Super–Lax Operator Embedded in
Self–Dual Supersymmetric Yang–Mills Theory\textsuperscript{1}

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Abstract

We show that the super-Lax operator for $N = 1$ supersymmetric Kadomtsev-Petviashvili equation of Manin and Radul in three-dimensions can be embedded into recently developed self-dual supersymmetric Yang-Mills theory in 2 + 2-dimensions, based on general features of its underlying super-Lax equation. The differential geometrical relationship in superspace between the embedding principle of the super-Lax operator and its associated super-Sato equation is clarified. This result provides a good guiding principle for the embedding of other integrable sub-systems in the super-Lax equation into the four-dimensional self-dual supersymmetric Yang-Mills theory, which is the consistent background for $N = 2$ super-string theory, and potentially generates other unknown supersymmetric integrable models in lower-dimensions.

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1. Introduction

The conjecture [1] that self-dual Yang-Mills (SDYM) theory in four-dimensions \((D = 4)\) will generate (possibly all) integrable systems in lower-dimensions has attracted much attention, not only from purely mathematical interest, but also from the practical applications in many useful models and systems in physics.

It is not merely a coincidence that the recent development of superstrings has also provided the important insight to the SDYM theory, which is nothing but the consistent background for the \(N = 2\) superstring [2]. In the recent formulation of the open \(N = 2\) superstring [3], it has been shown that if the supermultiplet for the background is to be described by an irreducible supermultiplet, it must necessarily be the \(N = 4\) self-dual supersymmetric Yang-Mills (SDSYM) theory. The usage of the superstring theory has provoked a new insight that the quantum aspects of the S newspaper the theory should be studied in the context of the \(N = 2\) superstring.

In a recent series of papers [4-8], we have studied various aspects of the S newspaper the theories and self-dual supergravity theories, and we have also shown [5-8] some explicit examples of embedding typical supersymmetric integrable systems into the \(D = 4\) S newspaper the, such as the supersymmetric KdV (SKdV) equations [9], supersymmetric Toda theory in \(D = 2\) or supersymmetric Kadomtsev-Petviashvili (SKP) equation of Manin and Radul [10] in \(D = 3\), some topological field theory and \(W_\infty\)-gravity [7]. We also showed [8] that such embeddings are also possible for the case of Wess-Zumino-Novikov-Witten models, which are closely related to supersymmetric integrable models or even more closely to \(N = 1\) superstrings. The success of these embeddings suggests that there must be some universal principle underlying them. Even though some ansatzes are presented in refs. [5,6] for explicit lower flows, more general embedding principles for the entire integrable hierarchy seem rather obscure. Motivated by this observation, we try in this Letter to show how the super-Lax operator for the \(N = 1\) SKP equation of Manin and Radul [10], which possesses one of the most common fundamental aspects for supersymmetric integrable systems, can be embedded into the S newspaper the theory in \(D = 4\). Since the \(D = 2\) supersymmetric integrable models such as SKdV equations [9] are generated by further dimensional reduction of the \(D = 3\) SKP equation, our result automatically applies also to the embedding of the formers.

We first show how the super-Lax operator for \(N = 1\) SKP equation of Manin and Radul [10] can be embedded into the \(D = 4\) S newspaper the. For this purpose we use a generalized ring for super-microdifferential operators\(^3\) instead of the usual finite-dimensional gauge Lie algebra. We next show the geometrical significance of our embedding, relating it to what is

\(^3\)The term “super-pseudodifferential operator” is also used. In this Letter, we try to accord with mathematical terminology in ref. [11], as much as possible.
called super-Sato equation for a wave operator [11]. We also show the meaning of infinite number of conserved charges in terms of our superspace formulation.

