HIDDEN SYMMETRIES, SPECIAL GEOMETRY AND QUATERNIONIC MANIFOLDS†

B. de Wit
Institute for Theoretical Physics
Utrecht University
Princetonplein 5, 3508 TA Utrecht, The Netherlands

A. Van Proeyen
Instituut voor theoretische fysica
Universiteit Leuven, B-3001 Leuven, Belgium

ABSTRACT

The moduli space of the Calabi-Yau three-folds, which play a role as superstring ground states, exhibits the same special geometry that is known from nonlinear sigma models in $N = 2$ supergravity theories. We discuss the symmetry structure of special real, complex and quaternionic spaces. Maps between these spaces are implemented via dimensional reduction. We analyze the emergence of extra and hidden symmetries. This analysis is then applied to homogeneous special spaces and the implications for the classification of homogeneous quaternionic spaces are discussed.

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The moduli space of the Calabi-Yau three-folds, which play a role as superstring ground states, exhibits the same special geometry that is known from nonlinear sigma models in \( N = 2 \) supergravity theories. We discuss the symmetry structure of special real, complex and quaternionic spaces. Maps between these spaces are implemented via dimensional reduction. We analyze the emergence of extra and hidden symmetries. This analysis is then applied to homogeneous special spaces and the implications for the classification of homogeneous quaternionic spaces are discussed.

1. Introduction

Upon dimensional reduction a field theory may exhibit certain unexpected symmetries. These symmetries are often called hidden symmetries. Some of them are clearly related to the symmetries of the original higher-dimensional theory, while for others there is no obvious explanation. A simple and well-known example of this phenomenon is Einstein gravity, which leads to a nonlinear sigma model coupled to gravity upon reduction to three space-time dimensions. This sigma model is associated with the symmetric space \( SO(2,1)/SO(2) \), which is invariant under an \( SO(2,1) \) group of isometries.

Hidden symmetries play, for instance, a role in Kaluza-Klein theories, in the study of lower-dimensional solutions of the Einstein equation, and in supergravity. In the latter dimensional reduction has been used as a convenient method for obtaining information about these theories in various dimensions. This often led to the discovery of new structures and unexpected connections with intriguing mathematics. The latter is especially the case when the dimensional reduction preserves supersymmetry. Supersymmetry poses restrictions on the symmetries of the matter sector of the lower-dimensional theory. In the context of this talk we are interested in the Kähler geometry associated with \( N = 2 \) vector multiplets coupled to supergravity (supergravity invariant under two independent local supersymmetries) [1]. This geometry is called special geometry [2]. New impetus for studying hidden symmetries
and special geometry came from superstring compactifications on Calabi-Yau manifolds. Again the method of dimensional reduction turns out to play a useful role here, which motivated us to try and explain the structure of hidden symmetries in the context of special geometry. Somewhat surprisingly, this study led to implications on the classification of homogeneous quaternionic spaces.

In order to clarify the connections between special geometry, hidden symmetries, Calabi-Yau manifolds and superstrings, let us first explain a few facts concerning superstring compactifications. Because the superstring lives in ten space-time dimensions, realistic theories require six of the spatial dimensions to be compactified. The corresponding superstring ground states can be described in terms of conformal field theories on the superstring worldsheet. In many cases the relation between such a conformal field theory and the compactification of the six coordinates is obvious, but there exist conformal theories without a space-time interpretation. In order to obtain realistic low-energy field theories one requires the compactification to be supersymmetric \[3\]. Supersymmetry implies that the compactified dimensions should constitute a so-called Calabi-Yau three-fold (a compact Kähler manifold of vanishing Chern class and complex dimension three; these spaces have a unique Ricci-flat metric). In terms of conformal field theories, the class of \((2, 2)\) superconformal field theories with central charge \(c = 9\) is relevant, which contains the Calabi-Yau three-folds, but possibly also other solutions without a corresponding space-time interpretation. In the context of this work we ignore this aspect and generically denote this class of ground states as Calabi-Yau manifolds. It is important to note that these manifolds can serve as a ground state for each of the three types of superstrings:

- For the so-called heterotic string they give rise to low-energy effective theories that exhibit space-time \(N = 1\) supersymmetry. All phenomenologically viable models belong to this class.
- For the so-called IIA and IIB superstrings they give rise to theories with space-time \(N = 2\) supersymmetry.

The type-II superstrings have a more restrictive symmetry structure. They are not phenomenologically viable, but the fact that they can be compactified on the same Calabi-Yau manifold implies that many of their systematic features carry over to the compactifications of the heterotic string. Our strategy is to make maximal use of this fact (following \([4]\)) and base our study on \(N = 2\) space-time supersymmetry, in spite of the fact that realistic low-energy effective theories exhibit only \(N = 1\) supersymmetry.

There exists a huge variety of string ground states corresponding to Calabi-Yau spaces, parametrized by parameters called moduli. The moduli thus correspond to the independent deformations of the Ricci-flat metric up to reparametrizations (see, for instance, \([5]\)). The mixed deformations (i.e., related to the components of the metric with one holomorphic and one anti-holomorphic index) are related to the real

\[^{1}\]The supersymmetry will eventually be broken at a distance scale much larger than the compactification scale. The precise mechanism for supersymmetry breaking is as yet not clear.
harmonic (1, 1) forms, while the pure (anti-)holomorphic deformations are related to the complex harmonic (2, 1) forms on the Calabi-Yau space. For given topology these spaces are thus described by $h_{11}$ real and $h_{12}$ complex parameters, where the Hodge numbers $h_{pq}$ define the number of independent harmonic $(p, q)$ forms. The $(1, 1)$ moduli parametrize the deformations of the Kähler class, which characterize the size of the Calabi-Yau manifold. The $(2, 1)$ moduli parametrize the deformations of the complex structure and characterize the shape of the Calabi-Yau manifold.

