Covariant differential complexes on quantum linear groups. *

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Abstract

We consider the possible covariant external algebra structures for Cartan’s 1-forms (Ω) on $GL_q(N)$ and $SL_q(N)$. Our starting points are that Ω’s realize an adjoint representation of quantum group, and all monomials of Ω’s possess the unique ordering. For the obtained external algebras we define the differential mapping $d$ possessing the usual nilpotence condition, and the generally deformed version of Leibniz rules. The status of the known examples of $GL_q(N)$-differential calculi in the proposed classification scheme, and the problems of $SL_q(N)$-reduction are discussed.

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1 Introduction

Since Woronowicz have formulated the general scheme for the constructing of differential calculi on quantum matrix groups [1], the most publications on this theme were appealed more or less to it (see e.g. [3]-[17]). This scheme has the following structure: the first order differential calculus is defined in axiomatic way and, once it is fixed, the higher order differential calculus can be constructed uniquely. The underlying quantum group structure is taking into account by the bicovariance condition.

The principal problem of the Woronowicz’s approach, which have been mentioned already in [1], but still remains unsolved, is that the scheme possesses variety of differential calculi for each quantum group, and there is no criteria to choose the most appropriate one.

From the other hand, the $R$-matrix formalism (see [2] and references therein), initially motivated by quantum inverse scattering method, appears to be an extremely useful tool in dealing with quantum groups and essentially with differential calculus on them. So it is not surprisingly that some papers relating Woronowicz’s scheme and $R$-matrix formalism were appeared [3]-[5]. One may hopes, starting from the differential calculus on quantum hyperplane and applying $R$-matrix formulation to construct finally the most natural differential calculus on quantum group (see [6]-[9]). This program have been realized for $GL_q(N)$-case in [10]-[15], but when restricting to $SL_q(N)$ the calculus obtained reveals some unfavourable properties (see discussion in Section 5), which force us to search for other possibilities. So the classification of differential calculi on linear quantum groups remains an actual problem up to now.

In the present paper we make an attempt to approach this problem from an opposite direction, i.e. to construct firstly the higher order differential calculus. Here, the key role is played by the conditions:

a.) Cartan’s 1-forms realize the adjoint representation of $GL_q(N)$;
b.) all the higher order invariant forms, being polynomials of the Cartan’s 1-forms, can be ordered (say, lexicographically) to unique expression.

The paper is organised as follows: all the preliminary information and notations are collected in Section 2. In Section 3, developing the ideas of Ref.[14] we consider $GL_q(N)$-covariant quantum algebras (CA). Arranging them into two classes, the $q$-symmetrical (SCA) and $q$-antisymmetrical (ACA) ones, we then concentrate on the studying the homogeneous ACA’s, which could be interpreted as the external algebras of Cartan’s 1-forms. We find four one-parametric families of such algebras. Section 4 is devoted to the construction of differential complexes on homogeneous ACA’s. In doing so we allow the deformation of Leibniz rule and, thus, extend the class of the permitted complexes. We conclude the paper by the considering how the known $GL_q(N)$-differential calculi is included in this scheme and discuss the problems of the $SL_q(N)$-reduction.
2 Notations

We consider the Hopf algebra $\text{Fun}(GL_q(N))$ which is generated by the elements of the $N \times N$ matrix $T = |T_{ij}|$, $i, j = 1, \ldots, N$ obeying the following relations:

$$\text{RTT'} = \text{TT' R}.$$  

(2.1)

Here $T \equiv T_1 \equiv T \otimes I$, $T' \equiv T_2 \equiv I \otimes T$, $I$ is $N \otimes N$ identity matrix, $R \equiv \hat{R}_{12} \equiv P_{12}R_{12}$, $P_{12}$ is the permutation matrix and $R_{12}$ is $GL_q(N)$ $R$-matrix\(^1\) satisfying quantum Yang-Baxter equations and Hecke condition respectively

$$RR'R = R'RR',$$  

(2.2)

$$R^2 - \lambda R + 1 = 0,$$  

(2.3)

where $\lambda = q-q^{-1}$, $R' \equiv \hat{R}_{23} \equiv P_{23}R_{23}$ and $I$ is $N^2 \times N^2$ identity matrix. In accordance with (2.3), for $q^2 \neq -1$ the matrix $R$ decomposes as

$$R = qP^+ - q^{-1}P^-,$$

$$P^\pm = (q + q^{-1})^{-1}\{q^\mp 1 \pm R\},$$  

(2.4)

where the projectors $P^+$ and $P^-$ are quantum analogues of antisymmetrizer and symmetrizer respectively.

The comultiplication for the algebra $\text{Fun}(GL_q(N))$ is defined as $\Delta T_{ij} = T_{ik} \otimes T_{kj}$, and the antipode $S(.)$\(^2\) obeys the conditions $S(T_{ij})T_{ji} = T_{ij}S(T_{ji}) = \delta_{ij}1$, so in what follows we use the notation $T^{-1}$ instead of $S(T)$.

3 $GL_q(N)$-covariant Quantum Algebras.

Consider the $N^2$-dimensional adjoint $\text{Fun}(GL_q(N))$-comodule $\mathcal{A}$. We arrange it's basic elements into $N \times N$ matrix $A = |A_{ij}|$, $i, j = 1, \ldots, N$. The adjoint coaction is

$$A^i_j \rightarrow T^i_kS(T)^j'_k \otimes A^i_j \equiv (TAT^{-1})^i_j,$$  

(3.1)

where in the last part of the formula (3.1) the standard notation is introduced to be used below.