2. Embedding of $N = 1$ SKP Equation

We start with the embedding of $D = 3$, $N = 1$ SKP equation by Manin and Radul [10] into the $D = 4$, $N = 1$ SSYM. The super-Lax equation for even time flows for the $N = 1$ SKP hierarchy [10,11] is given in terms of the super-microdifferential operator $L$:

$$\frac{\partial L}{\partial t_{2n}} = \left[ (L_{2n})^+, L \right], \quad (t_2 \equiv x, \ t_4 \equiv y, \ t_6 \equiv t), \quad (2.1)$$

where

$$L \equiv \sum_{m=0}^{\infty} u_m D^{1-m}, \quad u_m \equiv u_m(x, y, t, \theta), \quad u_0 = 1, \quad (Du_1) + 2u_2 = 0,$$

$$D^{-1} \equiv D\partial_x^{-1}, \quad D \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}, \quad D^2 \equiv \frac{\partial}{\partial x} \equiv \partial_x,$$  \quad (2.2)

$$\partial_x^{-m} f(x, y, t) \equiv \sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{n} \left( \frac{\partial^n f}{\partial x^n} \right) \partial_x^{-m-n}.$$

As usual [9,10], the subscript $+$ denotes the projection onto the class of terms with non-negative powers of $D$. The Lax operator $L$ is fermionic (anti-commuting). Unless braced by parentheses, $\partial_x^m$, $D^m$, $\partial_x^{-m}$ and $D^{-m}$ are to be regarded as operators, like $\partial_x f \equiv (\partial f/\partial x) + f \partial_x$. The symbol $\binom{m}{n}$ denotes the usual binomial coefficients. In this Letter we use first only $t_2, t_4, t_6$ out of infinitely many bosonic time variables $t_2, t_4, \cdots$, and only $\theta$ out of infinitely many fermionic variables $\theta, t_1, t_3, \cdots$ in the formalism of ref. [11], in order to see in particular the $N = 1$ SKP equation of Manin and Radul in $D = 3$ [10], as an example.

Since the super-Lax operator contains infinitely many fields $u_m$, as well as non-local operators like $\partial_x^{-1}$, the space of the algebra is expected to be infinite-dimensional, and we have to generalize the concept of the usual finite-dimensional Yang-Mills gauge group. Intuitively we can supersymmetrize the non-supersymmetric embedding by Schiff [12], and consider a “graded” ring $\mathcal{E}$ of the super-microdifferential operators. This $\mathcal{E}$ is a kind of gauge algebra, but it can be also identified with its universal enveloping algebra with the usual multiplication properties with gradations, and the unique splitting property $P =$

\footnote{In general we can choose any arbitrary three bosonic coordinates out of $t_2, t_4, t_6, \cdots$ for embedding of other sub-systems contained in the SKP super-Lax equation into $D = 3$. See eq. (2.13).}
\[ P_+ + P_- \] into the polynomials of respectively non-negative and negative powers of \( D \) for an arbitrary element \( P \in \mathcal{E} \). This is an intuitive description of the graded ring \( \mathcal{E} \) of the super-microdifferential operators.

Before performing our embedding, let us specify the ring \( \mathcal{E} \) in mathematical terms more precisely. For this purpose, we follow the terminology in ref. [11]. Consider the formal completion \( \mathcal{S} \):

\[
\mathcal{S} \simeq \mathcal{A} \otimes \mathbb{R}[[x, y, t, \theta]]
\]

(2.3)

