Symplectic Manifolds, Coherent States and Semiclassical Approximation

S. G. Rajeev
Department of Physics and Astronomy,
University of Rochester, Rochester, N.Y. 14627, U. S. A.

S. Kalyana Rama and Siddhartha Sen
School of Mathematics, Trinity College, Dublin 2, Ireland.

Email: rajeev@uorhep.bitnet; kalyan,sen@maths.tcd.ie

ABSTRACT. We describe the symplectic structure and Hamiltonian dynamics for a class of Grassmannian manifolds. Using the two dimensional sphere ($S^2$) and disc ($D^2$) as illustrative cases, we write their path integral representations using coherent state techniques. These path integrals can be evaluated exactly by semiclassical methods, thus providing examples of localisation formula. Along the way, we also give a local coordinate description for a class of Grassmannians.
1. Introduction

Recently there has been some interest in applying “localisation” theorems to quantum field theories [1]-[6]. The best known localisation formula for evaluating the integral of a $p$-form $\alpha$ on a $p$-dimensional symplectic manifold $M$ with closed, non-degenerate two form $\omega$ is given by the Duistermaat–Heckmann (DH) theorem [7]. The theorem states that if the form $\alpha$ is equivariantly closed, i.e. $d_{\chi}\alpha \equiv (d + i_{\chi})\alpha = 0$ and $\mathcal{L}_{\chi}\omega = 0$, then the value of the integral $\int_{M}\alpha$ is given in terms of contributions from points in $M$ which are fixed points of the vector field $\chi$. In the above, $d$ is the exterior differential operator, $i_{\chi}$ is the inner multiplication with respect to the vector field $\chi$ [8], and $\mathcal{L}_{\chi}$ is the Lie derivative.

A loop space extension of the DH theorem in which the localisation result involves contributions from classical trajectories relates the quantum mechanical path integral in phase space to the WKB approximation. The content of the theorem is: when this approximation is exact? It was subsequently suggested that other localisation results are possible. In particular by choosing the vector field $\chi$ appropriately a localisation result was obtained in which only time-independent solutions to the equations of motion were involved in the localisation result. See [1] and references therein.

In this paper we consider two general classes of symplectic manifolds where Hamiltonian dynamics can be defined. One involves the infinite dimensional Grassmannian $Gr$ and the other involves the non compact group $U(n,m)$ [9]. We will describe how these manifolds have coset space description and arise naturally from fermionic and bosonic systems, and how they also provide interesting examples of localisation formula. Specifically, we show that path integral representation for certain traces of operators, related to the Cartan subalgebra (CSA) of a relevant Lie group system, can be obtained using coherent state techniques. An evaluation of these path integrals using semiclassical methods give rise to the Weyl character formula and are then exact. The examples are shown to satisfy the conditions of DH theorem hence the exactness of the semiclassical methods is to be expected.

In section 2 we set up the symplectic structures. We consider Grassmannian manifolds, the Siegel disc and flag manifolds of a certain type and construct the symplectic structure and the Hamiltonian $H$. In section 3 we give a local coordinate description for these Grassmannian. In sections 4 and
5, a path integral formulation for $\text{tr} e^{-\beta H}$ is constructed using coherent state techniques [10] for the special cases given by the coset spaces $U(\mathfrak{g})/U(\mathfrak{h})$ and $U(1,1)/U(1) \times U(1)$. We then evaluate the partition function using semiclassical methods and show that the Weyl character formula is obtained. We note that each term in our semiclassical evaluation corresponds to a term in the Weyl character formula. In section 6 we summarise our results and comment on them.

