Symmetry transformations in Batalin-Vilkovisky formalism.

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Abstract

Let us suppose that the functional $S$ on an odd symplectic manifold satisfies the quantum master equation $\Delta_\rho e^S = 0$. We prove that in some sense every quantum observable (i.e. every function $H$ obeying $\Delta_\rho (He^S) = 0$) determines a symmetry of the theory with the action functional $S$.

This short note was inspired by the paper [2] where the results of [1] were applied to obtain the description of the gauge transformations in Batalin-Vilkovisky theory. We begin with the observation that [1] contains conditions of physical equivalence of different solutions to the master equivalence and use these conditions to give a very transparent analysis of symmetry transformations in BV-approach. Let us recall some notions and results of [1].

Let us fix a manifold $M$ provided with odd symplectic structure ($P$-structure). Let us suppose that the volume element in $M$ is specified by the density $\rho$. We say that this volume element determines an $SP$-structure in $M$ if $\Delta_\rho^2 = 0$. (Here the operator $\Delta_\rho$ acts by the formula

$$\Delta_\rho A = \frac{1}{2} \text{div}_\rho K_A,$$

(1)

where $K_A$ denotes the hamiltonian vector field corresponding to $A$ and the divergence $\text{div}_\rho$ is calculated with respect to the density $\rho$.) By definition a function $S$ satisfies the quantum master equation on an $SP$-manifold $M$ if

$$\Delta_\rho e^S = 0.$$

(2)

Such a function $S$ can be considered as an action functional and determines physical quantitional by means of integration over Lagrangian submanifolds of $M$.

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The following statement was proven in [1](see Lemma 4 and Equ.(32)):

The density \( \tilde{\rho} = e^\sigma \rho \) determines a new \( SP \)-structure in \( M \) if and only if

\[
\Delta_\rho \sigma + \frac{1}{4} \{ \sigma, \sigma \} = 0. \tag{3}
\]

If \( S \) is a solution to the quantum master equation (1) then \( \tilde{S} = S - \frac{1}{2} \sigma \) satisfies the quantum master equation

\[
\Delta_\rho e^{\tilde{S}} = 0 \tag{4}
\]

corresponding to the new \( SP \)-structure. The action functional \( \tilde{S} = S - \frac{1}{2} \sigma \) on the manifold \( M \) with the new \( SP \)-structure describes the same physics as the action functional \( S \) on the manifold \( M \) with the old \( SP \)-structure. In particular

\[
\int_L e^{\tilde{S}} d\lambda = \int_L e^{S} d\lambda \tag{5}
\]

for every Lagrangian submanifold \( L \subset M \). (Here \( d\lambda \) and \( d\tilde{\lambda} \) denote the volume elements on \( L \) determined by new and old \( SP \)-structure correspondingly.)

One can introduce the notion of quantum observable for the theory with the action \( S \) on an \( SP \)-manifold \( M \) in the following way. We will say that an even function \( A \) determines a quantum observable if

\[
\Delta_\rho (A e^{\tilde{S}}) = 0 \tag{6}
\]

or, equivalently, if \( \Delta_\rho A + \{ A, S \} = 0 \). It is easy to check that for every observable \( A \) we have also \( \Delta_\rho (A e^S) = 0 \); in other words the quantum observables for the action functionals \( S \) and \( \tilde{S} \) coincide. Moreover

\[
\int_L A e^{\tilde{S}} d\tilde{\lambda} = \int_L A e^{S} d\lambda \tag{7}
\]

for every observable \( A \) and for every Lagrangian manifold \( L \). This equation can be considered as a little bit more precise expression of physical equivalence of action functionals \( S \) and \( \tilde{S} \) than (5). The condition (6) is equivalent to the requirement that \( S + \varepsilon A \) is a solution to the quantum master equation for infinitesimal \( \varepsilon \); therefore we can derive (7) applying (4) to the functionals \( S + \varepsilon A, \tilde{S} + \varepsilon A \) where \( \varepsilon \to 0 \).

Let us use the statement above in the case \( \sigma = 2S \). As was mentioned in [1], the equation (2) can be represented in the form \( \Delta_\rho e^{\sigma/2} = 0 \); therefore it is satisfied for \( \sigma = 2S \). We arrive to the following conclusion:

The theory with the action \( \rho \) is physically equivalent to the theory with the trivial action functional \( \tilde{S} = 0 \) and the \( SP \)-structure determined by the density \( \tilde{\rho} = e^{2S} \rho \).
The statement above can be used to analyze the symmetry transformations in Batalin-Vilkovisky approach. If $\tilde{S} = 0$ then the symmetries can be characterized as transformations of $M$ preserving the symplectic structure and the density $\tilde{\rho} = e^{2S}\rho$. Infinitesimal symmetry transformation correspond therefore to Hamiltonian vector fields with zero divergence with respect to $\tilde{\rho}$, i.e. to functions $H$ obeying $\Delta_\epsilon H = 0$. One can say therefore that infinitesimal symmetry transformations are in one-to-one correspondence with quantum observables of our theory. It is important to emphasize that the notion of observable does not change when we replace $S$ by $\tilde{S} = 0$. Therefore we can describe symmetries also in terms of the original action $S$ and the original measure $d\mu = \rho dx$. (This description follows immediately from the description of symmetry transformations in the formulation with $\tilde{S} = 0$; one can prove it directly using (5),(6),(7).)

We obtain that every function $H$ satisfying $\Delta_\epsilon H + \{H, S\} = 0$ (every quantum observable) determines a symmetry in the following sense. Neither the action functional $S$, nor the density $\rho$ are invariant with respect to an infinitesimal transformation with the Hamiltonian $H$, however the new action functional

$$\tilde{S} = S + \varepsilon \{H, S\}$$  \hspace{1cm} (8)

and the new measure

$$d\tilde{\mu} = d\mu (1 + 2\varepsilon \Delta_\epsilon H)$$  \hspace{1cm} (9)

describe the same physics as the old action functional $S$ and the old measure $d\mu = \rho dx$.

Here $\varepsilon$ is an infinitesimal parameter. The formula (9) follows immediately from (1) and from the standard formula

$$d\tilde{\mu} = d\mu + \varepsilon d\mu \ div_v K$$  \hspace{1cm} (10)

for the variation of measure by the infinitesimal transformation $x \to x + \varepsilon K$, where $K$ is an arbitrary vector field.

It is essential to stress that quantum observables don’t remain intact by the symmetry transformation (8),(9). Namely, one should replace a quantum observable $A$ by the observable $\tilde{A} = A + \varepsilon \{H, A\}$.

References