The Hodge diamond for a Calabi-Yau manifold is shown on the right. The only Hodge numbers that are not uniquely determined are $h_{12} = h_{21}$ and $h_{11} = h_{22}$. Knowing the numbers of independent harmonic forms allows us to count the number of massless Kaluza-Klein modes for each of the various fields that occur in supergravity in ten dimensions, compactified on a Calabi-Yau manifold (see, e.g. [6]). For instance, the ten-dimensional metric decomposes into a four-dimensional tensor, a tensor with mixed indices, and a six-dimensional tensor. The four-dimensional tensor is associated with the massless spin-2 graviton. The mixed components could in principle describe spin-1 states, but as there are no $(1, 0)$ or $(0, 1)$ harmonic forms, these are absent. Finally, the massless spin-0 states associated with the pure six-dimensional components are precisely related to the modular parameters of the Calabi-Yau space. Consequently, they correspond to $h_{11}$ real and $h_{12}$ complex states. Using some elementary knowledge of the various field equations, we can thus generally analyze the metric $g_{MN}$, a scalar field $\phi$, a vector gauge field $A_M$, antisymmetric tensor gauge fields $A_{MN}$ and $A_{MNP}$ and a four-rank antisymmetric gauge field $A_{MNPQ}$ with (anti)selfdual field strength and determine the number of massless Kaluza-Klein states. With these results, summarized in table 1, it is easy to count the number of bosonic massless states that emerge in the compactification of IIA and IIB supergravity on a Calabi-Yau manifold:

On the right we grouped the fields into $N = 2$ vector and scalar supermultiplets in four space-time dimensions (note that the extra vector states belong to the graviphoton of the supergravity multiplet). The bosonic fields of a vector multiplet are a vector

$$
\begin{array}{c}
\begin{align*}
\text{nonchiral IIA SG :} & \quad h_{11} + 1 \text{ spin-1} \\
& \quad h_{11} \text{ complex spin-0} \\
& \quad h_{12} + 1 \text{ quaternionic spin-0} \\
\end{align*}
\end{array}
$$

$$
\begin{array}{c}
\begin{align*}
\text{chiral IIB SG :} & \quad h_{12} + 1 \text{ spin-1} \\
& \quad h_{12} \text{ complex spin-0} \\
& \quad h_{11} + 1 \text{ quaternionic spin-0} \\
\end{align*}
\end{array}
$$

$$
\begin{array}{c}
\begin{align*}
\text{1} & \quad h_{11} \text{ vector supermultiplets} \\
0 & \quad h_{12} \text{ scalar supermultiplets} \\
1 & \quad 1 \\
0 & \quad 0 \\
0 & \quad 0
\end{align*}
\end{array}
$$
gauge field and a complex scalar field. These complex scalar fields are described by a nonlinear sigma model associated with a Kähler space. Each vector multiplet thus gives rise to one spin-1 and two spin-0 states. The bosonic fields of a scalar multiplet are just four scalar fields, described by a nonlinear sigma model associated with a quaternionic space. The bosonic fields of a scalar multiplet give rise to four spin-0 states.

There is a subtle relation between the moduli space of Calabi-Yau manifolds and these Kählerian and quaternionic sigma models. In the effective low-energy theory corresponding to a string compactification, the sigma-model fields have no potential. Therefore their vacuum-expectation values are undetermined and parametrize the (classical) ground states states of this field theory, up to certain equivalence transformations. The metric of the non-linear sigma model (in four space-time dimensions) is therefore related to the metric on moduli space [4]. This implies that the moduli space of the superstring groundstates associated with Calabi-Yau spaces must exhibit special geometry. Here we extend the notion of special geometry to the quaternionic manifolds that emerge in this context. A more precise definition will be given shortly.

An intriguing feature is the existence of mirror pairs [7]. Under the interchange of $h_{11}$ and $h_{12}$ the Hodge diamond changes into its mirror image by reflecting about a diagonal. Manifolds related by such a reflection form a mirror pair of two topologically different spaces. Nevertheless they correspond to the same conformal field theory (except that some of the $U(1)$ charge assignments have been reversed when identifying the geometrical objects). From (1) and (2) it is clear that this change is related to a change of chirality of the corresponding supergravity theory, converting IIA and IIB supergravity. This suggests that there must exist a map between $n$-dimensional Kähler manifolds and $(n+1)$-dimensional quaternionic manifolds. As was demonstrated by [8], this map can be induced at the level of $N = 2$ supergravity by dimensional re-

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>field</th>
<th>spin-2</th>
<th>spin-1</th>
<th>spin-0</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>$g_{MN}$</td>
<td>1</td>
<td>0</td>
<td>$h_{11}$ real + $h_{12}$ complex</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\phi$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$A_M$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$A_{MN}$</td>
<td>0</td>
<td>0</td>
<td>$(h_{11} + 1)$ real</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$A_{MNP}$</td>
<td>0</td>
<td>$h_{11}$</td>
<td>$(h_{12} + 1)$ complex</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$[A_{MNPQ}]_{\pm}$</td>
<td>0</td>
<td>$h_{12} + 1$</td>
<td>$h_{11}$ real</td>
</tr>
</tbody>
</table>

Table 1: Massless Kaluza-Klein modes associated with various fields in ten dimensions, compactified on a Calabi-Yau space. The first two columns specify the number of these fields contained in IIA or IIB supergravity in ten space-time dimensions.
duction from four to three dimensions. Under this reduction the bosonic degrees of freedom associated with the vector multiplets whose scalar fields parametrize a special Kähler manifold are all converted into scalar fields, which parametrize a quaternionic manifold. The quaternionic manifolds that emerge through this so-called c map are called special quaternionic manifolds.