The comodule $\mathcal{A}$ is reducible, and the irreducible subspaces in $\mathcal{A}$ can be extracted by use of the so called quantum trace ($q$-trace) \cite{2,19} (see also \cite{5,13,20}). In the case of $\text{Fun}(GL_q(N))$ it has the form:

$$\text{Tr}_qA \equiv \text{Tr}(DA) \equiv \sum_{i=1}^{N} q^{-N-1+2i}A^i_i,$$  

(3.2)

\[^1\text{For the explicit form of } GL_q(N) R \text{-matrix see Refs.} [18, 2] .\]

\[^2\text{Strictly speaking in order to define the antipodal mapping on } \text{Fun}(GL_q(N)) \text{ we must add one more generator } (det_q T)^{-1} \text{ to the initial set } \{T_{ij}\} \text{ (see } [2]).\]
and possess the following invariance property:

\[ Tr_q(TAT^{-1}) = Tr_q(A) , \]

i.e. \( Tr_q(A) \) is the scalar part of the comodule \( \mathcal{A} \), while the \( q \)-traceless part of \( A \) forms the basis of \( (N^2 - 1) \)-dimensional irreducible \( Fun(GL_q(N)) \)-adjoint comodule. Let us note also the following helpful formulas:

\[ Tr_{q(2)}(RAR^{-1}) = Tr_{q(2)}(R^{-1}AR) = Tr_q A I_{(1)} , \]

\[ Tr_{q(2)} R^\pm = q^{\pm N} I_{(1)}, \quad Tr_q I = [N]_q , \]

where \( A \equiv A_1 \equiv A \otimes I, \ [N]_q = \frac{q^N - q^{-N}}{q^2 - 1} \), and by \( X_{(i)} \) we denote quantities (operators) \( X \) living (acting) in the \( i \)-th space.

Consider now the associative unital \( \mathbb{C} \)-algebra \( C < A_{ij} > \) freely generated by the basic elements of \( \mathcal{A} \). As a vector space \( C < A_{ij} > \) naturally carries the \( Fun(GL_q(N)) \)-covariant quantum algebra (CA) as the factoralgebra of \( C < A_{ij} > \), possessing the following properties [14]:

(A) The multiplication in this algebra is defined by a set \( \{ \alpha \} \) of quadratic in \( A_{ij} \) polynomial identities:

\[ C_{ijkl}^\alpha A_{ij} A_{kl} = C_{ij}^\alpha A_{ij} + C^\alpha . \quad (3.3) \]

In other words, CA is the factor algebra of \( C < A_{ij} > \) by the biideal generated by (3.3).

(B) Considered as a vector space CA is a \( GL_q(N) \)-adjoint comodule, so the coefficients \( C_{ijkl}^\alpha \) in (3.3) are \( q \)-analogues of the Clebsch-Gordon coefficients coupling two adjoint representations, and the set of the relations (3.3) is divided into several subsets corresponding to different irreducible \( Fun(GL_q(N)) \)-comodules in \( \mathcal{A} \otimes \mathcal{A} \). Parameters \( C_{ij}^\alpha \) are not equal to zero when \( C_{ijkl}^\alpha \) couple \( \mathcal{A} \otimes \mathcal{A} \) into the adjoint \( GL_q(N) \)-comodule again, while \( C^\alpha \neq 0 \) only if \( C_{ijkl}^\alpha A_{ij} A_{kl} \) are the scalars.

(C) All the monomials in CA can be ordered lexicographically due to (3.3).

(D) All the nonvanishing ordered monomials in CA are linearly independent and form the basis in CA.

Now we recall that for the classical case \( (q = 1) \) the dimensions of the irreducible \( Fun(GL(N)) \)-subcomodules in \( \mathcal{A} \otimes \mathcal{A} \) are given by Weyl formula [21]:

\[ \dim \mathcal{A} \otimes \mathcal{A} = \left[ (N^2 - 1) + 1 \right]^2 = 2 \cdot [1] \oplus (3 + \theta_{N,2}) \cdot \left[ N^2 - 1 \right] \oplus 2\theta_{N,2} \cdot \left[ \frac{(N^2 - 1)(N^2 - 4)}{4} \right] \oplus \left[ \frac{N^2(N + 3)(N - 1)}{4} \right] \oplus \theta_{N,3} \cdot \left[ \frac{N^2(N + 1)(N - 3)}{4} \right] , \quad (3.4) \]

where \( \theta_{N,M} = \{ 1 \text{ for } N > M; 0 \text{ for } N \leq M \} \). Thus, \( \mathcal{A} \otimes \mathcal{A} \) splits into 2 scalar subcomodules, 4 (3 for \( N = 2 \)) adjoint (traceless) subcomodules and 4 (1 for \( N = 2 \) and 3 for \( N = 3 \)) higher-dimensional mutually inequivalent subcomodules. In quantum case according to the results of Ref’s., [22] the situation generally is not changed (the exception is for \( q \) being root of unity). Below we employ the \( q \)-(anti)symmetrization
projectors $P^\pm$ and $q$-trace to extract the irreducible submodules in $\mathcal{A} \otimes \mathcal{A}$, therefore supposing from the initial that $q \neq -1$ and $Tr_q I = [N]_q \neq 0$.

