi$$

This situation may be visualized by the following diagram

\[ \begin{array}{ccc} 
\mathbb{R}^3 & \rightarrow & \mathbb{R}^3 \\
\pi' & \uparrow \phi & \\
\mathbb{R}^4 & \rightarrow & \mathbb{R}^3 
\end{array} \]

The map $\pi'$ is what we call the deformed Jordan–Schwinger map. Starting with a q-deformed algebra we consider the associated commutator algebra and realize it in terms of Poisson brackets looking for a map $\phi$ to be composed with the classical Jordan–Schwinger map $\pi$. Of course in some cases one may find convenient to look for $\pi'$ directly, without bothering with the factorization. We give a solution, that is a deformed Jordan–Schwinger map, for $SL_q(2, \mathbb{R})$, $E_q(2)$ and for $\mathcal{H}_q(1)$ ($\mathcal{H}(1)$ is the Lie algebra of the Heisenberg–Weyl group).

Let us consider the Lie algebra $SL(2)$ given by relations (2). A deformation of this realization is obtained ([3], [4]) by setting

$$f(\hat{n}_i) = \sqrt{\frac{\sinh \lambda \hat{n}_i}{\hat{n}_i \sinh \lambda}} \quad \lambda \in \mathbb{R} - \{0\}$$

and defining

$$X_{q^+} = f(\hat{n}_1) a_1^\dagger a_2 f(\hat{n}_2) = a_1^\dagger a_2 q$$
$$X_{q^-} = f(\hat{n}_2) a_2^\dagger a_1 f(\hat{n}_1) = a_2^\dagger a_1 q$$
$$X_{q^3} = \frac{1}{2}(\hat{n}_1 - \hat{n}_2), \quad q = e^\lambda$$

we get a realization of $SL_q(2)$. This is defined to be

$$[X_{q^+}, X_{q^3}] = -X_{q^-}$$
$$[X_{q^-}, X_{q^3}] = X_{q^-}$$
$$[X_{q^+}, X_{q^-}] = \frac{\sinh 2\lambda X_{q^3}}{\sinh \lambda}.$$  \hfill (45)

This realization is a generalization of the Jordan–Schwinger map to $SL_q(2)$. 

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Let us look now at the classical counterpart and consider, to be definite, $\mathcal{SL}(2, \mathbb{R})$. What we mean by classical version of the commutators algebra associated with $\mathcal{SL}_q(2)$ is a Poisson algebra of functions such that Poisson brackets have the same form of commutators (45)

$$\{X_{q\pm}, X_{q\beta}\} = \mp \tilde{V}_{q\pm} \quad \{X_{q+}, X_{q-}\} = \frac{\sinh 2\lambda X_3}{\sinh \lambda}. \quad (46)$$

It can be checked that these brackets are in fact Poisson brackets, evaluating the associated Casimir 1-form, $\alpha_q$, and checking the Jacobi identity, which, as we said in section 3, is equivalent to show that $d\alpha_q \wedge \alpha_q = 0$. The Casimir 1-form for the Poisson structure given by relations (46) is

$$\alpha_q = d \{X_{q+}, X_{q-}\} + \frac{1}{\lambda \sinh \lambda} (\sinh \lambda X_3)^2 \quad (47)$$

verifying

$$d\alpha_q \wedge \alpha_q = 0$$

Let us look now at a realization of this Poisson algebra in terms of functions of the undeformed algebra. A classical realization of $\mathcal{SL}(2, \mathbb{R})$, is given by relations (6) or by their real version (8). We will indicate with the subscript 0 the undeformed Poisson structure.