Likewise we can define special real geometry as the geometry associated with the sigma models that arise in $N = 2$ supergravity coupled to vector multiplets in five space-time dimensions. Upon dimensional reduction these real manifolds give rise to a subclass of the special Kähler manifolds. The corresponding map is called the r map. We should emphasize that supersymmetry is the crucial ingredient in this construction. As dimensional reduction preserves supersymmetry, the manifolds that emerge under these maps must satisfy the restrictions appropriate to $N = 2$ supersymmetric matter. That means that the scalar fields of vector multiplets parametrize a real manifold in five [9] and a Kähler manifold in four [1], while scalar multiplets parametrize quaternionic manifolds in four [10] and in three dimensions [11]. This ensures the existence of the r and c maps, which act according to

$$\mathbb{R}_{n-1} \xrightarrow{r} \mathbb{C}_n, \quad \mathbb{Q}_n \xrightarrow{c} \mathbb{H}_{n+1},$$

where $n - 1$, $n$ and $n + 1$ denote the real, complex and quaternionic dimension of the real, Kähler and quaternionic spaces, respectively.

The r and c maps are convenient tools for studying the symmetry structure of the various special geometries. It is here that the hidden symmetries enter. As we discuss below, we make a distinction between extra symmetries that can be understood from the invariances of the higher-dimensional theory and hidden symmetries that have no immediate explanation. The maps are particularly useful for studying homogeneous special spaces, because a homogeneous space in the image of one of these maps must originate from a theory in which the combined transformations of scalar and vector fields act transitively on the scalars and vice versa. Motivated by this relation we present a classification of the real homogeneous spaces, whose image under the r and c◎r map leads to corresponding homogeneous Kähler and quaternionic spaces. The latter are then confronted with Alekseevskii’s classification of the normal quaternionic spaces [12] and their related Kähler spaces discussed in [13]. Our analysis allows a complete determination of the isometry and isotropy groups of the various spaces [14].

2. Special Kähler geometry

As discussed above, special Kähler geometry is the geometry that arises when coupling $n$ vector supermultiplets to $N = 2$ supergravity in $d = 4$ space-time dimensions. A characteristic feature is that the geometry is encoded in a single holomorphic function $F(X)$ of $n + 1$ complex parameters $X^I$, where $I = 0, 1, \ldots, n$, which is homogeneous of second degree. Therefore it satisfies identities such as $F = \frac{1}{2} F_I X^I$, $F_I = F_{IJ} X^J$, $X^I F_{IJK} = 0$, where the subscripts $I, J, \ldots$ denote differentiation with respect to $X^I$. 
Furthermore it is convenient to define the tensor $N_{I,J} \equiv \frac{1}{2} \text{Re} F_{I,J}$. The $X^I$ are scalar fields, which appear in the Lagrangian according to

$$\mathcal{L} \propto \left( N_{I,J} - \frac{(N\bar{X})_{I} (NX)_{J}}{XN\bar{X}} \right) \partial_{\mu}X^{I} \partial^{\mu}\bar{X}^{J},$$

(4)

where we used an obvious notation where $(NX)_{I} = N_{I,J}X^{J}$, $\bar{X}NX = \bar{X}^{I}N_{I,J}X^{J}$, etc.. Although the metric in (4) is degenerate, this poses no problems because the overall scale of the fields is fixed by the condition $N_{I,J}\bar{X}^{I}X^{J} = \text{constant}$. Furthermore the Lagrangian does not depend on the overall phase of the fields. (In other words (4) is invariant under local phase transformations.) Therefore it depends only on $n$ complex fields $z^A$, which can be used to parametrize the $X^I$. Introducing $n + 1$ unrestricted functions $X^I(z)$, we substitute everywhere for $X^I$,

$$X^{I} \rightarrow \frac{X^{I}(z)}{\sqrt{N_{KL}\bar{X}^{K}(z)X^{L}(\bar{z})}}.$$ 

(5)

The kinetic term for the fields $z^A$ then takes the form of a non-linear sigma model corresponding to a Kähler manifold. Its Kähler potential is equal to

$$K(z, \bar{z}) = \ln N_{I,J}X^{I}(z)\bar{X}^{J}(\bar{z}) = \ln \frac{1}{2} \text{Re} F_{I}(X(z))\bar{X}^{I}(z),$$

(6)

and the sigma model metric is given by

$$g_{\bar{A}\bar{B}} = \frac{\partial^2 K(z, \bar{z})}{\partial z^A \partial \bar{z}^B}.$$ 

(7)

The curvature tensor corresponding to this metric equals [15]

$$R^{A}_{\quad BC}D = -2\delta^A_B \delta^D_C - e^{-2K} Q_{BCE} \bar{Q}^{EAD},$$

(8)

where

$$Q_{ABC}(z) \equiv \frac{1}{4} F_{JKL}(X(z)) \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial \bar{z}^B} \frac{\partial X^K(z)}{\partial \bar{z}^C}, \quad \bar{Q}^{ABC} = g^{D\bar{A}} g^{E\bar{B}} g^{F\bar{C}} Q_{DEF}.$$ 

(9)

Two different functions $F(X)$ can correspond to supergravity theories with equivalent equations of motion. Some of the equivalence transformations involve duality transformations on the electric and magnetic field strengths of the gauge fields. The reparametrizations constitute a representation of $Sp(2n + 2, \mathbb{R})$. For the corresponding Calabi-Yau manifolds these symplectic reparametrizations are naturally induced

\[2\]The so-called special coordinates are defined by $X^0(z) = 1, X^A(z) = z^A$.

\[3\]The connection field associated with the local phase transformations takes the form $A_\mu = \text{Im}(\partial_\mu z^A \partial_A K(z, \bar{z}))$. This quantity acts also as a connection for Kähler transformations. Under these transformations the Kähler potential is changed according to $K(z, \bar{z}) \rightarrow K(z, \bar{z}) + f(z) + \bar{f}(\bar{z})$, so that the metric remains invariant, while $A_\mu$ changes into $A_\mu + \partial_\mu (\text{Im} f(z))$. 