First, we shall obtain the sets of quadratic in $A_{ij}$ combinations, that correspond to the left hand side of (3.3) and contain four higher dimensional $Fun(\text{GL}_q(N))$-submodules (see (3.4)). Let us start with $N^2 \times N^2$ matrix $\text{ARA}$ containing all the $N^4$ independent quadratic in $A_{ij}$ combinations and having convenient comodule transformation properties:

$$\text{ARA} \rightarrow (TT')\text{ARA}(TT')^{-1}.$$  \hspace{1cm} (3.5)

From (2.1), (2.4) it follows that $P^\pm TT' = TT'P^\pm$, hence we can split $\text{ARA}$ into four independently transforming (for $N \geq 3$) parts:

$$X^{\pm\pm} \equiv P^\pm \text{ARA} P^\pm, \quad X^{\pm\mp} \equiv P^\pm \text{ARA} P^{\mp}.$$  \hspace{1cm} (3.6)

Namely the $q$-traceless (in both 1st- and 2nd- spaces) parts of $X^{++}$, $X^{--}$ and $X^{\pm\mp}$ are the four higher dimensional submodules in $\mathcal{A} \otimes \mathcal{A}$ with dimensions: $\frac{N^4}{4}$, $\frac{N^2(N-2)(N+1)}{4}$ and $\frac{N^2(N-1)(N^2-4)}{4}$ respectively.

Now acting on $X$’s by $Tr_q$-operation we obtain (for $N \neq 2$ and $q$ not being a root of unity) four independent bilinear in $A_{ij}$ combinations transforming as adjoints:

$$A^2, \quad (Tr_q A)A, \quad A(Tr_q A), \quad A \ast A \equiv Tr_q [R^{-1}\text{ARA} R^{-1}] .$$  \hspace{1cm} (3.7)

The $q$-traceless parts of these combinations correspond to the irreducible adjoint subcomodules in $\mathcal{A} \otimes \mathcal{A}$. Applying $Tr_q$ to the Eqs.(3.7) once again we result in two independent expressions

$$\left(Tr_q A^2\right), \quad Tr_q (A^2),$$  \hspace{1cm} (3.8)

corresponding to the scalar submodules. We refer to the expressions (3.6), (3.7) and (3.8) as higher-dimensional, adjoint and scalar terms respectively.

As it was argued in [14], in order to satisfy the condition (C) for CA, the left hand side of the relations (3.3) must contain independently either $X^{++}$ with $X^{--}$, or $X^{+-}$ with $X^{-+}$. One can combine these pairs into single expressions:

$$(q + q^{-1})(X^{++} - X^{--}) = \text{RARA} + \text{ARAR}^{-1},$$  \hspace{1cm} (3.9)

$$(q + q^{-1})(X^{+-} - X^{-+}) = \text{RARA} - \text{ARAR}.$$  \hspace{1cm} (3.10)

The way of the combining the quantities (3.6) is not important. We choose the concise forms (3.9), (3.10) because in the classical limit they are nothing but the anticommutator $[A_2, A_1]_+$ and commutator $[A_2, A_1]_-$. So it is natural to call (3.9) and (3.10) $q$-anticommutator and $q$-commutator respectively. In view of this all the CA’s with the defining relation (3.3) are classified into two types depending on whether their defining relations contain $q$-anticommutator or $q$-commutator. The first will be called further \textit{antisymmetric} CA (ACA) and the last – \textit{symmetric} CA (SCA).
At the moment we still fix the higher-dimensional terms in a quadratic part of the
relations (3.3), but there remains an uncertainty in the choice of the adjoint and the
scalar terms. Let us show it explicitly. First of all we employ the simple dimensional
arguments. In order to satisfy the ordering condition (C) at a quadratic level we must
include at least \( \frac{N^2(N^2-1)}{2} \) independent relations in (3.3) (e.g. for the classical case
of \( gl(N) \) it corresponds to the number of the commutators \([A_{ij}, A_{kl}]\)). Since the h-
dimensional terms (3.10) for SCA contain \( \frac{(N^2-1)(N^2-2)}{2} \) independent combinations, we
must add to them at least \( 2 \cdot (N^2 - 1) \) independent combinations, i.e. two \( q \)-traceless
adjoint terms. Actually estimation is precise: including any other additional adjoint or
scalar terms in (3.3) would result in a linear dependence of quadratic ordered monomials
and, thus, contradict with (D). As regards the ACA’s, \( ^2 \) from the \( \frac{N^2(N^2-2)}{2} \) independent
combinations, contained in h-dimensional terms (3.9), the \( N^2 \) combinations lead to the
relations (3.3) of the type: \( A^2_{ij} = 0 \) \( (i \neq j) \), \( A^2_{ii} = \sum_{kl} f^{kl}_{i} A_{kl} \) (where \( f^{kl}_{i} \) are some
constants), i.e. they are useless in ordering procedure. Hence we have the deficit of the
2\( N^2 \) independent quadratic combinations in \( N \geq 3 \) case (5 combinations for \( N = 2 \))
and are forced to include in (3.3) \( 2(1 \) for \( N = 2 \)) independent \( q \)-traceless adjoint terms
and a pair of scalar terms. With this inclusion ACA’s are defined by the set of \( \frac{N^2(N^2-3)}{2} \)
relations.

