Liouville Theory and Special Coadjoint Virasoro Orbits

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Abstract

We describe the Hamiltonian reduction of the coadjoint Kac-Moody orbits to the Virasoro coadjoint orbits explicitly in terms of the Lagrangian approach for the Wess-Zumino-Novikov-Witten theory. While a relation of the coadjoint Virasoro orbit \( \text{Diff } S^1/SL(2,R) \) to the Liouville theory has been already studied we analyse the role of special coadjoint Virasoro orbits \( \text{Diff } S^1/T_{\pm,n} \) corresponding to stabilizers generated by the vector fields with double zeros. The orbits with stabilizers with single zeros do not appear in the model. We find an interpretation of zeros \( z_i \) of the vector field of stabilizer \( T_{\pm,n} \) and additional parameters \( q_i \), \( i = 1, \ldots, n \), in terms of quantum mechanics for \( n \) point particles on the circle. We argue that the special orbits are generated by insertions of "wrong sign" Liouville exponential into the path integral. The additional parameters \( q_i \) are naturally interpreted as accessory parameters for the uniformization map. Summing up the contributions of the special Virasoro orbits we get the integrable sinh-Gordon type theory.

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1 Introduction

The representations of the affine algebras and the Virasoro algebra play the crucial role in the modern approaches to the 2D conformal and integrable theories. The physical operators are classified in terms of the representations and moreover the partition functions can be found in terms of the characters of the representations relevant for a particular system. A popular example is the partition function of the 2D Yang-Mills theory which can be expressed in terms of the characters of the representations for the corresponding finite dimensional algebra which classifies them for the affine case. There is a rich structure for the representations of the Virasoro algebra. Having in mind Kirillov-Konstant construction which allows to find the representations of the algebra starting from the quantization of the coadjoint orbits we can reduce the problem to the classification of the orbits. For the Virasoro algebra the classification can be obtained in terms of the stabilizer vector fields (see as a review ref.[1]). There are four types of the stabilizers: $S^1$, $SL_n(2, R)$, $T_{\pm, n}$ and $T_{\Delta, n}$. First two types of the orbits admit a transparent interpretation: they correspond to the representation with the highest vector for $S^1$ and the representations with the null vectors for $SL_n(2, R)$. The last two types of the orbits did not attract much attention of physicists. The point is that when quantizing the corresponding orbits one deals with an unbounded Hamiltonian. But these types of representations are interesting because they introduce some non-trivial effects into the game. Namely it was shown in refs.[3, 2] that these orbits can be described via the Hamiltonian reduction procedure starting from the affine $SL(2, R)$ group if one performs the reduction of an element of $SL(2, R)$ with a nontrivial winding number.

In what follows we shall discuss the Lagrangian approach to the special Virasoro orbits. A natural object associated with a coadjoint orbit of the affine algebra is the Wess-Zumino-Novikov-Witten (WZNW) action which provides the symplectic structure on this orbit. To describe the reduction procedure it is sufficient to couple the WZNW theory to a gauge field. In particular the reduction to the Virasoro algebra or to the associated Liouville action results from the gauging of the Borel subgroup of the affine algebra [4, 5]. We will try to argue that the special Virasoro orbits appear if one takes into consideration the classical background of the gauge field which reminds the vortex configuration and has a non-trivial winding number. In this description each special coadjoint orbit is naturally enlarged to a finite set of orbits of singular 2-differentials on $S^1$ with single poles which are stabilized by regular vector fields with double zeros. We find an interpretation of residues of these poles in terms of quantum mechanics on $S^1$ and from the point of view of the uniformization problem of Riemann surfaces.

The paper is organized as follows. In sections 2 and 3 we will describe the reduction of the WZNW theory to the geometric action which corresponds to the special Virasoro orbits. In section 4 a possible relation to the uniformization problem will be discussed. In Conclusion we collect main statements of the paper.
2 Hamiltonian Reduction to the Special Virasoro Orbits

We consider the $SL(2,R)/B \times B$ coset conformal theory where $B \times B$ group corresponds to the left and right rotations by the Borel subgroups of $SL(2,R)$. This theory can be described by the gauged Wess-Zumino-Novikov-Witten (WZNW) theory with the following Lagrangian

$$L = L_{WZNW} - \frac{k}{4\pi} \tilde{A} \text{Tr} g^{-1} g' t_+ - \frac{k}{4\pi} \text{Tr} t_+ \dot{g} \dot{g}^{-1} - \frac{k}{4\pi} A \tilde{A} \text{Tr} g^{-1} \dot{g} t_+ - \bar{\mu} A - \mu \tilde{A}. \quad (2.1)$$

Here $k$ is a level of the Kac-Moody algebra, $t_-$ and $t_+$ are the matrices

$$t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.2)$$

$A$ and $\tilde{A}$ are the corresponding gauge fields, $\mu$ and $\bar{\mu}$ are constant parameters and $g'$ and $\dot{g}$ are the derivatives of the group element $g$ with respect to the light-cone coordinates $x = x_1 + x_2$ and $t = x_1 - x_2$, respectively (in this section we consider the Minkovskian version of the theory with respect to the 2D world sheet). The light-cone coordinate $x = x_1 + x_2$ is compactified on a circle of unity radius, while the second coordinate $t = x_1 - x_2$ is for definitness non-compactified.