6
on the periods by changes in the corresponding cohomology basis (see, e.g. [5, 7]). A subclass of the reparametrizations may correspond to an invariance of the equations of motions. For the scalar fields these transformations constitute isometries of the Kähler manifold.

3. Special real geometry

Special real geometry is related to the functions

$$F(X) = i d_{ABC} \frac{X^AX^BX^C}{X^0},$$  \hspace{1cm} (10)

with $d_{ABC}$ a symmetric real tensor. This tensor defines a Maxwell-Einstein supergravity theory in five space-time dimensions [9], which contains $n$ real scalar fields $h^A$ and $n$ vector fields $A^A_\mu$ (one of them corresponding to the graviphoton). The relevant bosonic Lagrangian is

$$e^{-1} \mathcal{L} = -\frac{1}{2} R - \frac{3}{2} (d^h)_{AB} \partial_\mu h^A \partial^\mu h^B + \frac{1}{2} (6 (d^h)_{AB} - 9 (dhh)_A (dhh)_B) F^A_{\mu \nu}(A) F^{B \mu \nu}(A) + e^{-1} i \epsilon^{\mu \rho \sigma \lambda} d_{ABC} F^A_{\mu \nu}(A) F^B_{\rho \sigma}(A) A^C_\lambda,$$  \hspace{1cm} (11)

where $(d^h)_{AB} = d_{ABC} h^C$, $(dhh)_A = d_{ABC} h^B h^C$, $R$ is the Ricci scalar and $F^A_{\mu \nu}(A)$ are the abelian field-strength tensors corresponding to $A^A_\mu$. The scalar fields $h^A$ are subject to the condition

$$d_{ABC} h^A h^B h^C = 1,$$  \hspace{1cm} (12)

and parametrize an $(n-1)$-dimensional special real space whose metric follows from the sigma-model metric in (11). The Lagrangian (11) is invariant under linear transformations of the fields

$$h^A \to \tilde{B}^A_B h^B, \quad A^A_\mu \to \tilde{B}^A_B A^B_\mu,$$  \hspace{1cm} (13)

that leave the tensor $d_{ABC}$ invariant. These transformations induce isometries on the special real space.

After reduction to four dimensions, the Lagrangian (11) leads to a nonlinear sigma model associated with a special Kähler space corresponding to (10); the imaginary parts of the four-dimensional scalar fields $z^A$ (in special coordinates) originate from the fifth component of the gauge fields $A^A_\mu$, while their real part corresponds to the $n-1$ independent fields $h^A$ and the component $g_{55}$ of the metric. The $n+1$ gauge fields in four dimensions are related to the $n$ gauge fields and the off-diagonal components $g_{\mu 5}$ of the metric in five dimensions. The conversion of the bosonic fields is summarized in table 2. Clearly in four dimensions we have $n + 1$ extra scalar fields.

The resulting theory in four space-time dimensions exhibits the same symmetry (13) as the original theory in five dimensions. Besides, a number of extra symmetries emerge that also find their origin in the five-dimensional theory. First of all, the extra
vector field emerging from the five-dimensional metric has a corresponding gauge invariance related to reparametrizations of the extra fifth coordinate by functions that depend only on the four space-time coordinates. Then there are special gauge transformations of the \( n \) vector fields with gauge functions that depend exclusively and linearly on the fifth coordinate. Under these transformations the fifth component of each gauge field transforms with a constant translation, whereas the remaining four-dimensional gauge fields transform linearly into the gauge field originating from the five-dimensional metric. Finally there are the scale transformations of the fifth coordinate. The five-dimensional origin of the duality invariances in four dimensions and their parameters is concisely summarized by

\[
\tilde{B}_A^B \quad \Rightarrow \quad \tilde{B}_B^A
\]

gauge transformations \( \propto x^5 \quad \Rightarrow \quad b^A \)

scale transformation of \( x^5 \quad \Rightarrow \quad \beta \)

Hence altogether we have \( n + 1 \) extra symmetries associated with the parameters \( b^A \) and \( \beta \), and \( n + 1 \) extra scalar fields. In addition to that, there may be hidden symmetries for which there exists no explanation in terms of the underlying higher-dimensional theory. In terms of special coordinates \( z^A \) the combined transformations take the form

\[
\delta z^A = b^A - \frac{2}{3} \beta z^A + \tilde{B}_B^A z^B + \frac{1}{2} R^A_{BCD} a_D z^B z^C ,
\]

where the possible hidden symmetries are characterized by the parameters \( a_A \). Hidden symmetries exist for those independent parameters \( a_A \) for which \( R^A_{BCD} a_D \) is constant (in special coordinates). The maximal number of hidden symmetries is thus realized whenever the curvature tensor itself is constant. In that case the corresponding Kähler space is symmetric [15, 16].

The symmetries (14) correspond to invariances of the full supergravity theory in four dimensions and it is known that they comprise the full isometry group of the Kähler metric for this class of spaces [17]. The root lattice corresponding to these transformations consists of the root lattice for the subgroup corresponding to \( \tilde{B}_B^A \) extended with one dimension associated with the eigenvalue of the roots under the \( \beta \)-symmetry. This leads to a characteristic lattice such as shown on the next page for a particular example with \( n = 3 \) and \( F(X) = 3i (X^2/X^0)(X^2X^1 - (X^3)^2) \). In this case the subgroup associated with the parameters \( \tilde{B}_B^A \) has two generators denoted by \( \tilde{B}_2 \) and \( \tilde{B}_3 \). Their roots correspond to the solvable algebra of \( SU(1,1) \), which is ex-

<table>
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</tr>
<tr>
<td>( n ) vectors</td>
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<td>( n )</td>
<td>( n )</td>
</tr>
<tr>
<td>( n - 1 ) scalars</td>
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<td>0</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td>( n + 1 )</td>
<td>2( n )</td>
</tr>
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</table>

Table 2: Decomposition of the \( d = 5 \) bosonic fields into \( d = 4 \) fields.
tended to a six-dimensional solvable algebra by the roots associated with the parameters $\beta, b_1, b_2$ and $b_3$. In this case there is one hidden symmetry associated with the parameter $a_2$, indicated in the diagram by $\diamond$.