2. Grassmannian as Symplectic Manifolds

We now turn to a number of examples of symplectic manifolds. We start with Grassmannians. These provide examples of compact manifolds and are, as shown in [9], related to an underlying free fermion theory. The Grassmannian $Gr$ can be defined as

$$Gr = \cup_m Gr_m$$  \hspace{1cm} (1)

where $Gr_m$ is a set of all hermitian matrices satisfying a quadratic constraint:

$$Gr_m = \{ P | P^\dagger = P, \ P^2 = P, \ trP = m \}. \hspace{1cm} (2)$$

The quadratic constraint implies that the eigenvalues of the operator $P$ is 0 or 1. Thus it can be interpreted as occupation number and the trace condition as the total number of fermions. Each component $Gr_m$ of the Grassmannian $Gr$ can also be viewed as a coset space of the unitary group

$$Gr_m = \frac{U(H)}{U(m) \times U(H^\perp)} \hspace{1cm} (3)$$

where $H^\perp \subset H$ is orthogonal to $U(m)$ and the group $U(H)$ acts transitively on each component of $Gr_m$ by the action

$$P \rightarrow gPg^\dagger. \hspace{1cm} (4)$$

To see this note that any hermitian matrix $P$ can be diagonalised by a unitary transformation. There will be $m$ eigenvalues equal to 1 and the rest equal to 0. Thus for each $P \in Gr_m$, there is a $g \in U(H)$ such that

$$P = gP_0g^\dagger \hspace{1cm} (5)$$

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where $P_0 = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ and $I_m$ is an $m \times m$ unit matrix. Furthermore, if $h$ commutes with $P_0$ then $g$ and $gh$ correspond to the same $P$. The subgroup of elements of $U(H)$ that commute with $P_0$ is $U(m) \times U(H^\perp)$ consisting of unitary matrices that are block diagonal:

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$$  

(6)

Thus there is a one to one correspondence between $P \in Gr_m$ and the coset space $\frac{U(m \times U(H^\perp))}{U(H)}$.

We will be interested in physical systems for which the Grassmannian is a phase space. To this end one can define a symplectic form — a closed non degenerate 2-form — which is invariant under the action of $U(H)$. A unique such homogeneous symplectic form is given, up to an overall constant, by

$$\omega = -\frac{1}{2} tr P (dP)^2.$$  

(7)

It is invariant under $P \to g P g^\dagger$, where $g$ is a constant unitary matrix, and is a closed 2-form. A quick way to see that $\omega$ is closed is to define $\Phi = (1 - 2P)$. Note that $\omega \propto tr \Phi (d\Phi)^2$ and $\Phi^2 = 1$ which implies that $d\Phi \Phi + \Phi d\Phi = 0$. Then,

$$d\omega \propto tr (d\Phi)^3 = tr (d\Phi)^3 \Phi^2 = tr \Phi (d\Phi)^3 \Phi = -tr (d\Phi)^3 \Phi^2 = 0$$

(8)

where the first equality above is due to $\Phi^2 = 1$, second due to cyclicity of the trace, third due to $d\Phi \Phi + \Phi d\Phi = 0$ and the last one is obvious.

Since $\omega$ is homogeneous, it will be non degenerate everywhere if it is so at one point, say $P = P_0 \in Gr_m$. A tangent vector $U$ at this point is given by

$$U^\dagger = U, \quad P_0 U + U P_0 = 0$$

and hence is of the form $U = (0 \ u) \ (u^\dagger \ 0)$. Then

$$\omega(U, V) = -\frac{1}{2} tr P_0 [U, V] = tr (uv^\dagger - vu^\dagger).$$

(9)

If $\omega(U, V) = 0$ for all $V$, then $U = 0$ implying that $\omega$ is non degenerate. Thus $\omega$ is a symplectic form on $Gr$. 

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The symplectic structure can be represented in terms of the Poisson algebra of smooth functions. For a constant hermitian matrix $\xi$ define a function

$$F_\xi = \text{tr}(\xi P).$$

It can then be shown that

$$[F_\xi, F_\eta]_{PB} = F_{[\xi, \eta]}$$

where $[\cdot, \cdot]_{PB}$ is the Poisson bracket given in terms of the symplectic form by

$$[F_1, F_2]_{PB} = -\omega^{-1}(dF_1, dF_2).$$

The Jacobi Identity follows when $\omega$ is a closed form. Upon quantisation, the Poisson brackets are replaced by the commutators and the symplectic form $\omega$ gives the relevant commutation relations between the operators in the quantum theory.