To obtain the wanted realization in terms of functions on $\mathbb{R}^3$ while using the Poisson brackets (7), i.e. to find the map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, we have to solve the three differential equations

$$\{X_{q\pm}, X_{q\beta}\}_0 = \mp X_{q\pm} \quad \{X_{q+}, X_{q-}\}_0 = \frac{\sinh 2\lambda X_3}{\sinh \lambda}. \quad (48)$$

It can be checked that a solution is given by

$$X_{q\pm} = \frac{sh \lambda (J + X_3) sh \lambda (J - X_3)}{(J + X_3)(J - X_3) sh^2 \lambda} \sqrt{\frac{sh \lambda}{\lambda}} X_{q\pm}, \quad X_{q\beta} = X_{q\beta} \quad (49)$$

where

$$J = \sqrt{(X_+ X_- + X_3^2)).$$

Composing the map $\phi$ with the classical Jordan–Schwinger map, $\pi$, given by (6), we obtain the wanted map $\pi'$

$$\pi'(X_{q+}) = i \zeta_1^* \zeta_2, \quad \pi'(X_{q-}) = i \zeta_2^* \zeta_1, \quad \pi'(X_{q3}) = \pi'(X_3) \quad (50)$$

with

$$\zeta_i = \tilde{f}(i z_i z_i^*) z_i \quad i \in \{1, 2\}, \quad (51)$$

$$\tilde{f}(x) = \left\{ \frac{\sinh \lambda}{\lambda} \right\}^{\frac{1}{2}} f(x) \quad f(x) = \sqrt{\frac{sh \lambda}{x sh \lambda}}. \quad (52)$$

The map $\pi'$ is what we call classical Jordan–Schwinger map for the deformed case.
In comparing the two realizations (44) and (50), we notice that in the last one the factor \((sh\lambda)\lambda\) appears signalling that in the deformation process also the Dirac map is deformed. A similar construction can be done with real functions. Let us introduce new real variables in \(\mathbb{R}^4\)

\[
\pi_i = \tilde{f}(p_i q_i) p_i, \quad \xi_i = \tilde{f}(p_i q_i) q_i
\]

where the function \(\tilde{f}\) is given by relation (52). Then the map \(\pi'\) is given by:

\[
\mathcal{Y}_{q+} = \pi_1 \xi_2, \quad \mathcal{Y}_{q-} = \pi_2 \xi_1, \quad \mathcal{Y}_{q0} = \mathcal{Y}^3
\]

One can check that the Poisson algebra so obtained is again the q–deformed algebra \(\mathcal{S}\mathcal{L}(2, \mathbb{R})\) defined by relations (45).

Let us now consider the euclidean algebra \(\mathcal{E}(2)\). The Lie algebra commutation relations are

\[
[P_x, P_y] = 0, \quad [J, P_x] = P_y, \quad [J, P_y] = -P_x.
\]

(55)

A deformation of this algebra which has been shown to be a non commutative Hopf algebra has been given in [14], the commutation relations being

\[
[P_{xq}, P_{yq}] = 0, \quad [J_q, P_{xq}] = P_{yq}, \quad [J_q, P_{yq}] = -\frac{1}{\lambda} sh(\lambda P_x).
\]

(56)

Also in this case it can be checked that the classical analogue of (56)

\[
\{\mathcal{P}_{xq}, \mathcal{P}_{yq}\} = 0, \quad \{\mathcal{J}_q, \mathcal{P}_{xq}\} = \mathcal{P}_{yq}, \quad \{\mathcal{J}_q, \mathcal{P}_{yq}\} = -\frac{1}{\lambda} sh(\lambda \mathcal{P}_x).
\]

(57)

satisfies the Jacobi identity, by evaluating the Casimir 1–form \(\alpha_q\) and by checking that \(d\alpha_q \wedge \alpha_q = 0\). As it can be verified the 1-form is

\[
\alpha_q = \frac{1}{2} d \{ \mathcal{P}_y^2 + \frac{2}{\lambda^2} (sinh \frac{\lambda}{2} \mathcal{P}_x)^2 \}
\]

(58)

and it satisfies the wanted equation. For the undeformed algebra a classical Jordan–Schwinger map is given by relations (27), where we make the following identification

\[
\mathcal{P}_x = y, \quad \mathcal{P}_y = w, \quad \mathcal{J} = x.
\]