which is the algebra of analytic functions on a super affine space \( B^{(1|1)}_\mathcal{A} \) of dimension \( (1|1) \) over \( \mathcal{A} \) which is Grassmann algebra \( \Lambda(\mathbb{R}) \otimes \mathbb{R}^2 \) generated by a three-dimensional vector space \( \mathbb{R} \otimes \mathbb{R}^2 \) with the respective coordinates \( x \) and \( (y, t) \). The \( \mathbb{Z}_2 \)-gradation in \( \mathcal{A} \) is applied only to the first \( \mathbb{R} \) out of \( \mathbb{R} \otimes \mathbb{R}^2 \) with the variables \( (x, \theta) \), where \( \theta \) is anti-commuting (fermionic). In terminology for physicists, this vector space is nothing but what is called \( N = 1 \) superspace in \( D = 3 \).\(^5\) The \( \mathbb{R}[[x, y, t, \theta]] \) is the algebra of formal power series with \( x, y, t \) and \( \theta \). The algebra \( \mathcal{S} \) is a super-commutative algebra with the \( \mathbb{Z}_2 \)-gradation: \( \mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1 \), where the suffix \( o \) (or \( 1 \)) denotes the bosonic (or fermionic) grading. Let \( \mathcal{E}_\mathcal{S}^{(1|1)} \) denote the ring of super-microdifferential operators over \( \mathcal{S} \):

\[
\mathcal{E}_\mathcal{S}^{(1|1)} \equiv \left\{ \sum_{-\infty < n < \infty} u_n(x, y, t, \theta) D^n \left| u_n(x, y, t, \theta) \in \mathcal{S} \right\} \right.
\]

(2.4)

\[
= \bigcup_{m=-\infty}^{\infty} \mathcal{E}_\mathcal{S}^{(1|1)}(m) \; .
\]

The last line is for a filtration compatible with the \( \mathbb{Z}_2 \)-gradation: \( \mathcal{E}_\mathcal{S}^{(1|1)}(m) = \mathcal{S}[[D^{-1}]] D^m \), whose typical element is

\[
P = \sum_{n=-\infty}^{m} p_n(x, y, t, \theta) D^n \in \mathcal{E}_\mathcal{S}^{(1|1)}(m) \; , \quad (p_m \neq 0) \; .
\]

(2.5)

Notice also that \( \mathcal{E}_\mathcal{S}^{(1|1)} = \mathcal{S}(D) \oplus \mathcal{E}_\mathcal{S}^{(1|1)}(-1) \), where \( \mathcal{S}(D) \) is the ring of super-differential operators. Accordingly, any element \( P \in \mathcal{E}_\mathcal{S}^{(1|1)}(m) \) is uniquely projected into

\[
P = P_+ + P_- \; , \quad P_+ \in \mathcal{S}(D) \; , \quad P_- \in \mathcal{E}_\mathcal{S}^{(1|1)}(-1) \; .
\]

(2.6)

Reviewing \( L \) in (2.2), we see that

\[
L \in \mathcal{E}_\mathcal{S}^{(1|1)}(1) \cap (\mathcal{E}_\mathcal{S}^{(1|1)})_1 \; ,
\]

(2.7)

\(^5\)Even though we write \( D = 3 \), our base space is more precisely \( (D = 1, N = 1 \text{ superspace}) \otimes \mathbb{R}^2 \), because we have no fermionic \( \theta \)-coordinates for the \( y \) and \( t \)-coordinates. In this sense our \( \mathcal{E}_\mathcal{S}^{(1|1)} \) differs from \( \mathcal{E}_\mathcal{S}^{1|1} \) by Ueno and Yamada [11]. Relevantly, our coordinates \( x, y, t \) are all real.
namely the fermionic operator $L$ has the powers of $D$ between $-\infty$ and $+1$. Now the practical significance of these mathematical terms is self-explanatory.

We now come to our formulation of supersymmetric Yang-Mills theory in $N = 1$ superspace $(Z^M) = (x, y, t, \theta)$ with a potential superfields $A_M(x, y, t, \theta)$ regarded as a connection superfields on the base superspace $(Z^M)$ with their values in the ring $\mathcal{E}_s^{(1|1)}$. This is nothing else than a generalization of the finite-dimensional Lie algebra used in supersymmetric Yang-Mills theory. It is now clear that the usual Lie ring of a gauge Lie group is replaced by the ring $\mathcal{E}_s^{(1|1)}$ of super-microdifferential operators. By construction, the Grassmann parities (gradations) for the coordinates $(x, y, t, \theta)$ are identified with those for $\mathcal{E}_s^{(1|1)}$.