It is convenient to use the Gauss parametrization for the group element

$$g = (1 + \alpha t_-) e^{\alpha \phi} (1 + \beta t_+). \quad (2.3)$$

Here $\alpha$, $\beta$ and $\phi$ are real scalar fields. This representation is not quite well defined on all the group manifold. Actually it is necessary to use four copies of such a representation to cover all the group manifold [5]. However we can consider the part of the group manifold where this representation is correct.

In terms of the fields $\alpha$, $\beta$ and $\phi$ the Lagrangian of the model reads as

$$L = -\dot{\phi} \phi' + e^{2\phi} (\alpha' + A) (\beta + \tilde{A}) - \bar{\mu} A - \mu \tilde{A}. \quad (2.4)$$

Integrating over the gauge fields $A$ and $\tilde{A}$ we easily get the Lagrangian of the Liouville theory with the cosmological term (see, e.g. [6] and refs. there)

$$L = -\dot{\phi} \phi' - \mu \bar{\mu} e^{-2\phi}. \quad (2.5)$$

We ignored here the volume of the gauge group $B \times B$. The normalization of the measure in the functional integral is determined by the metric on the group manifold

$$||\delta g||^2 = \int (-\delta \phi^2 + e^{2\phi} \delta \alpha \delta \beta). \quad (2.6)$$
This fact should be taken into account when we calculate the quantum corrections to the Lagrangian.

Now we want to interpret this Liouville theory in terms of the coadjoint Virasoro orbits. To this aim we split the functional space of the gauge fields $A$ into classes of equivalence as follows. We integrate over all $\bar{A}$ gauge fields with a fixed $A$ field. In this case the field $A$ can be considered as a representative of a coadjoint orbit of Kac-Moody algebra while the gauging of the Borel subgroup acting from the right can be considered as the Hamiltonian reduction [4]. Actually in this situation we have the Lagrangian

$$ L = -\dot{\phi} \phi' - \mu A, \quad (2.7) $$

with the following constraint

$$ \partial \alpha + A = \mu e^{-2\phi}. \quad (2.8) $$

For convenience we omitted here the factor $k/4\pi$ which can be easily inserted when it is necessary.

Now we need to make a comment on restrictions to the field space. The group element $g$ is assumed to be smooth and periodic with respect to the $x$ coordinate. The point is that the classification of coadjoint Virasoro orbits corresponds to a splitting of the space of $(2,0)$-forms $b$ into classes of equivalence with respect to smooth deformations with a central extension, i.e. $Diff S^1$. The condition of the smoothness of the group element $g$ is introduced because we want to include all the homotopically non-trivial information into the field $A$ which can so have singularities. In this way we will classify the coadjoint Virasoro orbits in terms of homotopic equivalence classes of $A$. The field $A$ is assumed to be periodic and single-valued but it may have poles.

To make a reduction from the coadjoint Kac-Moody orbit corresponding to the representative $At_-$ to a Virasoro coadjoint orbit it is convenient to start once again from the beginning, i.e. with eq. (2.1). First we should put the matrix $At_-$ into the following form

$$ \tilde{A} = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \quad (2.9) $$

by a Kac-Moody transformation

$$ g \rightarrow hg, \quad (2.10) $$

with a regular group element $h$.

Using the Polyakov-Wiegman formula it is easy to calculate the change of the Lagrangian under the transformation with the group element $h$

$$ L = L_{WZW} - \frac{k}{4\pi} Tr \tilde{A} g^{-1} \tilde{g} - \frac{k}{4\pi} Tr \tilde{A} g^{-1} \tilde{g} + \frac{k}{4\pi} Tr g^{-1} \tilde{g} \tilde{A} - \frac{k}{4\pi} Tr At_+ \tilde{h} h^{-1}. \quad (2.11) $$

Here

$$ \tilde{A} = h^{-1} h' + h^{-1} At_- h. \quad (2.12) $$
It is convenient to use the Gauss representation for the group element $h$

$$
h = (1 + at_-)e^{\omega^2}(1 + bt_+) =$$

$$
= \left( \begin{array}{cc} e^p & be^p \\
 a e^p & a b e^p + e^{-p} \end{array} \right).
$$