The root lattice in the above example reflects the general situation. Decomposing the full symmetry algebra $\mathcal{W}$ into eigenspaces of the generator associated with the $\beta$ symmetry (in the adjoint representation), we always have [15]

$$\mathcal{W} = \mathcal{W}_{-2/3} + \mathcal{W}_0 + \mathcal{W}_{2/3},$$

where the subscript denotes the eigenvalue with respect to the $\beta$ symmetry. As it turns out, $\mathcal{W}_0$ corresponds to the subalgebra associated with the parameters $\tilde{B}_A^\beta$ and $\beta$, $\mathcal{W}_{2/3}$ contains the generators corresponding to the parameters $b^A_B$, and all possible generators corresponding to the hidden symmetries belong to $\mathcal{W}_{-2/3}$. The dimension of $\mathcal{W}_{-2/3}$ is thus at most equal to $n$, whereas the dimension of $\mathcal{W}_{2/3}$ is always equal to $n$. Unless we have maximal symmetry (i.e. unless there are $n$ independent symmetries associated with the parameters $a_A$, in which case the space is symmetric) the isometry group of the corresponding Kähler space is not semi-simple.

4. Special quaternionic manifolds

We now return to the $\mathcal{N} = 2$ Maxwell-Einstein Lagrangian in four dimensions based on a general (holomorphic and homogeneous) function $F(X)$ and consider its reduction to three space-time dimensions. In this case an extra feature is present, because the standard (abelian) gauge-field Lagrangian in three dimensions can be converted to a scalar-field Lagrangian by means of a duality transformation. Each four-dimensional gauge field thus gives rise to two scalar fields, one of which its component in the fourth dimension and the other a scalar field resulting from the conversion of the three-dimensional gauge field. Up to total divergences only derivatives of the new scalar field (introduced as a Lagrange multiplier to impose the Bianchi identity) appear in the Lagrangian, so that in addition to the original gauge transformations there is a second symmetry corresponding to constant shifts of the new field. These two invariances have parameters denoted by $\alpha_I$ and $\beta_I$. The same conversion can be carried out for the vector field that emerges from the three-dimensional metric, so that the four-dimensional metric gives rise to a three-dimensional metric and two scalar fields. These scalar fields are also subject to two invariances, one related to the scale transformation of the extra coordinate with parameter $\epsilon^0$ and another one corresponding to the converted three-dimensional vector field with parameter $\epsilon^+$. Altogether, the Lagrangian (4) thus gives rise to $4(n + 1)$ scalar fields, as shown in
Table 3: Decomposition of the $d = 4$ bosonic fields into $d = 3$ fields.

<table>
<thead>
<tr>
<th></th>
<th>$d = 4$</th>
<th>metric</th>
<th>scalars</th>
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<td></td>
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<tr>
<td>$n + 1$ vectors</td>
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<tr>
<td>$2n$ scalars</td>
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</tr>
<tr>
<td>total</td>
<td>1</td>
<td>$4n + 4$</td>
<td></td>
</tr>
</tbody>
</table>

table 3, coupled to gravity with $2n + 4$ additional invariances. The four-dimensional origin of these invariances can be summarized as follows:

Kähler isometries $\Rightarrow$ Kähler isometries

gauge transformations $\propto x^4 \Rightarrow \alpha'$

Lagrange multiplier shifts $\Rightarrow \beta_I$, $\epsilon^+$

scale transformation of $x^4 \Rightarrow \epsilon^0$

As the Lagrangian is still supersymmetric the scalar fields must define a quaternionic non-linear sigma model of quaternionic dimension $n + 1$ [11]. The corresponding quaternionic spaces are called special quaternionic spaces and obviously depend on a homogeneous holomorphic function of second degree. Like all quaternionic spaces, they are irreducible Einstein spaces. The explicit Lagrangian was determined in [18], where the quaternionic structure was explicitly verified, and in [19], where the complete set of isometries was determined.

Again we consider the root lattice corresponding to the isometries of the quaternionic space. It consists of the root lattice associated with the invariance of the corresponding special Kähler space extended with one dimension associated with the eigenvalue of the roots under the scale symmetry with parameter $\epsilon^0$. The roots of the extra symmetries noted above take a characteristic position, as can be shown from the example below, where we have exhibited the $n = 1$ case with $F(X) = i(X^1)^3/X^0$. The roots corresponding to the $SU(1, 1)$ isometries of the Kähler manifold, denoted by $b^1$, $\beta$ and $a_1$, are extended with the roots belonging to the extra transformations that emerge from the reduction to three dimensions. Observe that there are no Kähler isometries associated with the matrices $\tilde{B}^1_1$ in this case, because the corresponding five-dimensional theory is pure supergravity. Just as before hidden symmetries emerge, which take
characteristic positions in the root lattice on the left half-plane. In the case at hand, there are five such symmetries indicated by ☆, which extend this diagram to the root lattice of $G_{2(+2)}$. Its solvable subalgebra consists of the solvable subalgebra of $SU(1,1)$, associated with the parameters $\beta$ and $b^1$, extended by the generators of the six extra symmetries corresponding to $\epsilon^0$, $\epsilon^+$, $\alpha^I$ and $\beta_I$.