We now turn to our original purpose, namely the embedding of the super-Lax operator for the SKP equation of Manin and Radul into the SDSYM theory. In our recent papers [5,6], it has been shown that the system of vanishing Yang-Mills superfield strength in $D \leq 3$ can be obtained as a “descendant” theory of the $D = 4$ SDSYM theory after appropriate dimensional reductions. It is to be understood also that our embedding is equivalent to a further embedding into the $N = 2$ superstring theory [2,3] as a more fundamental theory, since the $D = 4$ SDSYM fields are nothing but the consistent backgrounds for the $N = 2$ superstring theory [2,3].

As a matter of fact, in the superspace formulation of the $\mathcal{E}_s^{(1|1)}$-ring algebra-valued supersymmetric Yang-Mills theory above, we can set up such a system of vanishing superfield strength in superspace. Namely we can show that

$$F_{AB} \equiv \{ \nabla_A, \nabla_B \} - T_{AB}^C \nabla_C = 0 \quad ,$$

for all the supercoordinate indices of $A, B, \ldots = x, y, t, \theta$.

Our embedding satisfying (2.8) can be performed by the ansätze:

\begin{align}
\nabla_\theta & \equiv D + A_\theta \equiv L \quad , \\
\nabla_x & \equiv \partial_x + A_x \equiv \nabla_\theta^2 \equiv L^2 \quad , \\
\nabla_y & \equiv \partial_y + A_y \equiv \partial_y - (L^4)_+ + L^4 \quad , \\
\nabla_t & \equiv \partial_t + A_t \equiv \partial_t - (L^6)_+ + L^6 \quad ,
\end{align}

(2.9a)

where $\partial_y \equiv \partial/\partial y$ and $\partial_t \equiv \partial/\partial t$.

It is now easy to show that (2.8) holds for $AB = \theta \theta, x \theta, y \theta, t \theta, a b$, where $a, b, \ldots = x, y, t$.
For example,
\[
F_{\theta \theta} = \{L, L\} - 2\nabla_x \equiv 0 ,
\]
\[
F_{x \theta} = \left\{ \nabla_x, \nabla_\theta \right\} = \left[ \partial_x - (L^2)^+, L \right]
= \frac{\partial L}{\partial x} - \left[ (L^2)^+, L \right] = 0 ,
\]
\[
F_{y \theta} = \left[ \partial_y - (L^4)^+ + L^4, L \right] = 0 ,
\]
\[
F_{x y} = \left[ L^2, \partial_y - (L^4)^+ + L^4 \right]
= - \left\{ L, \left[ \partial_y - (L^4)^+, L \right] \right\} = 0 ,
\]
\[
F_{y t} = \left[ \partial_y - (L^4)^+ + L^4, \partial_t - (L^6)^+ + L^6 \right]
= \left[ \partial_y - (L^4)^+, L^6 \right] - \left[ \partial_t - (L^6)^+, L^4 \right] + \left[ \partial_y - (L^4)^+, \partial_t - (L^6)^+ \right] = 0 .
\]

We have made use of the relations
\[
\frac{\partial L_{2p}^2}{\partial t_{2n}} = \left[ (L^{2n})^+, L^{2p} \right] ,
\]
\[
\frac{\partial (L^{2n})^+}{\partial t_{2m}} - \frac{\partial (L^{2m})^+}{\partial t_{2n}} = \left[ (L^{2m})^+, (L^{2n})^+ \right] ,
\]
which can be easily shown to hold for arbitrary natural numbers \( m, n \) and \( p \). Similarly we can get \( F_{t \theta} = F_{x t} = 0 \). It is interesting to see that \( A_a (a = t_{2n}) \) can be rewritten as \( A_{t_{2n}} = (L^{2n})^- \).