Then it is easy to find the equations for the parameters $a, b$ and $p$

$$p' - b = 0, \quad e^{2p}a' + e^{2p}A = 1. \quad (2.14)$$

The off-diagonal matrix element of $\hat{A}$ corresponding to $u$ in eq.(2.9) is a representative of the Virasoro coadjoint orbit

$$u = p'' + (p')^2. \quad (2.15)$$

Let us emphasize that the matrix elements of $h$ are assumed to be smooth and periodic with respect to $x$. However it is easy to see that it is not possible in general to put the Kac-Moody representative $\hat{A}t_-$ into the form (2.9) for a generic 1-form $A$. Actually a short analysis shows that it can be done only if the function $A$ does not have any poles of order higher than 1. Therefore one can interpret this theory in terms of the coadjoint Virasoro orbits only if we limit the functional space of $A$'s to contain fields not very singular, i.e. with poles at most of order 1. It is interesting that namely this constraint corresponds to (almost) finite action (2.7) for non-vanishing cosmological constant. Indeed with an appropriate regularization of a singularity in $\alpha$ the cosmological term in eq.(2.7) has only logarithmical divergence:

$$\int A = \int (e^{-2\psi} - \alpha'). \quad (2.16)$$

We also see that the contributions of fields $A$ with stronger singularities are suppressed due to a stronger divergence of classical action.

It is also clear that the gauge field $A$ does not have any zeros since the exponential $p$ is regular.

However a problem appears now that the representative of Virasoro orbit in eq. (2.15) can be singular. Indeed let $A = h/x$ where $q$ is a non-zero constant near the pole $x = 0$. Then let us find an appropriate solution to eq.(2.14) near $x = 0$

$$p = \frac{1}{2} \ln \nu + \ln x - qx/2 = \ldots, \quad a = \frac{r}{x}(1 + sx) + \ldots, \quad b = \partial p = \frac{1}{x} - q/2 + \ldots, \quad (2.17)$$

where $\nu, h, r, s$ are real parameters, $\nu > 0$. From eqs. (2.14) it is easy to see that

$$q = h\nu. \quad (2.18)$$

In turn a direct calculation shows that

$$u = \frac{q}{x}, \quad (2.19)$$

4
Moreover \( \exp(-2p) \) has no zeros since we assume that \( \exp p \) is regular. Hence the poles of the 2-differential \( u \) are determined only by poles of \( A \).

Now we want to show that in a sense such a representation corresponds to a special coadjoint Virasoro orbit with a stabilizer generated by a vector field with double zeros. Indeed, the coadjoint action of the Virasoro group (see, e.g. ref. [1]) reads
\[
Ad \ast (F)(u(x), c) = (u(F(x)) \cdot F''(x) - (c/24\pi)S(F), c),
\]
where \( c \) is a central charge, \( F \) is an element of \( Diff S^1 \) and \( S(F) \) is the Schwartzian derivative
\[
S(F) = \frac{F'''}{F'} - \frac{3}{2} \left( \frac{F''}{F'} \right)^2.
\]
Let us consider the equation for a stabilizer near \( x = 0 \) where the gauge field \( A \) has a pole \( A \propto 1/x \). The Virasoro representative is \( u = q/x \). We have
\[
-\frac{1}{2} \frac{e'''}{x} - \frac{2q}{x} e' + \frac{q}{x^2} e = 0.
\]
A regular solution to this equation (up to a normalization) is
\[
e = x^2 - qx^3.
\]
Therefore \( e \) can not have a single zero at this point. Instead we get a double zero of the stabilizer in the same point where the gauge field \( A \) has a pole. It is worth to noticing that at \( q \neq 0 \) \( e''' \) is not zero at zeros of \( e \) in contrast to usual description of the special \( Diff S^1/\hat{T}_{\pm,n} \) Virasoro orbits [1].

It is necessary to emphasize here that we did not claim that the vector field \( e \) of stabilizer has no additional zeros. Here we only point out that there are no single zeros of the vector field \( e \). Indeed the stabilizing vector field is given by \( e = \exp 2p \) and hence a single zero would lead to a double pole in the function \( u \) without any single pole at the same point. In turn according to eq.(2.15) this would lead to an appearance of a single pole in the field \( \exp(-2p) \) which can not of course have a single pole without a double pole at the same point (this is a consequence of the regularity of the group element \( h \) in eq.(2.12)). Therefore we conclude that the vector field \( e \) can not have a single zero, and hence the orbits \( Diff S^1/T_{\Delta,n} \) do not appear in this model. As to \( Diff S^1/SL_n(2,R) \) we shall see in next section that these orbits can appear while \( Diff S^1/S^1 \) do not enter the model.

Another important observation is that if we consider a limit of vanishing residues we have still the vector field \( e \) with double zeros that is a stabilizer of a regular 2-differential representing the special Virasoro orbit with a stabilizer \( \hat{T}_{\pm,n} \). Therefore we may consider these special Virasoro orbits as a limit of vanishing residues.