Furthermore, a vector field $V_\xi$ can be defined for a given smooth function $F_\xi$ by

$$i_{V_\xi}\omega = dF_\xi.$$ 

Noting that $i_{V}\omega = -\frac{1}{2}\text{tr}P[V, dP]$, one obtains after a few simple steps

$$V_\xi = [\xi, P].$$

This is just an infinitesimal action of $U(H)$ on $Gr_m$. Hence,

$$[F_\xi, F_\eta]_{PB} = \mathcal{L}_{V_\xi} F_\eta F_{[\xi, \eta]}.$$ 

As an example, consider the simplest special case of a Grassmannian when $H = C^2$, $m = 1$. Then

$$Gr = \frac{U(2)}{U(1) \times U(1)} = S^2.$$ 

The projection operator $P$ is a $2 \times 2$ hermitian matrix given by $P = \frac{1}{2} I_2 + \beta_i \sigma^i$ where $\sigma^i$ are the Pauli matrices. The quadratic condition $P^2 = P$ implies that $\sum \beta^2 = \frac{1}{2}$ which describes a sphere $S^2$. If we choose the constant vector $\xi = \sigma^3$ then the smooth function $F_\xi$ is given by

$$F_\xi = \text{tr}(\xi P) = \hat{z} \cdot \hat{r} = \cos \theta$$

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where \( \mathbf{r} \) is the unit radial vector and \( \theta \) is the polar angle in polar coordinates. The symplectic two form \( \omega \) is given by \( \omega = \sin \theta d\theta \wedge d\phi \). Thus \( F_\xi \) is the height function for \( S^2 \). Upon quantisation, it can be replaced by the operator \( J_3 \), the component of the angular momentum operator \( J \) along a given direction, since the remaining coordinates of \( S^2 \) are given by \( \sin \theta \cos \phi \) and \( \sin \theta \sin \phi \), which correspond to the operators \( J_1 \) and \( J_2 \) in quantum theory.

It is sometimes convenient to solve for \( \Phi = 1 - 2P \), where \( P \) is defined in equation (2), by introducing explicit coordinates as in [9]. For \( S^2 \) a convenient choice would be

\[
\Phi = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \frac{2}{1+zz^*} \left( \begin{array}{cc} 1 & z \\ z^* & zz^* \end{array} \right). \tag{10}
\]

The corresponding symplectic form \( \omega \) is given by

\[
\omega_{zz^*} = (1 + zz^*)^{-2}.
\]

3. Local Coordinates on Grassmannian

We now give an alternate set of natural coordinate on Grassmannian manifold and also on certain flag manifolds which highlight global geometrical features. For this we first develop a method to parametrise any element \( u_n \in SU(n) \). We do it inductively using the fact that (i) \( u_n \) can be thought of as inducing a “rotation” in \( n \)-dimensional complex space \( C^n \) and (ii) any \( n \)-dimensional complex rotation can be composed by a rotation in \( C^{n-1} \) followed by a rotation around \( n \)th complex direction in \( C^n \). An element of \( SU(2) \) can be characterised by

\[
u_2 = \left( \begin{array}{cc} a & b \\ -b^* & a^* \end{array} \right) \tag{11}
\]

where the complex numbers \( a \) and \( b \) obey \( aa^* + bb^* = 1 \). The above parametrisation is obtained straight from the definition of \( SU(2) \), \( uu_2 + u_2u = 1 \). From this we explicitly see the isomorphism \( SU(2) \simeq S^3 \). For \( SU(n) \), note that [12] rotation about \( n \)th complex direction is generated by \( e^A \) where the \( n \times n \) matrix

\[
A = \left( \begin{array}{cc} 0 & Y \\ Y^\dagger & 0 \end{array} \right) \tag{12}
\]
belongs to the Lie algebra of $SU(n)$ and $Y$ is a $1 \times (n - 1)$ dimensional complex matrix. Motivated by the above facts and the form of $e^A$ we seek a parametrization of $u_n \in SU(n)$ in the form

$$u_n = \begin{pmatrix} 1 & 0 & \Sigma \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} a & Y \\ -\Sigma Y^\dagger & a^* S \end{pmatrix}$$