In the above formulation, we have only dealt with the embedding into the \( D = 4 \) SDSYM. In fact, based on the more general formulation in ref. \[11\], it is expected that the embedding can be generalized to any even-dimensional (generalized) SDSYM theories. However, the \( D = 4 \) SDSYM is to be the most well-motivated from the viewpoint of \( N = 2 \) superstring, as its effective background.

We mention the arbitrariness related to the identification (2.9a). There are infinitely many alternative identifications, such as
\[
\nabla^{(k)}_\theta \equiv L^{2k+1} , \quad (k = 0, 1, \cdots) .
\]

For the embedding into \( D = 3 \), we choose any arbitrary three bosonic coordinates such as \( x \equiv t_{2(2k+1)}, y \equiv t_{2l}, t \equiv t_{2m} \), containing \( x \equiv t_{2(2k+1)} \) out of the set \( \{ t_2, t_4, t_6, \cdots \} \), with other coordinates truncated. Now we can confirm the set
\[
\nabla^{(k)}_\theta \equiv L^{2k+1} ,
\]
\[
\nabla_x \equiv \partial_x + A_x \equiv L^{2(2k+1)} ,
\]
\[
\nabla_y \equiv \partial_y + A_y \equiv \partial_y - (L^{2l})^+ + L^{2l} , \quad (l, m = 1, 2, \cdots) ,
\]
\[
\nabla_t \equiv \partial_t + A_t \equiv \partial_t - (L^{2m})^+ + L^{2m} , \quad (l \neq m \neq 2k + 1 \neq l) ,
\]
\[
\frac{\partial L_{2p}^2}{\partial t_{2n}} = \left[ (L^{2n})^+, L^{2p} \right] ,
\]
\[
\frac{\partial (L^{2n})^+}{\partial t_{2m}} - \frac{\partial (L^{2m})^+}{\partial t_{2n}} = \left[ (L^{2m})^+, (L^{2n})^+ \right] ,
\]
satisfies our fundamental equation (2.8) again, by the help of (2.11). This sort of arbitrariness in our superspace, or in other words, the arbitrariness for choosing a superspace is closely related to the existence of infinitely many conserved charges discussed later.

3. Relationship with Super-Sato Equation

The vanishing superfield strength suggests that all the potential superfields must be pure-gauge in our generalized algebra space (ring) $\mathcal{E}^{(1|1)}_S$. We can show this is indeed the case, and it has more significance, when we review the super-Sato equation [11] associated with the super-Lax equation (2.1).

The super-Sato equation [11] is just supersymmetric generalization of what is called Sato equation [13] for the bosonic Lax equation. It has been known that the super-Lax equation (2.1) can be solved, if and only if there exists a bosonic super-microdifferential operator

$$W = \sum_{n=-\infty}^{0} w_n(x, y, t, \theta) D^n \in \mathcal{E}^{(1|1)}_S(0) \cap (\mathcal{E}^{(1|1)}_S)_0 ,$$  \hspace{1cm} (3.1)

satisfying

$$L = WDW^{-1} ,$$ \hspace{1cm} (3.2a)

$$\frac{\partial W}{\partial t_{2n}} = (L^{2n})_+ W - WD^{2n} .$$ \hspace{1cm} (3.2b)

This $W$ is usually called super wave operator, and (3.2b) is the super-Sato equation [11]. In the purely bosonic system, it is known that $W$ can be regarded as a gauge transformation parameter [12]. We can easily see that this is also the case with the super-Lax equations after simple algebra. In fact, inserting (3.2a) into our ansätze (2.9), and using also (3.2b), we get

$$\nabla_\theta = WDW^{-1} , \quad \nabla_x = W \partial_x W^{-1} ,$$ \hspace{1cm} (3.3a)

$$\nabla_y = \partial_y - (L^4)_+ + WD^4W^{-1} = W \partial_y W^{-1} ,$$

$$\nabla_t = \partial_t - (L^6)_+ + WD^6W^{-1} = W \partial_t W^{-1} .$$ \hspace{1cm} (3.3b)