Notice that the residue of the pole in the Virasoro representative \( u \) is not uniquely defined even if the residue in the gauge field \( A \) is fixed. The freedom for a choice of
the residue in the 2-differential $u$ corresponds to a possibility to change this residue by diffeomorphisms while it is not possible for 1-form $A$. However no diffeomorphism with positive orientation can change the sign of the residue. Hence the space of all orbits of 2-differentials corresponding to stabilizers $v$ with $n$ double zeros can be splitted into $2 \cdot 3^n$ subspaces classified by sets of $n$ numbers (0, +1 or −1) (an additional doubling corresponds to ± signature of a special orbit $\text{Diff } S^1/\hat{T}_{n,\pm}$). These classes are not intersecting and are reduced to the special Virasoro orbits (with regular representatives) in the limit of vanishing residues. Moreover such a singular 2-differential is additionally characterized by $n$ real numbers because the value $v''/(v'')^2$ at a zero of the vector field $v$ is invariant under the action of $\text{Diff } S^1$.

3 Geometric Action for the Special Virasoro Orbits

Now we want to put the action into the form which is most close to the form of the geometrical action for the Virasoro orbit. To this aim we change a parametrization of the functional space.

Let us come back to the Lagrangian (2.7). Since the gauge field can have only single poles the residues of the double poles in the exponential $\exp \phi$ and in the 1-form $\alpha'$ should be correlated so that to satisfy eq.(2.8). Therefore we can identify the pair of fields $(\alpha, \phi)$ with the pair $(\alpha, p)$ (eq. (2.14)).

We can introduce the following parametrization of the space of fields $A$. Let us choose a particular representative $A_0$ of an orbit of gauge fields $A$. Then for this field $A_0$ we can choose a pair of fields $\alpha_0$ and $\phi_0$ which provide us with a 2-form $u(x)$ representing the corresponding coadjoint Virasoro orbit. The existence of this pair is guaranteed by the constraint (2.8).

Taking into account that $\exp(-2\phi)$ is 1-form we now can get any point of the Virasoro orbit using the following equation

$$e^{-2\phi(x)} = F'(x) e^{-2\phi_0(F(x))}, \quad (3.1)$$

where $F$ is a diffeomorphism of $S^1$. This new field $\phi$ corresponds of course to the gauge field $A(x)$ given by

$$A(x) = F'(x)A_0(F(x)). \quad (3.2)$$

Therefore integrating over smooth fields $F$ we cover all space of fields related to $A_0$, $\alpha_0$ and $\phi_0$ by the action of $\text{Diff } S^1$.

Actually the equation (2.8) has generically an infinite set of solutions for a given field $A$. First, there is a continuous degeneracy which corresponds to the gauge
invariance of the theory under transformation

\[ A \to A + f' \quad \alpha \to \alpha - f \]  

(3.3)

with a regular function \( f \). However this gives simply the volume of the gauge group in the path integral. Second, there is a discrete degeneracy which is discussed below.

Let us first consider the sector of the gauge fields \( A \) without poles. In this case the field \( A \) can be represented in the following form

\[ A = < A > F', \]  

(3.4)

where \( F \) is a diffeomorphism which can be represented as

\[ F = f + x, \]  

(3.5)

where \( f \) is a regular periodic function. If \( \alpha \) is a regular periodic function then after a rescaling of the field \( \alpha \) and a shift of \( \phi \) by a constant we get from eq. (2.8)

\[ \hat{F}' = e^{-2\phi}, \]  

(3.6)

where a diffeomorphism \( \hat{F} = \alpha + f + x \) obeys the same boundary condition as \( F \), \( \hat{F}(x + 2\pi) = \hat{F}(x) + 2\pi \). It is clear that this case corresponds to the coadjoint Virasoro orbit \( Diff S^1/SL(2, R) \) with a zero representative [4]. Actually there is an infinite set of solutions to eq.(2.8) for the gauge field (3.4). One can show that they are parametrized by the number of zeros of the exponential \( \exp \phi \). It is easy to find, for example, the following solution to eq.(2.8)

\[ \phi_0 = \ln \cos \frac{n x}{2} - \frac{1}{2} \ln (1 + \cos^2 \frac{n x}{2}), \]  

(3.7)

while

\[ \alpha_0 = \frac{2}{n} \tan \frac{n x}{2}. \]  

(3.8)

It can be verified that this orbit of the field \( \phi \) corresponds exactly to the Virasoro orbit with a stabilizer with \( n \) double zeros. This choice of the field \( \phi_0 \) corresponds to

\[ u = -\frac{n^2}{2} \frac{1 + \sin^2 \frac{n x}{2}}{(1 + \cos^2 \frac{n x}{2})^2}. \]  

(3.9)

In turn we can try to find a regular solution to eq. (2.15) for \( u = constant \). Then we easily see that such a solution exists only for \( u = -n^2/4 \) where \( n \) is an integer and has the following form

\[ \phi_0 = \ln \cos \frac{n x}{2}. \]  

(3.10)

This solution of course corresponds to the vanishing gauge field \( A \) and hence gives no contribution to the dependence of the Liouville partition function (for the Lagrangian (2.7)) on the cosmological constant \( \mu \bar{\mu} \). It is also clear that this Virasoro
orbit has the stabilizer $SL_n(2, R)$ [1] which is generated by a vector field $\epsilon$ without zeros and two vector fields with $n$ single zeros. It is worth emphasizing that this the only type of orbits with a constant representative.