As shown in [19] this example is completely characteristic for the general case. The hidden symmetries are always associated to roots with eigenvalues $-1$ or $-\frac{1}{2}$ under the $\epsilon^0$ symmetry. The generators corresponding to all isometries of the special quaternionic space decompose generally according to

$$V = V_{-1} + V_{-1/2} + V_0 + V_{1/2} + V_1. \quad (16)$$

As shown in [19], for symmetric spaces, the dimension of $V_{-1}$ and $V_{-1/2}$ is equal to 1 and $2n+2$, respectively. Otherwise $V_{-1}$ is empty and the dimension of $V_{-1/2}$ is less than or equal to $n+1$. The dimension of $V_{1/2}$ and $V_1$ is always equal to $2n+2$ and 1, respectively. The parameter associated with the new symmetry in $V_{-1}$ is denoted by $\epsilon^-$; those corresponding to the symmetries in $V_{-1/2}$ are denoted by $\hat{\alpha}^I$ and $\hat{\beta}_I$. The maximal number of symmetries exist if and only if $V_{-1}$ is not empty. In that case both the quaternionic manifold and the corresponding Kähler manifold are symmetric.

5. Homogeneous quaternionic spaces

As shown above the three types of special manifolds are related by dimensional reduction of the corresponding supergravity models. A special real $(n-1)$-dimensional manifold $\mathbb{R}^{n-1}$ is thus related to a special Kähler manifold $\mathbb{C}^n$ of complex dimension $n$. Likewise, a special Kähler manifold of complex dimension $n$ is related to a special quaternionic manifold $\mathbb{H}^{n+1}$ of quaternionic dimension $n+1$. These relations define the so-called $r$ map from special real to special Kähler manifolds, and the $c$ map from special Kähler to special quaternionic manifolds. Under both maps the isometry groups are enlarged and their rank is increased by precisely one unit. In fact one can show that the number of additional symmetries is at least as large as the number of new coordinates that emerge under the map. It should be clear that the inverse $r$ and $c$ maps do not always exist. For instance, only the Kähler manifolds based on the functions (10) (and the functions that lead to equivalent field equations) can be in the image of the $r$ map.

Another important property of the maps is that the homogeneity of the manifold is preserved, provided there is a transitive subgroup of the isometry group that can be extended to a full invariance group of the corresponding supergravity theory. Conversely, if a homogeneous manifold is in the image of one of the maps, then the original manifold must also be homogeneous [14]. Therefore the maps have particular relevance for homogeneous special spaces. Actually, already in [8] it was recognized that Alekseevskii’s classification of normal quaternionic spaces [12] amounts to the construction of an inverse $c$ map. Normal quaternionic spaces are quaternionic spaces that admit a transitive completely solvable group of motions. It was conjectured in
[12] that the homogeneous quaternionic spaces consist of compact symmetric quaternionic and normal quaternionic spaces. The algebra corresponding to the group of solvable motions for the normal spaces can be decomposed according to the eigenvalues of one of its generators $e_0$. This generator can be identified with the generator associated with the $e^0$ symmetry of the previous section, and we find that the solvable algebra can be embedded in $\mathcal{V}_0 + \mathcal{V}_{1/2} + \mathcal{V}_1$ (cf. (16)).

According to Alekseevskii there are two different types of normal quaternionic spaces characterized by their so-called canonical quaternionic subalgebra. The first type with subalgebra $C_1^1$ turns out to correspond to the quaternionic projective spaces $\text{USp}(2n+2,\mathbb{H})/(\text{USp}(2n+2) \otimes \text{SU}(2))$. Their solvable algebra contains only one element in $\mathcal{V}_0$ (namely the generator $e_0$), and $4n$ and $3$ elements in $\mathcal{V}_{1/2}$ and $\mathcal{V}_1$, respectively. However, from section 4 we know that the special quaternionic spaces contain only one generator in $\mathcal{V}_1$, so that the quaternionic projective spaces are not special. Their rank is equal to $1$ or $0$ (the latter corresponds to the empty space). Although they do not appear in the image of the $c$ map, they can be coupled to $N = 2$ supergravity in three or four dimensions. The rank-$0$ case corresponds to pure supergravity, as is indicated in table 4.

The second type of normal spaces has a canonical subalgebra $A_1^1$. The structure of the solvable algebra is now as follows: $\mathcal{V}_0$ contains the direct sum of $e_0$ and a normal Kähler algebra $\mathcal{W}^n$ of dimension $2n$, $\mathcal{V}_{1/2}$ contains $2n + 2$ generators and $\mathcal{V}_1$ contains precisely $1$ generator. In order to be quaternionic, the representation of $\mathcal{W}^n$ induced by the adjoint representation of the solvable algebra on the $2n + 2$ generators in $\mathcal{V}_{1/2}$ must generate a solvable subgroup of $\text{Sp}(2n+2,\mathbb{R})$. Therefore each normal quaternionic space of this type defines the basic ingredients of a special normal Kähler space, encoded in its solvable transitive group of Kähler isometries. Alekseevskii’s analysis thus strongly indicates that a corresponding four-dimensional $N = 2$ supergravity theory should exist, so that under the $c$ map one will recover the original normal quaternionic space. To establish the existence of the supergravity theory, one must prove that a corresponding holomorphic function $F(X)$ exists that allows for these Kähler isometries. This program was carried out by Cecotti [13], who explicitly constructed the function $F(X)$ corresponding to each of the normal quaternionic spaces with canonical subalgebra $A_1^1$. For rank $2$ one has the so-called minimal coupling, where $F(X)$ is a quadratic polynomial, and a special case corresponding to $F(X) = i(X^1)^3/X^0$. This is the example discussed in section 4. The corresponding Kähler space is the image under the $r$ map of the empty real space. In other words, it is the space one obtains after reducing pure $d = 5$ supergravity to four dimensions (see table 4). The surprising feature of Cecotti’s result is that all remaining normal quaternionic spaces (which are of rank $3$ or $4$) are indeed in the image of the $c$ map and are associated with functions $F(X)$ that can be brought into the form (10). These spaces are therefore also in the image of the $c \circ r$ map. If Alekseevskii’s classification is complete, there can be no other special real or Kähler spaces with solvable transitive groups of isometries that can be promoted to an invariance group.
Table 4: Normal quaternionic spaces with rank $R \leq 2$ and quaternionic dimension $n + 1$ and the corresponding special real and Kähler spaces (whenever they exist). "SG" means pure supergravity and thus corresponds to the empty space.

of the corresponding supergravity Lagrangian.