Thus, we have determined the number of independent adjoint and scalar terms in
symmetric and antisymmetric CA’s. Note that \( q \)-commutator and \( q \)-anticommutator
itself contain the true number of adjoints and scalars, which is demonstrated by the
following symmetry properties:

\[
P^{\pm}\{\text{RARA} - \text{ARAR}\}P^{\pm} = 0 , \quad (3.11)
\]

\[
P^{\pm}\{\text{RARA} + \text{ARAR}^{-1}\}P^{\mp} = 0 . \quad (3.12)
\]

But there is an opportunity to change the form of quadratic adjoint terms in the left
hand side of Eq.(3.3) without changing of their number. Indeed, consider the quantities

\[
\Delta_{\pm}(U_{ad}(A)) = RU_{ad}(A)R^{\pm1} \pm U_{ad}(A) , \quad (3.13)
\]

\[
U_{ad}(A) = u^1(R) \cdot A^2 + (u^2 - \epsilon(R)) \cdot (Tr_q A) A + (u^3 - \epsilon(R)) \cdot A(Tr_q A) + u^4(R) \cdot (A + A), \quad (3.14)
\]

where \( u^a(R) = u^1_a + u^2_a R, a = 1, 2, 3, 4, \) and \( \epsilon(R) = \frac{1}{[N_q]}(u^1(R) + q^{-N}u^4(R) - 1) \). We
make the \( \epsilon(R) \)-shift of the parameters \( u^a(R) \) and \( u^5(R) \) for the sake of future
convenience. Expressions \( \Delta_{\pm} \) are the most general covariant combinations which contain
only adjoint and scalar (for \( \Delta_+ \)) terms and satisfy symmetry properties

\[
P^{\pm}\Delta_{-}P^{\pm} = P^{\pm}\Delta_{+}P^{\mp} = 0 . \quad (3.15)
\]

Therefore we may use \( \Delta_+ \) and \( \Delta_- \) in varying of the quadratic part of the defining
relations (3.3) for ACA’s and SCA’s, respectively. Note, that in principal one could
add to the r.h.s. of (3.14) the scalar combination \( U_{\infty} = h(R)Tr_q(A^2) + g(R)(Tr_q A)^2 \),
where \( h \) and \( g \) are arbitrary functions of \( R \). This addition, obviously, does not affect \( \Delta_- \). As concerns \( \Delta_+ \), remember that defining relations for ACA must contain a pair of independent quadratic scalars represented in general as:

\[
Tr_q(A^2) = C_1 Tr_q(A) + C_2 , \quad (Tr_q A)^2 = C_3 Tr_q(A) + C_4 .
\]  
(3.16)

Here \( C_i \) are some constants. Thus, even changing the form of \( \Delta_+ \), the term \( U_{sc} \) can not change the content of the bilinear part of defining relations for ACA and we will omit this term in further considerations.

From until now we shall concentrate on studying the homogeneous (pure quadratic) ACA’s, which possess the natural \( Z_2 \)-grading and may be interpreted as an external algebras of the invariant forms on \( GL_q(N) \). To emphasize this step we change notations \( \dot{\omega} \) from \( A \) to \( \Omega \). All the other cases can be considered following the same lines.

As we have shown, the general defining relations for homogeneous ACA looks like

\[
R \Omega R \Omega + \Omega R \Omega R^{-1} = \Delta_+ .
\]  
(3.17)

These relations contain 8 random parameters \( u^a_i \), \( (a=1,2,3,4; \ i=1,2) \), but actually this parametrization of the whole variety of homogeneous ACA’s is redundant. To minimize the number of parameters in (3.17), let us pass to the new set of generators:

\[
\Omega \rightarrow \begin{cases} 
\omega = & Tr_q \Omega , \\
\hat{\Omega} = & \Omega - \frac{\omega}{N_q^4} I , \quad Tr_q \hat{\Omega} = 0 .
\end{cases}
\]  
(3.18)

Using these new variables one can extract the first scalar relation \( \omega^2 = 0 \) and (3.17) is changed slightly to

\[
R \hat{\Omega} R \hat{\Omega} + \hat{\Omega} R \hat{\Omega} R^{-1} = \Delta_+(U(\hat{\Omega})) ,
\]  
(3.19)

\[
\omega^2 = 0 ,
\]  
(3.20)

where \( \Delta_+(U) = RUR + U \) and

\[
U(\hat{\Omega}) = u^1(R) \hat{\Omega}^2 + u^2(R) \omega \hat{\Omega} + u^3(R) \hat{\Omega} \omega + u^4(R)(\hat{\Omega} \ast \hat{\Omega}) .
\]  
(3.21)

Here, as usual, \( \hat{\Omega} \equiv \hat{\Omega}_1 \equiv \hat{\Omega} \otimes I \).