Notice that the singular fields $\phi_0$ of the type of those in eqs. (3.7) and (3.10) are in fact reduced $SL(2, R)$ vortices as it was discussed in ref.[3]. To check this one should substitute the group element $g = \exp inx\sigma_2$ into eq.(2.3) and then rotate it into the form (2.9).

Thus it is clear that the space of solutions to eq.(2.8) for the regular fields $A$ covers an infinite set of orbits with stabilizers with double zeros. Actually as it is shown in the previous section each special Virasoro orbit associated with a stabilizer with $n$ double zeros corresponds to $n$-dimensional space of (singular) 2-differentials. The singular 2-differentials are shown to correspond to gauge fields $A$ with single poles. It is clear that for a given gauge field $A$ there is an infinite set of gauge non-equivalent solutions of eq.(2.8). These are parametrized by the number of double zeros $x_i$, $i = 1, ..., n$ of a stabilizer $\epsilon$ and $n$ parameters $q_i$ defined as the residues of 2-differential associated with the vector field $\epsilon$.

The considerations above for static fields can also be generalized to the fields depending on both light cone coordinates $x$ and $t$. In this situation we should consider diffeomorphisms $F(t, x)$ depending on "time", i.e. on $t$. A representative $\phi_0(t, x)$ of the orbit of the field $\phi$ can depend in general on $t$ explicitly. Then the variables of integration in the path integral are not exhausted by diffeomorphisms $F(t, x)$ and we should assume that the positions of zeros $x_i$ of a stabilizer and the parameters $q_i$ are also variables of integration. Thus we can introduce a dynamics on the space of singular 2-differentials.

To express the action in terms of new parametrization of functional space we can make a shift of the field $\phi$ in the equation above and hence in the Lagrangian (2.7) by $2\phi_0(t, F(t, x))$. Then we get an expression for the Lagrangian in terms of a diffeomorphism $F$

$$L = -\phi'\dot{\phi} - u(t, F) F'\dot{F} - \phi_0(t, F)\phi_0(t, F) F' - \bar{\mu}A(t, x),$$

where the smooth fields $\phi$ and $F$ are connected by the following constraint

$$F' = e^{-2\phi}.$$  \hspace{1cm} (3.12)

This Lagrangian has a form which is very close to the form of geometrical action [4]. In particular for $u = -n^2/4$ ($n \in \mathbb{Z}$) and $\phi_0$ given by eq.(3.10) this lagrangian gives the geometrical action for the $Diff S^1/SL_n(2, R)$ Virasoro orbit [4]

$$L = -\phi'^2 + n^2F'\dot{F}/4 - \mu\bar{\mu} < A > /2\pi.$$  \hspace{1cm} (3.13)

The third term in eq.(3.11) vanishes for this case.

It is necessary to emphasize however that the Lagrangian above contains also two additional terms as compared to the ordinary geometrical action [4]. The remarkable
fact is that both of them do not actually depend on the quantum field $F$ when integrated over $S^1$ and hence characterize only a choice of the orbit of the gauge field $A_0$.

It is easy to see that one may consider the values $x_i$ of positions of zeros of the stabilizer and the residues $q_i$ of poles in $u(t, x)$ as quantum mechanical variables, while the space of fields $F$ can be restricted to the diffeomorphisms which do not change the parameters $x_i$ and $q_i$. Thus we get a quantum mechanics of $n$ particles on a circle $S^1$ coupled to the quantum field theory.

To understand the meaning of the variables $x_i$ and $q_i$ one can try to extract an effective quantum mechanical Lagrangian for $x_i$ and $q_i$ integrating over field $F$ in semiclassical approximation at $k \to \infty$ (the Kac-Moody level $k$ is the coupling constant). However there is no regular solution to the classical equation of motion for the field $F$. Instead a configuration can be found that asymptotically satisfies the equation of motion for the field $F$. Such a configuration is equivalent to $F(x, t) = x$ with a particular choice of the classical background field $\phi_0$ which is concentrated near the positions of singularities. If $\phi_0$ is suppressed (for example, exponentially) at $|x - x_i| < \delta$ (where $\delta << 1$ is a parameter) then to the leading approximation the quantum mechanical action reads as

$$L_{QM} \sim \delta \sum_i \dot{x}_i q_i^2.$$  

Thus one can see that the variables $x_i$ and $q_i$ are dual each other.

### 4 Special Virasoro Orbits in the Liouville Theory

Now we want to find a relation of the considerations above to the Liouville theory formulated on the euclidean worldsheet. With this modification of the formulation of the theory the positions of singularities of the fields in the chiral sector are rather complex points on the disk. That means that in the antichiral sector there are singularities in the complex conjugated points.