(13)

where $a$, $Y$ and $S$ are complex matrices of dimension $1 \times 1$, $1 \times (n - 1)$ and $(n - 1) \times (n - 1)$ respectively and $\Sigma \in SU(n - 1)$ is the rotation in $C^{n-1}$. Demanding $u_n u_n^\dagger + u_n^\dagger u_n = 1$ we obtain

$$S = (aa^* 1_{n-1} + Y^\dagger Y)^{-\frac{1}{2}}$$

$$u_n = \begin{pmatrix} a & Y \\ -\Sigma Y^\dagger & a^* S \end{pmatrix}$$

(14)

and the condition

$$aa^* + YY^\dagger = 1$$

(15)

which describes the manifold $S^{2n-1}$. In $S$ above, $aa^*$ is multiplied by a $(n - 1) \times (n - 1)$ unit matrix $1_{n-1}$. One can evaluate $S$ by first diagonalising $(aa^* 1_{n-1} + Y^\dagger Y)$ by a matrix $\Delta$, i. e. ,

$$\Delta (aa^* 1_{n-1} + Y^\dagger Y) \Delta^{-1} = \Lambda$$

where the elements of $\Lambda$ are given by $\Lambda_{ij} = \delta_{ij} \lambda_i$. $S$ is then given by

$$S = \Delta \Lambda^{-\frac{1}{2}} \Delta^{-1}.$$ 

(16)

Note that since $(aa^* 1_{n-1} + Y^\dagger Y)$ is nondegenerate $S$ and hence $u_n$ is well defined. Thus, since $\Sigma \in SU(n - 1)$ we have

$$SU(n) \simeq S^{2n-1} \otimes SU(n - 1)$$

(17)

and since $SU(2) \simeq S^3$, it follows that

$$SU(n) \simeq S^{2n-1} \otimes S^{2n-3} \otimes \cdots \otimes S^3.$$ 

(18)

Thus we have obtained an explicit parametrization of $SU(n)$ manifestly showing the local isomorphism of $SU(n)$ to a product of odd dimensional spheres.
We also note that these coordinates of $SU(n)$ are sufficient to determine its cohomology $H^*(SU(n), R)$, which is the same as that of $S^{2n-1} \otimes S^{2n-3} \otimes \cdots \otimes S^3$, because of a theorem of Hopf [11].

From the above, a group element $U_n \in U(n)$ can be easily obtained as

$$U_n = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1_{n-1} \end{pmatrix} u_n$$

and we see that

$$U(n) \simeq S^{2n-1} \otimes \cdots \otimes S^3 \otimes S^1.$$  \hspace{1cm} (20)

This parametrisation of $U(n)$ groups can be used to give coordinates on a Grassmannian manifold $Gr_{m,M}$ if one is given the embedding of $U(M - m) \times U(m)$ in $U(M)$.

Similarly, one can also coordinatise the flag manifolds. The flag manifold $F(\{m_i\}; M)$ can be described by a coset as

$$F(\{m_i\}; M) = \frac{U(M)}{U(m_1) \times U(m_2) \times \cdots \times U(m_i)}$$

with the condition that $\sum_i m_i = M$. Thus given the nature of embedding of $U(m_1) \times U(m_2) \times \cdots \times U(m_i)$ in $U(M)$, and using our parametrisation for $U(n)$ groups, the coordinates of the flag manifold $F(\{m_i\}; M)$ can be obtained in a straightforward way.

The final class of symplectic manifolds we consider arise when the Siegel disc is the phase space. In this case the manifold is not compact and can arise from an underlying bosonic model. The Siegel disc $D_{m+n}$ can be defined as the space of hermitian matrices $\phi$ subject to a quadratic constraint involving an indefinite metric $\eta$. It is defined as

$$D_{m+n} = \{ \phi^\dagger = \phi \mid \phi^\dagger \eta \phi = \eta \}$$

where

$$\eta = \begin{pmatrix} -1_m & 0 \\ 0 & 1_n \end{pmatrix}.$$  \hspace{1cm} (21)