Note that the second equalities for $\nabla_y$ and $\nabla_t$ hold only on-shell, namely only after the use of the super-Sato equation (3.2b). In these forms of $\nabla_A$, it is clear that $F_{AB}$ vanishes in this system. To put it differently, the vanishing superfield strength (2.8) is a sufficient condition of the super-Sato equation (3.2b), via the identification (3.2a).
4. Infinite Number of Conserved Charges

We can also understand the infinitely many conservation laws [10] associated with the whole hierarchy contained in the super-Lax equation in our superspace formulation. These conservation laws can be formulated by the use of the arbitrariness we mentioned with (2.12). In fact, we can define the generalized fermionic charges [10]

\[ Q_F^{(k)} \equiv \int dx dy d\theta \, \text{Res} (L^{2k+1}) \equiv \text{Tr} (L^{2k+1}) = \text{Tr} (\nabla_\theta^{(k)}) \quad (k = 0, 1, \cdots) , \quad (4.1) \]

where the symbol \( \text{Res} \) implies the coefficient of the \( D^{-1} \)-term in an arbitrary super-microdifferential operator \( \Gamma \):

\[ \text{Res} \Gamma \equiv g_{-1}(x, y, t, \theta) \quad , \]

\[ \Gamma \equiv \sum_{n=\infty}^{m} \, g_n(x, y, t, \theta) D^n \quad , \quad (m = 1, 2, \cdots) . \quad (4.2) \]

Eq. (4.1) is interpreted as the generalization of

\[ Q_F = \int dx dy d\theta \, \text{Res} (\nabla \theta) \equiv \text{Tr} (\nabla \theta) \quad , \quad (4.3) \]

based on the arbitrariness mentioned in (2.12). We can show for arbitrary super-microdifferential operators \( \Gamma \) and \( \Lambda \) that [9]

\[ \int dx dy d\theta \, \text{Res} [ \Gamma, \Lambda ] = \text{Tr} [ \Gamma, \Lambda ] = 0 , \quad (4.4) \]

because \( \text{Res} [ \Gamma, \Lambda ] \) is always a total divergence with respect to \( x, y, \theta \). It is now straightforward to show for \( t \equiv t_0 \) that

\[ \frac{dQ_F^{(k)}}{dt} \equiv \frac{d}{dt} \int dx dy d\theta \, \text{Res} (L^{2k+1}) \]
\[ = \int dx dy d\theta \, \text{Res} [ (L^0)_+, L^{2k+1} ] = 0 . \quad (4.5) \]

Eq. (4.5) holds as a sufficient condition, when all the eigenvalues of the operator \( \nabla_\theta^{(k)} \) are isospectral, (i.e., \( t \)-independent), as is seen from the Adler-trace expression in (4.1). By appropriate truncation [14] to the \( N = 1 \) SKdV system [9], we easily see that \( Q_F^{(0)} \) agrees with the fermionic charge \( \int dx \xi(x) \), where \( \xi(x) \) is the fermionic component field in the lowest flow of \( N = 1 \) SKdV equation [9]. The \( Q_F^{(k)} \) with higher \( k \) seem to yield new fermionic charges.

The existence of the infinitely many conserved charges \( Q_F^{(k)} \) is the reflection of the arbitrariness in \( k \) in the identification \( \nabla_\theta^{(k)} \equiv L^{2k+1} \) in our superspace. This feature seems universal in any super-Lax equation with a fermionic super-Lax operator \( L \). We should also notice the close relationship of this arbitrariness with the vanishing superfield strength.
We can also consider the conserved \textit{bosonic} charges
\begin{equation}
Q_B^{(k)} \equiv \int dxdy d\theta \text{ Res} \left( L^{2k} \right) = \text{Tr} \left( \nabla t_{2k} \right), \quad (k = 1, 2, \cdots) .
\end{equation}

Here we used the general formula (2.13) for $\nabla t_{2k}$, and for appropriate coordinates $x \equiv t_{2(2k+1)}$, $y \equiv t_{2l}$, $t \equiv t_{2m}$. After the dimensional reduction into $D = 2$ as in ref. [14], eq. (4.6) is reduced to the infinite bosonic charges for $N = 1$ SKdV hierarchy [9].