Applying the operations \( Tr_{q(2)}[...] \), \( Tr_{q(2)}[R^{-1}...] \), and then \( Tr_{q(1)}[...] \) to Eq.(3.19) we extract adjoint relations and then obtain the second scalar relation

\[
Tr_q(\hat{\Omega}^2) = q^{-N} Tr_q(\hat{\Omega} \ast \hat{\Omega}) = 0 ,
\]  
(3.22)

The adjoint relations are represented in the form:

\[
v^1(R) \hat{\Omega}^2 + v^2(R) \omega \hat{\Omega} + v^3(R) \hat{\Omega} \omega + v^4(R)(\hat{\Omega} \ast \hat{\Omega}) = 0 ,
\]  
(3.23)

where

\[
v^a(R) = v^a_0 + v^a_1 R = x(R) u^a(R) - \delta_{a,1} q^N R^2 - \delta_{a,4} 1 ,
\]  
(3.24)

\[
x(R) = x_0 + x_1 R = (q^N + q^{-N}) + ([N]_q + \lambda q^N) R .
\]  
(3.25)
Here we arrange the pair of adjoints into a single matrix relation. Expanding (3.23) in a power series of \( \mathbf{R} \) one can obtain both the adjoint relations explicitly.

Now we can reduce the number of coefficients parametrizing ACA’s. Namely, we use Eqs. (3.23) to represent some pair of adjoint terms (3.7) as linear combinations of the other two adjoints. Let us denote \( 2 \times 2 \) minors of the system (3.23) as

\[
\gamma^{ab} = \det \begin{bmatrix} v_1^a & v_1^b \\ v_2^a & v_2^b \end{bmatrix}.
\]

Note, if \( \gamma^{34} = \gamma^{24} = 0 \), then we get from (3.23) that \( \hat{\Omega}^2 \) must be proportional to either \( \omega^3 \hat{\Omega} \), or \( \hat{\Omega} \omega \), which contradicts with condition (D). Hence, there are only two variants of solving (3.23) with respect to either \( \hat{\Omega} \ast \hat{\Omega} \) and \( \hat{\Omega} \omega \) (if \( \gamma^{34} \neq 0 \)), or \( \hat{\Omega} \ast \hat{\Omega} \) and \( \omega \hat{\Omega} \) (if \( \gamma^{24} \neq 0 \)). Both choises are quite natural since, first, we exclude the cumbersome expression \( \hat{\Omega} \ast \hat{\Omega} \) from further considerations and, second, we fix the order of quantities \( \omega \) and \( \hat{\Omega} \) in their monomials (turning \( \omega \), respectively, to the left, or to the right). In fact, as we shall see further (see remark 3 to Theorem 1), both these variants are equivalent and conditions \( \gamma^{34} \neq 0, \gamma^{24} \neq 0 \) are necessary in obtaining consistent ACA’s. So, we suppose from the initial that both \( \gamma^{34} \) and \( \gamma^{24} \) are not equal to zero, and choose solving (3.23) w.r.t. \( \hat{\Omega} \ast \hat{\Omega} \) and \( \hat{\Omega} \omega \). The result is

\[
\hat{\Omega} \ast \hat{\Omega} = \delta \hat{\Omega}^2 + \tau \omega \hat{\Omega}, \quad \hat{\Omega} \omega = -\rho \omega \hat{\Omega} + \sigma \hat{\Omega}^2,
\]

where

\[
\delta = \frac{\gamma^{13}}{\gamma^{34}}, \quad \tau = \frac{\gamma^{23}}{\gamma^{34}}, \quad \rho = \frac{\gamma^{24}}{\gamma^{34}} \neq 0, \quad \sigma = \frac{\gamma^{14}}{\gamma^{34}}
\]

are namely that minimal set of parameters which we have search for. In this parametrization the defining relations for ACA looks like

\[
\mathbf{R} \hat{\Omega} \mathbf{R} + \hat{\Omega} R \hat{\Omega} \mathbf{R}^{-1} = \tilde{x}(\mathbf{R}) \left\{ (\delta + q^N \mathbf{R}^2)(R \hat{\Omega}^2 \mathbf{R} + \hat{\Omega}) + \tau \omega (R \hat{\Omega} \mathbf{R} + \hat{\Omega}) \right\},
\]

\[
\hat{\Omega} \omega = -\rho \omega \hat{\Omega} + \sigma \hat{\Omega}^2,
\]

\[
\omega^2 = 0,
\]

where

\[
\tilde{x}(\mathbf{R}) \equiv \{x(\mathbf{R})\}^{-1} = \frac{1}{[N+2][N-2]q} \{-x_2 + x_1 \mathbf{R}\},
\]

\[
x_2 \equiv x_0 + \lambda x_1 = q^N (q^{-2} + q^2).
\]

To get the relations (3.28) we solve (3.24) with respect to \( w^a(\mathbf{R}) \), substitute the resulting expressions in (3.21), (3.19), and then use (3.23) and the first of Eq’s. (3.27). The systems of relations (3.28-3.30) and (3.19-3.21) are equivalent if the matrix \( x(\mathbf{R}) \) is invertible (i.e. if \( [N+2][N-2]q \neq 0 \), or, equivalently, \( [N]_q \neq \pm [2]_q \), or \( q^{2N+4} \neq 1 \)).

\[3\]C.f. with the remark in the brackets over Eq.(3.7).
Further we shall consider this nonsingular case. The case \( N = 2 \) will be treated in detail in the next Section.