This result suggests an alternative approach. Namely one can start from the real special spaces, and try to classify all the homogeneous spaces with an invariance group (13) that acts transitively on the sigma-model manifold. This classification was carried out in [20]. Subsequently one can apply the $r$ and the $c$ map to obtain all the corresponding homogeneous Kähler and quaternionic spaces. These spaces can then be confronted with those in the classification of [12, 13] with rank 3 or 4.

Let us therefore briefly describe the classification of the homogeneous real spaces. First one introduces the so-called canonical parametrization of the coefficients $d_{ABC}$ by choosing a reference point on the manifold within the domain where the kinetic terms for the various fields of the corresponding supergravity Lagrangian have the proper signs [15, 9]. After suitable reparametrizations this reference point equals $h^A = (1, 0, \ldots, 0)$ and the $d$-coefficients are restricted to $d_{111} = 1, d_{11a} = 0, d_{1ab} = -\frac{1}{2} \delta_{ab}$, with $d_{abc}$ unrestricted $(a, b = 2, \ldots, n)$. This parametrization is preserved under rotations of the fields $h^2, \ldots, h^n$, which leave the reference point invariant. A subgroup of these orthogonal transformations that leaves $d_{abc}$ invariant thus defines the isotropy group of the homogeneous space, while the isometry group is the invariance group of the $d_{ABC}$ tensor. Therefore the homogeneous space is just the coset space of these two groups. The condition that the isometries act transitively on the reference point takes the form

$$d_{(ab} d_{cd)e} - \frac{1}{2} \delta_{(ab} \delta_{cd)} = d_{(ab} A_{c)d}$$,

where $A_{abc}$ is an arbitrary tensor antisymmetric in the first two indices. The homogeneous equation (i.e., with $A_{abc} = 0$) was already solved in [15]. According to [20] the solution of the inhomogeneous equation can be expressed as follows. First we decompose the indices $A$ into $A = 1, 2, \mu, i$, with $\mu = 1, \ldots, q + 1$ and $i = 1, \ldots, r$, so that $n = 3 + q + r$. Hence we assume $n \geq 2$. We then establish that the general
solution of (17) takes the form (after some redefinitions, so that we are no longer in the canonical parametrization)

\[
d_{ABC} h^A h^B h^C = 3 \left\{ h^1 (h^2)^2 - h^1 (h^\mu)^2 - h^2 (h^i)^2 + \gamma_{\mu ij} h^\mu h^i h^j \right\}.
\]

Here the coefficients \( \gamma_{\mu ij} \) are \( r \times r \) gamma matrices that generate a real \( (q+1) \)-dimensional Clifford algebra of positive signature. This property severely constrains the possible values for \( q \) and \( r \). The gamma matrices do not necessarily correspond to irreducible representations of the Clifford algebra and are thus uniquely specified (up to similarity transformations) by the multiplicity \( P \) of the irreducible representations contained in them. However, when \( q \) is a multiple of 4, there exist two inequivalent irreducible representations and we have to specify two corresponding multiplicities, \( P \) and \( \dot{P} \). The polynomials (18) and their corresponding manifolds are denoted by \( L(q, P) \) or \( L(4m, P, \dot{P}) \), depending on whether there exist inequivalent Clifford algebra representations.

Most of the isometries\(^4\) correspond to invariance transformations of (18). They contain three obvious subgroups. There are the scale transformations

\[
h^1 \to e^{2\lambda} h^1, \quad (h^2, h^\mu) \to e^{-\lambda} (h^2, h^\mu), \quad h^i \to e^{\pm \lambda} h^i,
\]

the \( SO(q+1) \) rotations that act on \( h^\mu \) in the vector and on \( h^i \) in the spinor representation, and finally the metric-preserving elements in the centralizer of the Clifford algebra, which act exclusively on \( h^i \). The latter constitute a subgroup \( S_q(P, \dot{P}) \), which equals \( SO(P) \otimes SO(\dot{P}), U(P) \otimes U(\dot{P}) \) or \( USp(P) \otimes USp(\dot{P}) \), depending on the value for \( q \).

Even in the generic case, there are additional symmetries present. The corresponding algebra decomposes into eigenspaces of the generator associated with the scale transformations (19) according to

\[
\mathcal{X} = \mathcal{X}_{-3/2} + \mathcal{X}_0 + \mathcal{X}_{3/2}.
\]

Here \( \mathcal{X}_0 \) contains the generators of three groups mentioned above and \( q + 1 \) extra generators. The latter extend the algebra of \( SO(q+1) \) to that of \( SO(q+1, 1) \), so that \( \mathcal{X}_0 \) contains the generators belonging to \( SO(1,1) \otimes SO(q+1,1) \otimes S_q(P, \dot{P}) \), where \( SO(1,1) \) denotes the scale transformations (19). It thus follows that the solvable algebra contained in \( \mathcal{X}_0 \) has dimension \( q + 2 \) with rank 2, except for \( q = -1 \) when the rank equals 1. The \( r \) generators in \( \mathcal{X}_{3/2} \) are present for all cases and extend the dimension of the solvable subalgebra to \( q + 2 + r = n - 1 \), which equals the dimension of the corresponding real space.