Note that since $\phi$ is hermitian it can be brought to the diagonal form $\eta$ by a matrix $g \in U(m, n)$. Again $g$ and $gh$ correspond to the same $\phi$ if $h$ commutes with $\eta$. Hence we have

$$D_{m+n} = \frac{U(m, n)}{U(m) \times U(n)}.$$  \hspace{1cm} (22)

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$D_{m+n}$ is a symplectic manifold with symplectic form $\omega = tr(\phi \eta d\phi \eta d\phi \eta)$. This is easily established using arguments used for the Grassmannian case. Again a Hamiltonian $H = -tr(\eta \xi \xi \phi)$, where $\xi$ is a constant real diagonal matrix, can be defined. We will consider the simplest example of such manifolds, namely, $D_{1+1} = U(1) \times U(1)$.

4. Partition Function for $S^2$

We will now consider the path integral formulation of the partition function for the Hamiltonian associated with the Grassmannian $\frac{U(2)}{\mathbb{U}(1) \times \mathbb{U}(1)} \simeq S^2$. See also [3, 4, 5]. As remarked in section 2, upon quantisation of this system, its Hamiltonian is described by the operator $J_3$, the component of the angular momentum $J$ along a given direction. In quantum theory, the partition function is given by

$$Z = tr_{j} e^{-i T \gamma H} = \sum_{m=-j}^{j} \langle j, m \rangle e^{-i T \gamma m} \langle j, m \rangle$$

(23)

where $H = J_3$ is in the Cartan Subalgebra (CSA) of $SU(2)$ and the subscript $j$ labels the representation and $T$ represents the total time elapsed. Dividing $T$ into $N$ equal intervals $\delta = \frac{T}{N}$, the partition function can be written, suppressing the label $j$, as

$$tr_{j}(\ ) = \langle m \mid \prod_{k=1}^{N} e^{-i \delta m} \rangle m \rangle$$

(24)

Now introduce in appropriate places the factor 1 whose resolution is given by

$$1 = \int d\mu(\lambda) |\lambda > < \lambda|$$

(25)

where $|\lambda >$ are the coherent states [10] given by

$$|\lambda > = e^{i \lambda J_3} |j, -j >$$

(26)

with $|j, -j >$ being the “ground state” annihilated by $J_-$, i.e. $J_- |j, -j > = 0$. 

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Now consider
\[ \frac{\langle \lambda_k(k\delta)\mid e^{-i\delta g J_3}\mid \lambda_{k+1}(k+1)\delta \rangle}{\langle \lambda_k\mid \lambda_k \rangle} . \]  

(27)

To order \( \delta \), we have
\[ \langle \lambda_k\mid \lambda_{k+1} \rangle = \langle \lambda_k\mid 1 + \delta(i\partial)\lambda_k \rangle e^{-i\delta g J_3} = 1 - i\delta g J_3 . \]

(28)

Representing \( |j, -j> \) as a tensor product of \( (2j) \) factors
\[ |j, -j> = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \cdots \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \]  

(29)

we obtain
\[ |\lambda> = \left( \frac{\lambda}{1} \right) \otimes \left( \frac{\lambda}{1} \right) \otimes \cdots \left( \frac{\lambda}{1} \right) \]

and
\[ \langle \lambda_k\mid \lambda_k \rangle = (1 + \lambda_k \lambda_k^*)^{2j} . \]

(31)

Similarly \( \langle \lambda_k\mid J_3\mid \lambda_k \rangle \) and \( \langle \lambda_k\mid i\partial\lambda_k \rangle \) can be evaluated to give
\[ \langle \lambda_k\mid J_3\mid \lambda_k \rangle = j(-1 + \lambda_k \lambda_k^*)(1 + \lambda_k \lambda_k^*)^{2j-1} \]
\[ \langle \lambda_k\mid i\partial\lambda_k \rangle = j(\lambda_k \lambda_k^* - \lambda_k \lambda_k^*)(1 + \lambda_k \lambda_k^*)^{2j-1} . \]

(32)

Using the above expressions, equation (27) becomes
\[ \frac{\langle \lambda_k\mid e^{-i\delta g J_3}\mid \lambda_{k+1} \rangle}{\langle \lambda_k\mid \lambda_k \rangle} = exp\{ -i\delta \langle \lambda_k\mid g J_3 + i\partial\mid \lambda_k \rangle \} \]

(33)

and hence
\[ tr_j e^{-i\tau gH} = \int d\mu(\lambda) exp\{ -i \int_0^\tau dt \langle \lambda_k\mid g J_3 + i\partial\mid \lambda_k \rangle \} \]

(34)

where \( d\mu(\lambda) \) is a path integral measure over \( \lambda \)-space.