Now let us discuss the symmetry properties of Eq. (3.28). Consider the following transformation

\[
\begin{align*}
q & \rightarrow q^{-1}, \quad \text{hence,} \quad R_q \rightarrow R_q^{-1}, \quad x_q(\cdot) \rightarrow x_q^{-1}(\cdot); \\
\Omega & \rightarrow \Omega' \equiv \Omega_2 \equiv I \otimes \Omega. \quad (3.32)
\end{align*}
\]

Here in the lower indices we type the values of quantization parameter for the considered quantities. Note, that using the symmetry property of \( GL_q(N) \) \( R \)-matrix:

\[
R_q = P_{12} R_q^{-1} P_{12}
\]

one can find that (3.32) is a product of two symmetries: the involution transformation of the operators \( B \rightarrow P_{12} B P_{12} \) and discrete symmetry

\[
\begin{align*}
q & \rightarrow q^{-1}, \quad x_q(\cdot) \rightarrow x_q^{-1}(\cdot), \quad \text{but} \quad R_q \rightarrow R_q^{-1}; \\
\Omega & \rightarrow \text{remains unchanged}. \quad (3.33)
\end{align*}
\]

It must be stressed that the replacement \( q \rightarrow q^{-1} \) doesn’t concerns the definition of \( \omega \), i.e. of the quantum trace. Otherwise, we would obtain an algebra with the different covariance properties, namely, the algebra of left-invariant (w.r.t. transitions in underlying quantum group \( GL_q(N) \)) objects.

Now using the identity

\[
x_q(R_q) R_q^{-1} = \frac{q^N R_q^2 - q^{-N} R_q^{-2}}{\lambda}
\]

we deduce the following properties of the matrix function \( \tilde{x}(R) \):

\[
\begin{align*}
\tilde{x}_q(R_q) &= R_q^{-2} \tilde{x}_q^{-1}(R_q) , \\
q^N R_q^2 \tilde{x}_q(R_q) &= q^{-N} R_q^{-2} \tilde{x}_q(R_q) + \lambda R_q^{-1}.
\end{align*}
\]

and, then, it is no hard to check that Eq. (3.28) is invariant under the substitution (3.33) and, therefore, under (3.32). In the classical limit \( q = 1 \) this transformation reduces to the identical, but in quantum case we get the discrete \( Z_2 \)-group of symmetries of Eq. (3.28). Namely this symmetry produce the doubling of differential calculi on \( GL_q(N) \) which was observed by many authors ( see e.g. [13]).

Thus, the most general form for the algebras, which admit an ordering for any quadratic monomial in their generators, is (3.28)-(3.30). The next step in finding out the consistent defining relations for homogeneous ACA’s is to consider the ordering of cubic polynomials. Let us present the result of our considerations in
Theorem 1: For a general values of quantization parameter $q$ there exist four one-parametric families of homogeneous ACA’s. The defining relations for the first pair of them looks like

\[
\begin{aligned}
\{ \mathbf{R} \dot{\mathbf{\Omega}} \mathbf{R} + \dot{\mathbf{\Omega}} \mathbf{R} \dot{\mathbf{\Omega}}^{-1} = \kappa_q \left( \dot{\mathbf{\Omega}}^2 + \mathbf{R} \dot{\mathbf{\Omega}}^2 \mathbf{R} \right) , \\
\omega^2 = 0 ,
\end{aligned}
\]

and

\[
\begin{aligned}
type I : \quad \dot{\mathbf{\Omega}} \omega &= -\rho \omega \dot{\mathbf{\Omega}} , \quad \rho \neq 0 ; \\
type II : \quad [\dot{\mathbf{\Omega}}, \omega]_+ &= \sigma \dot{\mathbf{\Omega}}^2 , \quad \sigma \neq 0 .
\end{aligned}
\]

Here $\kappa_q = \frac{\lambda^N}{Nq^3} \mathbf{R}$, and $q \neq -1$, $[N]_q \neq \{0, -\lambda q^N, -\lambda [2]_qq^{N+1}, \pm [2]_q \}$. For both cases the following remarkable relation holds:

\[
\mathbf{R} \dot{\mathbf{\Omega}}^2 \mathbf{R} - \dot{\mathbf{\Omega}} \mathbf{R} \dot{\mathbf{\Omega}}^2 \mathbf{R} = 0 .
\]

The resting pair of families can be obtained from the first one by the involution (3.33) or (3.33).

Finally in the classical limit ($q = 1$) there exists one more family of homogeneous ACA’s:

\[
\begin{aligned}
[\dot{\Omega}_1, \dot{\Omega}_2]_+ &= \tau' \left( P_{12} - \frac{2}{N} \right) \omega \left( \dot{\Omega}_1 + \dot{\Omega}_2 \right) , \\
[\dot{\Omega}_i, \omega]_+ &= 0 , \\
\omega^2 &= 0 .
\end{aligned}
\]

where $\tau' = \frac{\tau N}{2q} \neq 0$ (see Eq’s. (3.27), (3.28)).

**Proof:** We shall prove the Theorem for type I and II algebras. The results for the second pair of algebras are obviously obtained by applying transformation (3.33) to all the formulae below.

To check the ordering at a cubic level it is enough to consider two monomials: $(\mathbf{R} \dot{\mathbf{\Omega}})^2 \omega$ and $(\mathbf{R} \mathbf{R} \dot{\mathbf{\Omega}})^3$. In the classical limit these combinations become $\dot{\Omega}_2 \dot{\Omega}_1 \omega$ and $\dot{\Omega}_3 \dot{\Omega}_2 \dot{\Omega}_1$, respectively, and for the ordinary external algebra of invariant forms on $GL(N)$ the procedure of their ordering looks like $\dot{\Omega}_2 \dot{\Omega}_1 \omega \rightarrow \omega \dot{\Omega}_1 \dot{\Omega}_2$ and $\dot{\Omega}_3 \dot{\Omega}_2 \dot{\Omega}_1 \rightarrow \dot{\Omega}_1 \dot{\Omega}_2 \dot{\Omega}_3$.