Furthermore in four special cases there are yet another \( r \) generators contained in \( \mathcal{X}_{-3/2} \). These four cases are characterized by \( \frac{1}{2} r = q = 1, 2, 4 \) or 8, so that \( n = 6, 9, 15 \)

\(^4\)There may be additional isometries of the special real space, but those cannot be promoted to an invariance of the full corresponding supergravity theory [17].
or 27, respectively. (We remind the reader that the dimension of the real space equals \( n - 1 \).) The corresponding manifolds are symmetric and related to Jordan algebras and the magic square [9]. Their images under the \( r \) and \( c \) map are shown in table 5. Hence we are dealing with a root lattice for \( X \) that is rather similar to the root lattices encountered previously for the Kähler and quaternionic spaces. As an example consider the root lattice corresponding to \( L(1, 1) \), shown right. The roots on the vertical axis belong to \( SO(1, 1) \otimes SO(2, 1) \). The two roots denoted by \( \diamond \) exist only for the special case of \( L(1, 1) \) (and likewise for \( L(2, 1), L(4, 1) \) and \( L(8, 1) \), where we have 4, 8 and 16 such roots, respectively), unlike the other roots which exist for all generic homogeneous special spaces.

Hence we find the complete set of homogeneous real spaces, which are of rank 1 or 2 and shown in table 5. Upon application of the \( r \) and \( c \) maps they give rise to Kähler and quaternionic spaces of rank 2 or 3 and 3 or 4, respectively. Continuing this analysis one can systematically determine the isometry and isotropy groups of these spaces [14]. When comparing the results with the classification of [12, 13] one finds that there exist additional homogeneous spaces, which in table 5 have been indicated by a \( * \).

We close with a few comments on the spaces denoted by \( L(-1, P) \). The real spaces correspond to

\[
L(-1, P) : \quad \frac{SO(P + 1, 1)}{SO(P + 1)} \quad (n = P + 2)
\]

They were constructed in [21], but the corresponding Kähler spaces that emerge under the \( r \) map were not correctly identified. Moreover, the corresponding special Kähler and quaternionic spaces are not contained in the classifications of [12, 13]. The rank of these spaces equals 1, 2 and 3, for the real, Kähler and quaternionic versions, respectively. Only the case \( P = 0 \), corresponding to a \( n = 2 \) symmetric real, Kähler and quaternionic space, appeared in the classifications (see table 5). In spite of the fact that the real spaces are symmetric, the corresponding Kähler and quaternionic spaces are not symmetric for \( P > 0 \). The reason for the extra symmetry of the real sigma models is that they have extra invariances that are not preserved by the interactions with the vector fields in \( d = 5 \) supergravity [17]. Therefore these isometries are not preserved under the \( r \) map. Observe that the root lattice for the real and the Kähler spaces corresponding to \( L(-1, 1) \) was already discussed in section 3. For further details and an extensive list of references we refer to [14].
<table>
<thead>
<tr>
<th>$C'(h)$</th>
<th>real</th>
<th>Kähler</th>
<th>quaternionic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(-1, 0)$</td>
<td>$SO(1, 1)$</td>
<td>$[SU(1, 1)]^2$</td>
<td>$SO(3, 4) / (SU(2))^4$</td>
</tr>
<tr>
<td>$L(-1, P)$</td>
<td>$SO(P+1, 1) / SO(P+1)$</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$L(0, 0)$</td>
<td>$[SO(1, 1)]^2$</td>
<td>$[SU(1, 1)]^3$</td>
<td>$SO(4, 4) / SO(4) \otimes SO(4)$</td>
</tr>
<tr>
<td>$L(0, P)$</td>
<td>$SO(P+1, 1) / SO(P+1) \otimes SO(1, 1)$</td>
<td>$SU(1, 1) / U(1) \otimes SO(P+2, 2) / SO(P+4) \otimes SO(4)$</td>
<td></td>
</tr>
<tr>
<td>$L(0, P, \hat{P})$</td>
<td>$Y(P, \hat{P})$</td>
<td>$K(P, \hat{P})$</td>
<td>$W(P, \hat{P})$</td>
</tr>
<tr>
<td>$L(q, P)$</td>
<td>$X(P, q)$</td>
<td>$H(P, q)$</td>
<td>$V(P, q)$</td>
</tr>
<tr>
<td>$L(4m, P, \hat{P})$</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$L(1, 1)$</td>
<td>$SU(3) / SO(3)$</td>
<td>$Sp(6) / U(3)$</td>
<td>$F_4$ (\otimes SU(2))</td>
</tr>
<tr>
<td>$L(2, 1)$</td>
<td>$SU(3) / SO(3)$</td>
<td>$SU(3) \otimes SU(3) \otimes U(1)$</td>
<td>$SU(6) \otimes SU(2)$</td>
</tr>
<tr>
<td>$L(4, 1)$</td>
<td>$Sp(3) / SU(6)$</td>
<td>$SO^+(12)$</td>
<td>$E_7$ (\otimes SU(2))</td>
</tr>
<tr>
<td>$L(8, 1)$</td>
<td>$Sp(3)$</td>
<td>$SU(6) \otimes U(1)$</td>
<td>$E_8$ (\otimes SU(2))</td>
</tr>
</tbody>
</table>

Table 5: Homogeneous special real spaces and their corresponding Kähler and quaternionic spaces. The rank of the real spaces is equal to 1 (above the line) or 2 (below the line). The rank of the corresponding Kähler and quaternionic manifolds is increased by one or two units, respectively; $P$, $\hat{P}$, $q$ and $m$ are arbitrary positive integers.

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