Correspondingly the formulas related to the special Virasoro orbits are modified when translated to the euclidean formulation. In particular, near a singularity at $z = z_i$ of the classical field $\phi_0$ we have

$$\phi_0 = -\frac{1}{2} \ln \nu_i + \ln |z - z_i|^2 + \text{Re} h_i(z - z_i),$$

where $z$ is a complex coordinate on the disk, $\nu$ is a real number, $\nu > 0$, and $h_i$ is a complex constant. In turn the classical configuration $\phi_0$ determines a meromorphic 2-differential $\omega = (\partial \phi_0)^2 + \partial^2 \phi_0$ ($\partial$ stands for the derivative in $z$). This 2-differential is stabilized by a holomorphic vector field $e = \exp 2\phi_0$. Instead of Diff $S^1$ one has to consider here the algebra of conformal transformations. However if we draw
a smooth non-self-intersecting contour $K$ through the singularities of $\phi_0$ then the
generators of conformal transformations restricted to this contour correspond
to certain generators of the complexified algebra of vector fields on $S^1$, $c\ Vect\ S^1$,
while the contour $K$ can be understood as an image of $S^1$ under a holomorphic
function on a unit disk [7]. This translation of the vector field $\epsilon = \exp 2\phi$ gives
an element of $c\ Vect\ S^1$ with double zeros. Unfortunately such a correspondence
does not give yet any exact description of the special Virasoro orbits in terms of
the Liouville theory on a complex disk. Nevertheless we may assume that such a
relation exists.

Let us now consider the Liouville theory. As it is known the quantum corrections
coming form the integration measure in the path integral (eq.(2.6)) modify the
lagrangian [8]. Therefore at the quantum level we have

$$L = \sqrt{g} \left( \frac{1}{2\pi \gamma^2} (\nabla \phi)^2 - \frac{Q}{4\pi} \phi R(\tilde{g}) \right) + \frac{\mu \bar{\mu}}{8\pi \gamma^2} \sqrt{g} e^{-2\phi}. \quad (4.2)$$

Here we introduced a background metric on the Riemann surface. The parameter
$Q$ is related to the central charge of the theory by equation

$$c = 1 + 3Q^2, \quad Q = \frac{2}{\gamma} + \gamma. \quad (4.3)$$

In 2D gravity [6] the parameter $\gamma$ is determined by a requirement that the cosmo-
logical constant is represented by a marginal operator at the quantum level. In our
case its value follows from the condition that the theory is conformally invariant [8],
so that

$$Q = (k - 1)\gamma, \quad \gamma = \sqrt{2/(k - 2)}. \quad (4.4)$$

Notice that this value of a parameter $\gamma$ corresponds to the correct branch of solution
to eq.(4.3) (this is defined by the condition that a semiclassical limit of the theory
corresponds to $\gamma \to 0$ [6])

$$\gamma = \frac{Q_0}{2} - \frac{1}{2} \sqrt{Q^2 - 8} \quad (4.5)$$

only if $k > 3$, while for $2 < k < 3$ the parameter $\gamma = \sqrt{2/(k - 2)}$ obeys

$$\gamma = \frac{Q_0}{2} + \frac{1}{2} \sqrt{Q^2 - 8}. \quad (4.6)$$

After a rescaling $\phi \to -\gamma \phi/2$ we come to the standard expression for the Lagrangian

$$L = \sqrt{g} \left( \frac{1}{8\pi} (\tilde{\nabla} \phi)^2 + \frac{Q}{8\pi} \phi R(\tilde{g}) \right) + \frac{\mu \bar{\mu}}{8\pi \gamma^2} \sqrt{g} e^{\gamma \phi}. \quad (4.7)$$

The $zz$ component of the energy momentum tensor for this theory reads as ($T_{\alpha \beta} = 0$
since the theory is conformal invariant)

$$T_{zz} = -\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} Q \partial^2 \phi. \quad (4.8)$$
It is easy to see that the stress tensor given above is actually a modification (by quantum corrections) of the expression for the representative of the Virasoro orbit \( \text{(2.15)} \).

Now we want to use this formula to interpret the special Virasoro orbits as a presence of insertions of certain operators into the Riemann surface. In turn such an interpretation is related with the problem of uniformization (see, e.g. ref.\([6]\)).

As it is known the uniformization map \( \mathcal{J} \) of the Riemann sphere \( \mathbb{C} = \mathbb{C} \cup \{ \infty \} \) with \( n \) insertions of operators \( \exp \alpha_i \phi \) at points \( z_i, i = 1, ..., n \), can be determined by a ratio of solutions of the Fuchsian differential equation

\[
\frac{d^2 y}{dz^2} + \omega_X y = 0, \quad \omega_X = \frac{1}{2} S[\mathcal{J}^{-1}(z); z],
\]

where \( S[\mathcal{J}^{-1}(z); z] \) is the Schwarzian derivative. The inverse uniformization map \( \mathcal{J}^{-1} \) maps the hyperbolic plane \( H \) onto the Riemann sphere \( \mathbb{C} \) so that \( \mathbb{C} \simeq H/\Gamma \), where \( \Gamma \) is the Fuchsian group uniformizing \( \mathcal{C} \).