Before establishing the quantum analog of this procedure we have to choose the basis of ”ordered” cubic monomials. Here the notion ”ordered” is given in quotation-marks since we can’t achieve true lexicographic ordering of monomials without loosing the compact matrix form of our considerations and passing to cumbersome calculations in $\dot{\Omega}_i$-components. Such an in-component calculations, based on the use of Diamond Lemma (see [23]), were carried out for the case $N = 2$ in Ref’s. [9, 10, 16] and it seems doubtful that they could be repeated for general $N$. So, we use the basis of quasi-ordered cubic combinations, which are convenient in our matrix manipulations. Following this way we can not prove that we have exhausted all the possible types of
ACA’s, but the algebras obtained are shurely satisfy all the conditions for ACA and our conjecture is that the theorem 1 gives the all possible ACA’s.

Let us define some new symbols:

\[(A \circ B)_{12} = ARBR^{-1}, \quad (A \circ B)_{13} = R'(A \circ B)_{12}R', \quad (A \circ B)_{23} = RR'(A \circ B)_{12}R'R,
\]
\[(A)_1 = A, \quad (A)_2 = RAR, \quad (A)_3 = R'RARR'.\]

Here, as usual, \(B \equiv B_1 \equiv B \otimes I\). The lower indices in these notations are originated from the analogy with the classical case \((q = 1)\), where \((A \circ B)_{12} = A_1 B_2, \ (A \circ B)_{13} = A_1 B_3\) etc.

We choose the following basic set of cubic matrix combinations:

\[
\begin{align*}
(\hat{\Omega}^2 \circ \hat{\Omega})_{ij}, \quad (\hat{\Omega} \circ \hat{\Omega}^2)_{ij}, \quad \omega(\hat{\Omega} \circ \hat{\Omega})_{ij}, \quad (\hat{\Omega}^3)_{i}, \quad \omega(\hat{\Omega}^2)_{i}, \quad (3.39)
\end{align*}
\]

where \(i < j\) and \(i, j = 1, 2, 3\). We also imply that these basic combinations can be multiplied from the left by any matrix function \(f(R, R')\), but expressions produced from the combinations (3.39) by multiplication from the right are to be ordered yet.

Now in quantum case we order monomials \((R\hat{\Omega})^2\omega\) and \((R'R\hat{\Omega})^3\) in a following way:

\[
\begin{align*}
\begin{cases}
(R\hat{\Omega})^2\omega & \rightarrow -\omega R\hat{\Omega}R\hat{\Omega}^{-1} + ... \\
(R'R\hat{\Omega})^3 & \rightarrow -\hat{\Omega}RR'R\hat{\Omega}^{-1}\hat{\Omega}R^{-1}\hat{\Omega}^{-1} + ...
\end{cases}
\end{align*}
\]  \hspace{1cm} (3.40)

where by dots we denote some additional terms which are to be expressed in terms of the basic combinations (3.39). The point is that such an ordering can be performed by two different ways, depending on whether we first permute the left part of the generators, or the right one. According to the condition (D) both results must be identical, i.e. the additional terms in (3.40) calculated in two ways must coincide, otherwise the ordered cubic monomials would not be linearly independent. Checking this condition for the combination \((R\hat{\Omega})^2\omega\) we get the following relation:

\[
\begin{align*}
\sigma \left[ (\hat{\Omega} \circ \hat{\Omega}^2)_{12} + (\hat{\Omega} \circ \hat{\Omega}^2)_{21} - \rho((\hat{\Omega}^2 \circ \hat{\Omega})_{12} + (\hat{\Omega}^2 \circ \hat{\Omega})_{21}) \right] = \\
\sigma \hat{x}(R) \left[ (1 - \rho)(\delta + q^N R^2)((\hat{\Omega}^3)_1 + (\hat{\Omega}^3)_2) + \tau \omega((\hat{\Omega}^2)_1 + (\hat{\Omega}^2)_2) \right],
\end{align*}
\]  \hspace{1cm} (3.41)

where

\[
(\hat{\Omega}^2 \circ \hat{\Omega})_{21} \equiv R\hat{\Omega}^2 R\hat{\Omega}, \quad (\hat{\Omega} \circ \hat{\Omega}^2)_{21} \equiv R\hat{\Omega}R\hat{\Omega}^2 \quad (3.42)
\]

are the combinations to be expressed in terms of the basic ones (3.39). In doing so one can start with the relation

\[
R\hat{\Omega}^2 R\hat{\Omega} - \hat{\Omega}R\hat{\Omega}^2 R = R(\hat{\Omega}\Delta + \Delta\hat{\Omega})R
\]  \hspace{1cm} (3.43)

which directly follows from (3.28). Here \(\Delta\) is the shorthand notation for the r.h.s. of (3.28). Omitting the straightforward but rather tedious calculations we present the ’ordered’ expressions for \((\hat{\Omega}^2 \circ \hat{\Omega})_{21}\) and \((\hat{\Omega} \circ \hat{\Omega}^2)_{21}\) in the Appendix (see (A.4)-(A.6)). Substituting (A.4),(A.5) in (3.41) and considering carefully the conditions for vanishing
consequently $\omega(\hat{\Omega} \circ \hat{\Omega})_{12}$, $(\hat{\Omega} \circ \hat{\Omega}^2)_{12}$, $(\hat{\Omega}^2 \circ \hat{\Omega})_{12}$, $\omega(\hat{\Omega}^2)_{1,2}$ and $(\hat{\Omega}^3)_{1,2}$ -terms there we conclude that (3.41) is satisfied iff
\begin{align*}
a.) & \quad \sigma \quad = \quad 0 \; ; \\
b.) & \quad \sigma \quad \neq \quad 0 \quad \text{and} \quad \tau \quad = \quad 0 , \quad \rho \quad = \quad 1 \; . \quad (3.44)
\end{align*}