Introducing the stereographic coordinates, we write
\[ \lambda = e^{i\phi} \tan \frac{\theta}{2}, \quad \theta \neq \pi \]

(35)

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which gives

\[
\frac{1 - \lambda \lambda^*}{1 + \lambda \lambda^*} = \cos \theta
\]

\[
\frac{\lambda_k \lambda_k^* - \lambda_k' \lambda_k'^*}{1 + \lambda \lambda^*} = -i(1 - \cos \theta) \dot{\phi}.
\]  (36)

Setting further \(d\mu(\lambda)\) to be the path integral measure over \(S^2\) where

\[
d\mu(\lambda) = \sqrt{\frac{2\pi}{T}} d(\cos \theta) d\phi
\]  (37)

we finally obtain

\[
Z = tr e^{-i T J_3} = \sqrt{\frac{2\pi}{T}} \int d(\cos \theta) d\phi exp\left\{-i j \int_0^T dt ((1 - \cos \theta) \dot{\phi} - g \cos \theta)\right\}.
\]  (38)

Our above construction, which was carried out for the specific case of \(\frac{SU(2)}{U(1)} \simeq S^2\) can be carried out for any Grassmannian \(Gr_m\) associated with \(SU(n)\).

Now we consider evaluating the path integral given in equation (38). In a given representation \(j\), \(J_3\) takes values from \(-j\) to \(j\) in integer steps. Hence the trace formula in (38) just implies that

\[
Z = \sum_{j=0}^j e^{-i T J_3} = \frac{\sin(2j + 1) \frac{T g}{2}}{\sin \frac{T g}{2}}.
\]  (39)

On the other hand, the path integral formula can be evaluated using a semi-classical approximation as follows. We consider a particle that starts at a point, say \(\theta = \theta_0\) and \(\phi = 0\), at time \(t = 0\) and ends at the same point at time \(t = T\). We observe that to the classical action

\[
S_0 = j \int_0^T dt [(1 - \cos \theta) \dot{\phi} - g \cos \theta]
\]

can be added the “topological” term \(S_1 = n \int_0^T dt \dot{\phi}\) without changing the classical equations of motion. Furthermore, \(n\) has to be integer valued for consistency, namely, for \(\Delta \phi = \phi(T) - \phi(0)\) to be replaced by \(\Delta \phi + 2\pi\) and not change the partition function. Similarly \(j\) has to be an integer or half integer because

\[
e^{ij \int_{s^2} \omega} = e^{i4\pi j} = 1.
\]
Using the variable $z = \cos \theta$, the classical equations of motion are

\[
\begin{align*}
\dot{z} &= 0 \\
\dot{\phi} &= 1.
\end{align*}
\]  

(40)

Then with the topological term,

\[
S_{\text{classical}} = -(j + n) T.
\]

The fluctuations around the classical path $P_n$ also contribute to the partition function. This contribution can be evaluated by expanding the action around $S_n$ and using determinant formulas in $\zeta$-function regularisation scheme. After some calculations we find that these fluctuations contribute a factor of $\sqrt{\frac{T}{2\pi}}$ to the partition function which cancels against a similar factor in the path integral measure $d\mu$. Thus we have

\[
Z = \sum_n \int d(cos \theta ) d\phi e^{-i S_n}.
\]

We further note that the action as given in (38) is unbounded from below if $n$ is allowed to take negative integer values. Hence we restrict $n$ to positive values only, that is, $n \geq 0$. However in our formulation above, the analytic manifold we are considering is not the entire manifold $S^2$ itself but only an analytic patch covering part of it, namely its northern hemisphere. To cover $S^2$ fully we need another analytic patch similar to the one above obtained by $\lambda \to \frac{1}{\lambda}$ and in that patch the requirement of boundedness of action in $j$ restricts $n$ to be $n \leq 0$. Thus all possible paths on $S^2$ are taken into account if we evaluate the path integrals on both these patches with the corresponding restrictions on $n$.