Near the positions of the inserted operators we have

\[
\omega_X = \frac{\gamma^2}{2} T(z) = \sum_i \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} + \text{regular terms} \right)
\]

where

\[
\Delta_i = \frac{1}{4} (1 - (1 - \theta_i)^2), \quad \theta_i = \alpha_i \gamma,
\]

and \( c_i \) are the accessory parameters.

The exponential \( \exp \gamma \phi \) is related to the uniformization map \( \mathcal{J}^{-1} \) \([9]\)

\[
e^{\gamma \phi(z)} = \frac{|(\mathcal{J}^{-1})'(z)|^2}{(\text{Im} \mathcal{J}^{-1}(z))^2},
\]

so that

\[
T(z) = \frac{1}{\gamma^2} S[\mathcal{J}^{-1}(z); z].
\]

In the semiclassical limit \( \gamma \to 0 \) taking \( \phi = -\gamma \phi_0/2 \) (where \( \phi_0 \) is defined by eq.\((4.1)\)) we get from eq.\((4.8)\)

\[
T_{zz} = \frac{4 \hbar_i / \gamma^2}{z - z_i}.
\]

Comparing this expression with eq.\((2.19)\) we formally see that that our expression for the classical stress tensor for the case of special Virasoro orbits can be interpreted as an effect of inserted operators with \( \alpha_i = 2/\gamma \)

\[
\prod_i e^{(2/\gamma) \phi(z_i)},
\]

while \( 2 \hbar_i \) play the role of accessory parameters. In other terms we see that in the semiclassical limit the singularities in the classical field \( \phi_0 \) are generated by the
\(\delta\)-functional sources \(\sum_i \delta^2(z - z_i)\) for the field \(\phi\). Actually it is easy to check that these operators are also responsible for generating of the classical configuration \(\phi_0\) (eq.(4.1)) at the quantum level (in the model without the cosmological operator). However for finite values of \(\gamma\) the energy momentum tensor (4.6) with \(Q\) defined by eq.(4.3) has double poles at the singularities \(T_{zz} = 1/(z - z_i)^2 + \ldots\)

Notice that the parameter \(\alpha\) obeying \(\alpha\gamma = 2\) is actually another solution to the equation relating \(Q\) and \(\gamma\)

\[
\alpha = \frac{Q}{2} + \frac{1}{2} \sqrt{Q^2 - 8}, \quad \text{at} \quad k > 3, \quad (4.16)
\]

and

\[
\alpha = \frac{Q}{2} - \frac{1}{2} \sqrt{Q^2 - 8}, \quad \text{at} \quad 2 < k < 3. \quad (4.17)
\]

Hence the operator \(\exp(2\phi/\gamma)\) is also marginal.

Actually such an identification is not quite correct because the field \(\phi_0\) does not satisfy the classical equation of motion with sources in the presence of the cosmological term \(\exp \gamma \phi\). Moreover the expression for \(\phi_0\) does not satisfy to the boundary conditions at singularities fixed in refs. [9]. In turn in the case of insertions of operators \(e^{(2/\gamma)\phi(z_i)}\) there is no solution to the classical equation of motion in the presence of the cosmological term [6]. Therefore such a classical configuration is stable only if there is no cosmological term in the lagrangian.

On the other hand at \(k > 3\) this situation corresponds to the case when all the parameters are \(\theta_i = \alpha_i\gamma = 2\), i.e. higher than the Seiberg bound [10] \(\alpha = Q/2\). Recall that this limiting value of parameter \(\alpha\) is determined by the condition that the physical state corresponding to this operator can exist in the theory. In particular such an operator does not allow any smooth semiclassical limit \(D \to -\infty\) in the two-dimensional gravity, where \(D\) is a central charge of ‘matter’. Therefore the operators \(\exp(2\phi/\gamma)\) responsible for an appearence of the special orbits are rather ‘non-physical’.

In the presence of the cosmological operator \(\exp \gamma \phi\) in eq.(4.5) the semiclassical approximation for the correlators of the operators \(\exp 2\phi/\gamma\) is not well defined. However we conjecture here that at the quantum level the special Virasoro orbits correspond to insertions of the ‘wrong’ sign Liouville exponential (eq.(4.16)) at \(k > 3\), while for \(2 < k < 3\) the special orbits generate the usual cosmological operator (eq.(4.17)). Assuming the validity of the conformal Ward identities [11] with insertions of the ‘wrong’ sign exponentials we get the quantum analog of 2-differential of the special Virasoro orbit given by an expression without double poles at positions of the operators

\[
<T(z) \prod_i e^{(2/\gamma)\phi(z_i)} > = \sum_{i=1}^{\gamma-3} \left( \frac{1}{(z - z_i)^2} + \frac{z_i - 1}{z} \right) \frac{\partial}{\partial z_i} \ln < \prod_i e^{(2/\gamma)\phi(z_i)} >,
\]

\[12\]
where $SL(2, \mathbb{C})$ symmetry is assumed to be fixed by fixing points $z_n$, $z_{n-1}$ and $z_{n-2}$ to be equal to 0, 1 and $\infty$. This also implies that the values of accessory parameters are fixed by the positions of inserted operators and presumably corresponds to ‘vacuum’ values of residues $q_i$ in terms of geometric quantization in previous sections. From this point of view the parameters $h_i$ in eq.(4.1) are rather the variables of integration in the path integral near the classical configuration.