To find the map \(\phi\) such that

\[
\mathcal{P}_{xq} = \phi^*(\mathcal{P}_x), \quad \mathcal{P}_{yq} = \phi^*(\mathcal{P}_y), \quad \mathcal{J}_q = \phi^*(\mathcal{J})
\]

we have to solve the three differential equations

\[
\{\mathcal{P}_{xq}, \mathcal{P}_{yq}\}_0 = 0, \quad \{\mathcal{J}_q, \mathcal{P}_{xq}\}_0 = \mathcal{P}_{yq}, \quad \{\mathcal{J}_q, \mathcal{P}_{yq}\}_0 = -\frac{1}{\lambda} sh(\lambda \mathcal{P}_{xq}).
\]

(59)

It can be verified that a solution is

\[
\mathcal{P}_{xq} = \mathcal{P}_x, \quad \mathcal{P}_{yq} = \sqrt{-\frac{2}{\lambda^2} ch(\lambda \mathcal{P}_x)}, \quad \mathcal{J}_q = \frac{\mathcal{J}}{\mathcal{P}_y} \sqrt{-\frac{2}{\lambda^2} ch(\lambda \mathcal{P}_x)}.
\]

(60)
By composing the map \( \phi \) with the classical Jordan–Schwinger map (26) or (27) we obtain the Jordan–Schwinger map for the deformed algebra \( E_q(2) \)

\[
\pi^* p_{xq} = \pi^* p_x, \quad \pi^* p_{yq} = \sqrt{-\frac{2}{\lambda^2} \text{ch}(\lambda \pi^* p_x)}, \quad \pi^* J_q = \frac{\pi^* J}{\pi^* p_y} \sqrt{-\frac{2}{\lambda^2} \text{ch}(\lambda \pi^* p_x).}
\] (61)

Finally we consider the Heisenberg–Weyl group \( H(1) \). The Lie brackets for the algebra \( H(1) \) are given by

\[
[X, Y] = W, \quad [W, X] = 0, \quad [W, Y] = 0.
\]

A realization of this algebra as a Poisson subalgebra of \( \mathcal{F}(\mathbb{R}^4) \) is given by relations (30) and (31), which are respectively a real and a complex realization. A \( q \)-deformation of the Heisenberg–Weyl algebra, which turns out to be a noncommutative Hopf algebra has been given in [14], the deformed commutators being

\[
[X_q, Y_q] = \frac{sh \lambda W}{\lambda}, \quad [W_q, X_q] = 0, \quad [W_q, Y_q] = 0.
\] (62)

The classical analogue of (62), that is

\[
\{X_q, Y_q\} = \frac{sh \lambda W}{\lambda}, \quad \{W_q, X_q\} = 0, \quad \{W_q, Y_q\} = 0
\] (63)

can be shown to be a Poisson algebra by evaluating the associated Casimir 1-form, which turns out to be

\[
\alpha_q = \frac{d \text{cosh}(\lambda W)}{\lambda^2}
\] (64)

and verifying that \( d\alpha_q \wedge \alpha_q = 0 \). A realization of this algebra as a Poisson subalgebra of \( \mathcal{F}(\mathbb{R}^3) \), whose generators are functions of the undeformed ones

\[
X_q = \phi^*(X)
\]

can be verified to be

\[
X_q = X, \quad Y_q = \frac{sh(\lambda W)}{\lambda} Y, \quad W_q = \frac{sh(\lambda W)}{\lambda}.
\] (65)

Composing then the map \( \phi \) with the map \( \pi \) (the classical Jordan–Schwinger map), we obtain

\[
\pi^* (X_q) = iz_1, \quad \pi^* (Y_q) = i\frac{sh(\lambda z_2)}{\lambda}, \quad \pi^* (W_q) = i\frac{sh(-\lambda z_2)}{\lambda}
\] (66)

where we have used the complex realization of \( \pi \) given by relations (31). Relations(66) give then a possible choice for the classical deformed Jordan–Schwinger map of the Heisenberg–Weyl algebra.