We find that

\[
Z = e^{-iT_{\phi} j} \sum_{n \geq 0} e^{iT_{\phi} n} + e^{iT_{\phi} j} \sum_{n \leq 0} e^{-iT_{\phi} n}
\]

\[
= \frac{e^{-iT_{\phi} j}}{1 - e^{iT_{\phi}}} + \frac{e^{iT_{\phi} j}}{1 - e^{-iT_{\phi}}}.
\]  

(41)

The above summation results in

\[
Z = \frac{\sin(2j + 1)T_{\phi}}{\sin \frac{T_{\phi}}{2}}
\]  

(42)
which is the same answer that one gets by evaluating the trace formula.

It is also easy to check that if instead of working with coherent states arising from a lowest weight state, namely $|j, -j>$ we had started with an arbitrary state $|j, m>$, $-j \leq m \leq j$, all our results will still hold. The key remark is that the potential term in the Hamiltonian would now be $-mg \cos \theta$, while the symplectic potential would still remain $j(\cos \theta - 1)\dot{\phi}$. The classical equations for $\dot{\phi}$ would thus change.

5. Partition Function for $D^2$

We will now repeat the path integral formulation of the partition function for the Hamiltonian associated with the two dimensional disc $D^2$. As remarked in section 2, the Hamiltonian for the system is taken to be the operator $K_0$ belonging to the CSA of the non compact group $SU(1, 1)$, whose Lie algebra is given by

\[
[K_+, K_-] = -2K_0 \\
(K_0, K_\pm) = \pm K_\pm
\] (43)

with its representations labelled by an integer $k$. The Casimir invariant for $SU(1, 1)$ is given by

\[
C_2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) .
\] (44)

As remarked earlier in section 2, upon quantisation, these generators can be considered as quantum operators. Then the states are given by the eigenvectors of the operator $K_0$ and are labelled by the corresponding eigenvalues $\mu$, i.e.

\[
K_0|k; \mu > = \mu |k; \mu >
\] (45)

where $\mu = k + m$, $m = 0, 1, \cdots$, and $k = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$. Furthermore, the generators $K_\pm$ and $K_0$ can be represented in terms of harmonic oscillator operators:

\[
K_+ = a^\dagger b^\dagger \\
K_- = ab \\
K_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1)
\] (46)
with $a^\dagger$, $b^\dagger$, $a$, and $b$ being the creation and annihilation operators of two harmonic oscillators. The states can now be labelled by the occupation numbers $m$ and $n$ of these oscillators:

$$|m,n> = \frac{(a^\dagger)^m(b^\dagger)^n}{\sqrt{m!n!}}|0,0>.$$  

(47)

The states $|n+n_0,n>$ with fixed $n_0$ form a basis for the irreducible representations labelled by $K = \frac{1}{2}(1 + |n_0|)$.

Now, as before, we will define the coherent states as an example when $k = 1$

$$|\lambda> = e^{zK_+}|1,0>$$  

(48)

and calculate $<\lambda|O|\lambda>$, $O = 1$, $K_0$ and $\partial_n$. Noting that

$$<1,0|(K_-)^{n_1}(K_+)^{n_2}|1,0> = (n_1+1)!\delta_{n_1,n_2}$$  

(49)

we get

$$<\lambda|\lambda> = <1,0|e^{z^*K_-}e^{zK_+}|1,0> = \sum_0^{\infty} (n+1)(zz^*)^n = (1-zz^*)^{-2}.$$  

(50)

Similarly after a straightforward calculation we get

$$\frac{<\lambda|K_0|\lambda>}{<\lambda|\lambda>} = \frac{1+zz^*}{1-zz^*}$$  

(51)

and thus the partition function

$$Z = tr_k e^{-iTH}$$  

(52)

can be constructed as before where $H = gK_0$ and the eigenvalues of $K_0$ are given by $K_0 = n+k$ with $n = 0,1,2,\ldots$ and $k = 1,\frac{3}{2},2,\ldots$. We will discuss later the range of $k$.