It is worth noticing that the properties of the ‘unphysical’ operator $\exp(2\phi/\gamma)$ are softer here as compared to the usual Liouville theory. The point is that in the usual Liouville theory an insertion of operator with $\alpha > Q/2$ induces a curvature source for which the cosmological term is not integrable, since it gives $\int 1/|z|^3$. In the present formulation of the theory it is strictly speaking not the case because the cosmological operator is proportional to $\int A = \int \partial \alpha + \exp(-2\phi)$ (at least at the classical level). The restrictions to singularities of the fields $\phi$ and $\alpha$ are correlated so that the integral can have only logarithmical divergence as in the kinetic term.

It is interesting to sum up contributions due to arbitrary insertions of the ‘wrong’ sign exponentials to the path integral for the Liouville theory. In the absence of the cosmological operator this would correspond to a Coulomb gas. Such a model is very similar to the case of sine-Gordon model generated by contributions of vortices in the theory of real scalar field compactified to $S^1$. However in contrast to the latter model here all ‘vortices’ are of the same charge since they are generated by the same operator. Thus summing up the contributions of special Virasoro orbits the ‘wrong’ sign Liouville exponential appears in the lagrangian. Taking into account a cosmological term we get a gravitational version of the integrable sinh-Gordon model with a background charge

$$L = \sqrt{\bar{g}} \left( \frac{1}{8\pi} (\bar{\nabla} \phi)^2 + \frac{Q}{8\pi} \phi R(\bar{g}) \right) + \frac{\mu \bar{R}}{8\pi \gamma^2} \sqrt{\bar{g}} e^{\gamma \phi} + \frac{\tilde{\mu}}{8\pi \gamma^2} \sqrt{\bar{g}} e^{2\phi/\gamma}, \quad (4.19)$$

where $\tilde{\mu}$ is a parameter fixed by initial conditions. This Lagrangian is of course obtained in an approximation when the interaction between both exponentials is not taken into account. In the case $2 < k < 3$ and $\mu \bar{R} = 0$ one can see that the special orbits generate the usual cosmological operator.

At $k = 3$ the solutions (4.16) and (4.17) coincide: in this case the Virasoro central charge of the Liouville theory $c = 25$ and hence it corresponds to $c = 1$ two-dimensional gravity. It is tempting to assume that the value $k = 3$ would correspond to a phase transition in the gas of the wrong branch (at $k > 3$) vertex operators $\exp(2\phi/\gamma)$ since at this point the branch in eq.(4.4) changes.

## 5 Conclusion

It was shown that gauging Borel subgroup in the WZNW theory and taking into account singular configurations of the gauge field we naturally get the structures
assosiated with the special Virasoro coadjoint orbits. Quadratic differentials with
the single poles appear when the residues in the poles of the gauge field do not vanish.
Thus while the orbits with the stabilizers $\tilde{T}_{\pm,n}$ can be thought of as the perturbations
of orbits with $SL_n(2,R)$ ones the orbits with the singular representatives results from
the additional perturbation by the singular gauge configurations.

It is known [12] that the manifold $Diff S^1/S^1$ is relevant in the closed string
theory because of its close connection with the universal Teichmuller space of the
Riemann surfaces with genus $g > 1$. The consideration above leads to an assumption
that the parameters of the singular 2-differentials can be interpreted as the coordi-
nates on the moduli space of the spheres with the marked points. In this case the
orbits with $\tilde{T}_{\pm,n}$ stabilizers corresponding to vanishing residues are related with the
submanifold of moduli space with zero "momenta".

We can not add something to the open problem of quantization of the special
coadjoint orbits. Standard methods are not applicable here so new approaches
should be developed. But having in mind the relations with the moduli space for
the surfaces with the marked points we can conjecture that the conformal blocks for
the correlators on the sphere in the Liouville theory and the related tau functions
are relevant objects for the description of the Hilbert space in this hypothetical
quantization.

Finally we want to mention a possible connection of our results with dynamics of
Witten’s black hole which can be described as a gauged $SL(2,R)/SO(1,1)$ WZNW
model [13]. In this model there are vortices corresponding to non-trivial winding
numbers (see, e.g. ref. [8]) and we can study a problem of renormalization group
flow induced by a gas of such vortices. On the other hand the theory considered
in the present paper correspond to a factorization of $SL(2,R)$ by a Borel subgroup
and as we see the singularities corresponding to the special Virasoro orbits are
in fact generated by vortices in the group element of $SL(2,R)$. Therefore we can
conjecture that vortices in the $SL(2,R)/SO(1,1)$ model would generate the ‘non-
physical’ operators corresponding to the special Virasoro orbits.

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References