Before closing this section we would like to make a few remarks on the deformation procedure regarded in a classical setting.
As we have briefly sketched in section 3, Poisson structures on $\mathbb{R}^3$ can be characterized by their Casimir 1-form, $\alpha$, satisfying the condition $d\alpha \wedge \alpha = 0$ and their properties can be restated in terms of the 1-form. The most general expression for $\alpha$ is given by relation (18), which we write again for our convenience

$$\alpha = n(y dw - w dy) + d\frac{1}{2}(a x^2 + by^2 + c w^2).$$

If we consider now

$$\alpha = n( f \, dg - g \, df) + d\frac{1}{2}(a h^2 + b f^2 + c g^2)$$

with $f = f(x, y, w, \eta)$, $g = g(x, y, w, \lambda)$, $h = h(x, y, w, \tau)$, then (67) provides a general deformation of our algebras as long as $f(x, y, w, 0) = y$, $g(x, y, w, 0) = w$, $h(x, y, w, 0) = x$. If $df \wedge dg \wedge dh \neq 0$, we can think of our deformation as a change of coordinates on $\mathbb{R}^3$ associated with possible change of coordinates on $\mathbb{R}^4$. We can say that we are picking a different subalgebra of functions in the Poisson algebra $(\mathcal{F}(M), \{\cdot,\cdot\}_0)$, or we can say that we have performed a nonlinear noncanonical transformation taking us from one Poisson bracket to another. Both view points are equivalent and acceptable. Let us consider now our specific examples. A possible multiparametric deformation of the 1-form $\alpha$ would be

$$\alpha(\lambda, \eta, \tau) = n([y], d[w]_\lambda - [w], d[y]_\eta) + d\frac{1}{2}([a[x]]_\tau^2 + [b[y]]_\eta^2 + c[w]_\rho^2)$$

where $[r]_\rho$ are the Tchebyhev polynomials

$$[r]_\rho = \frac{\sigma^r - \sigma^{-r}}{\sigma - \sigma^{-1}}.$$

We find then that $\mathcal{SL}(2, \mathbb{R})$ algebra is obtained by setting $n = 0$ and choosing $\tau \rightarrow 1$, $\eta \rightarrow 1$, we get the deformed 1-form for $\mathcal{SL}(2, \mathbb{R})$ (4), which leads to the deformed Poisson brackets (46). For $n = 0, c = 0$ and $\tau \rightarrow 1$ we get the deformed 1-form (58) and then $\mathcal{E}_q(2)$; finally, for $n = 0, b = 0$ and $\tau \rightarrow 1$, we get the 1-form (64) which leads to $\mathcal{H}_q(1)$.

Despite of the fact that we have been able to obtain some known results, up to now this only provides a way to reinterpret the deformation procedure at the classical level, not to give new Hopf algebras, the reason being that we are not able to give a prescription which should be satisfied by the deformed Poisson algebras to make the corresponding operator algebras into Hopf algebras. We hope to come back to this issue in the near future.

5 Analogue of quantum and classical Jordan–Schwinger map for arbitrary Lie algebras

We make a few comments here on the possibility to realize commutation relations for a Lie algebra of higher dimensions, in terms of Poisson brackets. Realizations for an
arbitrary Lie algebra are based on a known procedure from second quantization ([15]).
To exemplify, let us consider three $N \times N$ matrices $A$, $B$, $C$ which satisfy the relation

$$[A, B] = C;$$

we can construct then three operators

$$\hat{A} = A_{ik} a_i^\dagger a_k \quad \hat{B} = B_{ik} a_i^\dagger a_k \quad \hat{C} = C_{ik} a_i^\dagger a_k$$

which reproduce the initial commutation relation

$$[\hat{A}, \hat{B}] = \hat{C}.$$

Here operators $a_i$, $a_i^\dagger$ are either bosonic ones i.e.

$$[a_i, a_k^\dagger] = \delta_{ik} \quad [a_i, a_k] = 0$$
or fermionic ones, i.e.