Introducing the coordinates $z = e^{i\phi}\tanh\frac{\theta}{2}$, the partition function can be written as

$$Z = \int \prod_i d\mu_i \exp\{-ik\int_0^T (\dot{\phi}(\cosh\theta - 1) - \cosh\theta) - iS_1\}$$  

(53)
where \( d\mu_t = \sqrt{\frac{2\pi}{T}} \frac{k}{4\pi} d(\cosh \theta_t) d\phi_t \) is the path integral measure and \( S_1 = n \int_0^T dt \phi \), \( n \) an integer, is a topological term analogous to the one introduced for the case of \( S^2 \). Note that \( \theta \) ranges over all real values. However, using the phase angle \( \phi \) in the definition of \( z \), the ranges of \( \theta \) and \( z \) can be restricted to \( 0 \leq \theta \leq \infty \) and \( 0 \leq z \leq 1 \) which describes a two dimensional disc \( D^2 \).

Now an analysis similar to that of previous section can be carried out for the case of \( SU(1,1) \) group which is isomorphic to the two dimensional disc \( D^2 \). This time there is only one patch to consider and the result again is exact and of the form expected from the Weyl character formula.

Evaluating the partition function given by the trace formula in (52) gives

\[
Z = \sum_n e^{-iT\theta(n+k)} = 2i \frac{e^{-iT\theta(k-\frac{1}{2})}}{\sin \frac{T\theta}{2}}.
\]

(54)

On the other hand, the path integral formula in (53) can be evaluated by semiclassical method as before. After some calculations we find that the fluctuations around the classical solutions contribute a factor of \( \sqrt{\frac{2\pi}{T}} \) to the partition function which cancels against a similar factor in the path integral measure \( d\mu_t \). Summing the contributions of classical solutions in various “winding sectors” finally gives

\[
Z = 2i \frac{e^{-iT\theta(k-\frac{1}{2})}}{\sin \frac{T\theta}{2}}.
\]

(55)

which is the same answer as before upto a constant factor.

The parameter \( k \) has to be an integer or half integer because of the fact that square integrable \( L^2 \) functions on \( D^2 \) do not exist but automorphic forms do; that is, the scalar product

\[
\int dz \overline{dz}(1 - zz^*) 4^{k-2} f^*(z) f(z)
\]

implies that \( 4k > 2 \) and \( k \) has to be integer or half integer valued if one requires single valuedness under the action of \( SU(1,1) \):

\[
f(z) \rightarrow (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right)
\]

with \( ad - bc = 1 \). For a discussion of these issues, we refer to [13].

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We observe that the expression obtained by semiclassical methods is exact. Moreover the form in which the result appears is what one expects from the Weyl character formula.

6. Discussions

In this work, we have considered the symplectic structure and the Hamiltonian for certain class of manifolds and described how they provide examples of localisation formula. This is achieved by obtaining the path integral representation for certain operators using coherent state techniques. Evaluating these path integrals by semi classical methods give rise to Weyl character formula and give exact results.

In particular, we consider in detail the manifolds $S^2$ and $D^2$. For the first case, we find it necessary to divide the manifold into two analytic patches and to restrict the windings in each sector to one particular direction. With these restrictions the semi classical method of evaluating the partition function gives an exact answer.

For $D^2$ we find the eigenvalues of the Hamiltonian to be labelled by $\mu = k + m$, $m = 0, 1, \cdots$, where $k = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$. If $k = \frac{1}{2}$ were allowed, it would have corresponded to the harmonic oscillator. As such, we do not fully understand the physical system that $D^2$ might correspond to.

We have obtained the Weyl character formula for the above two cases using coherent state techniques. It would be very interesting to extend these methods and to obtain the Weyl character formula for any given coset $G/H$ as well.

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