5. Concluding Remarks

In this Letter we have shown explicitly that the super-Lax operator for the $D = 3$, $N = 1$ SKP of Manin and Radul [10] is embeddable into the $D = 4$, $N = 1$ SDSYM theory, \textit{via} the super-Lax operator embedded in our superspace. We have also seen how the super-Sato equation is related to the super-Lax equation in a differential geometrical manner. The significance of infinitely many conserved charges is also clarified in our superspace. This is a good explicit indication that such an embedding is universally applicable to other super-Lax equation systems, and thus provides a strong support that any supersymmetric integrable model can be embedded into the $D = 4$ SDSYM theory, which is the supersymmetric version of the original Atiyah’s conjecture [1] for purely bosonic integrable models.

Understanding of the relationship between the super-Lax equation and the corresponding super-Sato equation in our embedding principle has elucidated the geometrical significance of the latter in terms of vanishing superfield strength in superspace. It is not a coincidence that there exists a super wave operator $W$ for each super-Lax operator $L$, because the former is nothing but a supersymmetric gauge transformation for a \textit{pure-gauge} potential superfield. Even though any system that has only pure gauge potential field is \textit{not} usually interesting physically except for non-trivial topology, we have seen that this observation is \textit{not} the case with the integrable systems in lower-dimensions, which provide many physically interesting models.

The new ingredient in this paper we emphasize is the \textit{superfield} strength formulation related to the $N = 2$ superstring theory applied to the super-Lax operator, which has not been presented in the past literature to our knowledge.\footnote{A prototype geometrical interpretation can be found in refs. [10,11], but no close relationship with the vanishing superfield strength in superspace, or with the more fundamental $N = 2$ superstring theory was pointed out.} It has been known that the \textit{bosonic} Lax equation can be reformulated in a zero-field strength system, either for the finite-dimensional graded Lie groups [14] or in the infinite-dimensional generalization [12]. Our
result that the super-Lax equation is closely related to the SDSYM theory in $D = 4$ via super-Sato equation for pure-gauge superpotential signals the fundamental importance of the SDSA theory [3,4] for supersymmetric integrable models.

We have seen that the infinitely many conserved charges associated with the super-Lax equation is the reflection of the arbitrary fermionic operators $\nabla_{\theta}^{(k)} \equiv L^{2k+1}$ in our superspace. It also corresponds to the alternative choices of three bosonic coordinates out of infinitely many time coordinates [11] in the whole SKP hierarchy.

The features of the super-Lax equation we utilized in our embedding are pretty common to any super-Lax hierarchy with a fermionic Lax operator. In particular, the patterns we have seen in sections 2 through 4 are universal, not limited to the SKP hierarchy. This implies that any super-Lax operator in the super-Lax equation (2.1) can follow the same embedding procedure into the $D = 4$ SDSA theory, via vanishing superfield strength (2.8) in superspace.

We have so far stressed the relationship between the super-Lax equation and the $D = 4$ SDSA. As some readers may have already noticed, our basic equation (2.8) in superspace has more to do with supersymmetric Chern-Simons theory of vanishing superfield strength in $D = 3$. It is no wonder that the three systems, namely the super-Lax equation [10,11], the SDSA theory in $D = 4$ [3,4], and supersymmetric Chern-Simons theory in $D = 3$ [15] have such close mutual relationships, which are dictated in terms of differential geometry in mathematics, and are motivated by $N = 2$ superstring theory [2-4] as their underlying “master theory”.

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