$$[a_i, a_k^\dagger]_+ = \delta_{ik} \quad [a_i, a_k]_+ = 0$$

Due to that, given $M$ generators-matrices of any $N$-dimensional Lie algebra representation $(L_\alpha)_{ik}, \quad i, k = 1, ..., N \quad \alpha = 1, ..., M$ with

$$[L_\alpha, L_\beta] = c_{\alpha \beta}^\gamma L_\gamma$$

then the operators

$$\hat{L}_\alpha = (L_\alpha)_{ik} a_i^\dagger a_k$$

realize the same Lie algebra in both cases of bosonic and fermionic nature of the operators $a_i, a_i^\dagger$. Below we will have in mind bosonic operators. Then it is easy to check that classical analogues of (70)

$$\tilde{L}_\alpha = (L_\alpha)_{ik} z_i^* z_k$$

give a realization of the same Lie algebra in terms of Poisson brackets. Namely we have

$$\{\tilde{L}_\alpha, \tilde{L}_\beta\} = c_{\alpha \beta}^\gamma \tilde{L}_\gamma .$$

In this sense we could always have Lie algebra realizations in terms of Poisson brackets of this type. Once we have this realization for Lie algebras, we can extend it to deformed algebras.

We can cast these comments in our geometrical framework by noticing that with any $N \times N$ matrix $A$ we can associate a vector field $X_A = A^j_i x^j \frac{\partial}{\partial x^i}$ acting on $\mathbb{R}^N$. By considering now $T^*\mathbb{R}^N$ we get a natural realization in terms of Poisson brackets by using functions $f_A = A^j_i x^j p_i$. If, on the other hand, we complexify $\mathbb{R}^N$ to get $\mathbb{C}^N$ we can consider a natural symplectic structure $\omega = \sum_a d\bar{z}_a^* \wedge dz_a \quad a \in \{1, ..., N\}$ and associate a complex valued function with any matrix, namely $\zeta_A = A^j_i z^j z^*_i$. It would be
possible to consider also Grassmann variables to get the classical analogue of fermionic variables.

It should be remarked that once we realize our algebra in terms of vector fields, we can perform any nonlinear coordinate transformation on $\mathbb{R}^N$ without spoiling the commutation relations of our vector fields.

From what we have said, it is clear that out of any finite dimensional representation of a Lie algebra we can construct a realization in terms of Poisson brackets and use this realization to undertake deformations.

6 Extension of Jordan-Schwinger map to $U_q(n)$

We will consider now the Lie algebra of $U(n)$ and its deformation, which is given in [16] and, in terms of q-oscillators, by [17]; we will look for an extension of previous results to it. The generators $T_{ik}$ of the Lie algebra $U(\lambda)$ may be constructed from the creation and annihilation operators $(a_i, a_i^\dagger)(i = 1, \ldots, n)$ in such a manner

$$T_{ik} = a_i^\dagger a_k$$

and they will satisfy the commutation relations

$$[T_{ik}, T_{mn}] = T_{im} \delta_{km} - T_{mk} \delta_{in}$$

A q-deformation of this algebra may be produced using the relations

$$a_{iq} = a_i f(\hat{a}_i)$$

$$T_{qik} = \frac{1}{\sinh \lambda} T_{ik} \frac{\sinh \lambda(T_{ii} + 1) \sinh \lambda T_{kk}}{(T_{ii} + 1) T_{kk}} = a_{iq} a_{kq} \quad i < k$$

$$T_{qii} = T_{ii}$$

with

$$T_{qik} = T_{qki} \quad for \quad k < i$$

and inverse

$$T_{ik} = (\sinh \lambda) T_{qik} \frac{T_{qkk}(T_{qii} + 1)}{\sinh(\lambda T_{qkk}) \sinh(\lambda T_{qii} + 1)} \quad i < k$$

$$T_{ik} = T_{k^i} \quad k < i$$

$$T_{ii} = T_{qii} .$$

The operators $T_{qik}$ satisfy the following relations

$$[T_{qik}, T_{qmn}] = 0, i \neq m, n \quad and \quad k \neq m, n$$

(78)
Since the relations (73) hold, the generators $T_{qik}$ given by (75) may be realized as functions of creation and annihilation operators. Let us specialize to the case $N=3$. The deformed $\mathcal{U}^{\text{H}}(\hat{\varepsilon})$ has three subalgebras $\mathcal{U}^{\text{H}}(\varepsilon)$ which are realized by

$$T_{ii}, \ T_{ik}, \ T_{kk}, \ T_{jk} \quad i, k = 1, \ldots, N \quad (79)$$

and happen to be the standard q-deformed $\mathcal{U}^{\text{H}}(\varepsilon)$, with commutation relations

$$[T_{i\overline{i}}, T_{j\overline{j}}] = -T_{ij} F(T_{i\overline{i}}) \quad (80)$$

The function $F(x)$ is given by

$$F(T_{i\overline{i}}) = \frac{sh\lambda(T_{i\overline{i}} + 1) - sh\lambda T_{i\overline{i}}}{sh\lambda}. \quad (81)$$

For $\lambda \to 0$ this function becomes

$$F_i(T_{i\overline{i}}) = \frac{ch\lambda T_{i\overline{i}}}{sh\lambda}. \quad (82)$$

This expression will be useful in the classical analogy which we are going to treat.

Let us consider now a classical realization of the q–deformed $\mathcal{U}(\hat{\varepsilon})$ algebra in terms of Poisson brackets. The classical Jordan–Schwinger map gives

$$T_{ik} = i\zeta_i^* \zeta_k$$

where

$$\zeta_i = \tilde{f}(in_i)z_i,$$

$\tilde{f}$ is given by (52), and

$$\{\zeta_{iq}, \zeta_{kq}^*\} = -\frac{i\lambda}{sh\lambda}ch(\lambda |\zeta_i|^2)\delta_{ik}. \quad (83)$$

As we can see from equations (80), (82) and (83), the realization in terms of Poisson brackets agrees with the one in terms of commutators only in the limit $\lambda \to 0$ or, which is the same, $q \to 1$. Namely we have

$$\{T_{i\overline{i}}, T_{j\overline{k}}\} = -T_{jk}F_i(T_{i\overline{i}}) + O(\lambda). \quad (84)$$

Thus, the q–deformation of the classical $\mathcal{U}^{(\hat{\varepsilon})}$ algebra with generators

$$T_{ik} = i\zeta_i^* \zeta_k \quad (85)$$

and commutation relations

$$\{T_{ik}, T_{kj}\} = (T_{km}\delta_{km} - T_{nk}\delta_{kn}) \quad (86)$$

gives back the subalgebras $\mathcal{U}^{\text{H}}(\varepsilon)$ in the limit of $\lambda$ small.
7 Conclusions

In this paper we have generalized the definition of Jordan–Schwinger map to any three-dimensional Lie algebra. Lie algebras are classically realized, in the sense that the Lie product is given by a Poisson bracket and the generators are realized as functions. We have considered then examples of deformations of these Poisson subalgebras, which give, through the Dirac map, non commutative Hopf algebras already known in the literature. In this classical setting we reinterpret the deformation procedure as a deformation of the Casimir 1–form associated to the Poisson structure characterizing the algebra. It is still missing how to implement in our classical procedure additional ingredients to be related to the notion of Hopf algebra, which appears once the algebras are realized in terms of operators. We hope to come on that in the next future. All this would allow thinking that new ways leading to quantum groups might exist starting at this classical level.

There is a second problem which is interesting in our opinion, but still open. We could think of extending the domain of the Jordan-Schwinger map to symplectic manifolds different from \( \mathbb{C}^n \). As we mentioned above, the given examples of Lie algebras realizations in terms of Poisson subalgebras might be related to the \( T^*G \) and \( T^*(G/K) \) structures. Some other examples of classical realizations of three dimensional Lie algebras already exist in the literature ([18]). Nevertheless the problem of an exhaustive classification of all possible symplectic realizations of Lie algebras needs extra investigation and we are working on that.

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