Now we repeat these considerations for $(R'^2 \hat{\Omega})^3$. Performing the ordering of this expression in two different ways we obtain the following condition
\begin{align*}
\Delta_+ R'^2 \hat{\Omega}^2 R'^{-1} & = \quad R'^2 \hat{\Omega}^2 R'^{-1} + R'^2 \hat{\Omega}^2 R'^{-1} - R \hat{\Omega} R' \Delta_+ R'^{-1} - \hat{\Omega} R' \Delta_+ R'^{-1} = 0 \; . \quad (3.45)
\end{align*}

Considering $(\hat{\Omega}^3)_{1,2,3}$-terms in decomposition of (3.45) over the basic set (3.39) (here the formulae (A.4),(A.5) are to be used) we get the condition on the parameter $\delta$
\begin{align*}
\delta = - \frac{q^N [N]_q - \lambda}{[N]_q + \lambda q^N} \quad \Leftrightarrow \quad \pi(R)(\delta + q^N R^2) = \frac{\lambda q^N}{[N]_q + \lambda q^N} 1 = \kappa_q 1 \; , \quad (3.46)
\end{align*}

where $[N]_q \neq -\lambda q^N$ is implied. The further restrictions on the possible values of quantization parameter $q$ are follows from the condition of invertibility of the matrix $E(R)$ (A.6), the inverse power of which enters through the formulae (A.4),(A.5) all our calculations. These restrictions are
\begin{equation*}
\kappa_q \neq 1 \; , \quad 1 + \kappa_q R^2 \notin P^\pm \quad \Leftrightarrow \quad [N]_q \neq 0 \; , \quad [N]_q \neq -\lambda [2] q^{N\pm 1} \; .
\end{equation*}

And finally, analizing the condition (3.45) for $\omega(\hat{\Omega} \circ \hat{\Omega})_{12,13,23}$-terms we obtain further restrictions on parameters for case a.) (3.44):
\begin{align*}
a1.) & \quad \sigma \quad = \quad 0 \quad \text{and} \quad \tau \quad = \quad 0 \; ; \\
a2.) & \quad \sigma \quad = \quad 0 \quad \text{and} \quad \tau \quad \neq \quad 0 , \quad \rho \quad = \quad 1 , \lambda \quad = \quad 0 \; .
\end{align*}

Checking the rest terms of Eq. (3.45) doesn’t lead to the further restrictions.

Thus, we prove the ordering conditions for cubic monomials for the algebras (3.31)-(3.36), (3.38). To conclude the proof of the Theorem we note that if the ordering condition is checked at a cubic level, then in accordance with Manin’s general remark [7], it automatically follows for all the higher power monomials. Finally, the relation (3.37) follows directly from (A.4), (A.5) under the restrictions on $\rho$, $\tau$, $\sigma$, $\delta$, that were obtained. Q.E.D.

In conclusion of the Section we make few remarks to the Theorem:

1. The parameters $\sigma \neq 0$ for type II algebra and $\tau \neq 0$ for the nonstandard classical algebra are inessential. They can be removed from the defining relations by simple rescalings of generators $\omega$ or $\hat{\Omega}$. 

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2. Note, that reproducing $q$ from the covariant relations (3.34)-(3.36) and (3.38) the explicit formulas for certain ordering prescriptions, one may obtain some additional limitations on the values of the parameters $\rho, \sigma, \tau$ (e.g. see below the $N = 2$ case).

3. One can directly check the requirements (3.44) assuming the following natural condition:

$$Tr_q(\hat{\Omega}^3) \neq 0$$

(3.47)

(this is true, e.g., for the classical case $q = 1$). Then, Eqs.(3.27) lead to the relations

$$[\hat{\Omega}, \hat{\Omega} + \hat{\Omega}] = \tau \left( \sigma \hat{\Omega}^3 - (1 + \rho)\omega \hat{\Omega}^2 \right),$$

$$\hat{\Omega}^2 \omega - \rho^2 \omega \hat{\Omega}^2 = \sigma (\rho - 1) \hat{\Omega}^3.$$ (3.48)

Applying to them the operation $Tr_q(.)$ and using (3.47), (3.22) we deduce

$$\sigma (\rho - 1) = 0 = \tau \sigma$$

which is equivalent to (3.44).

4. Finally, we present the defining relations for homogeneous ACA’s in terms of $\Omega$’s (see (3.18))

$$R \Omega R \Omega + \Omega R \Omega R^{-1} = \kappa_q \left( R \Omega^2 R + \Omega^2 \right) + \begin{cases} \text{type I:} & \frac{(1 - \rho)(1 - \rho)}{N^2} \omega \left( R \Omega R + \Omega \right) \\ \text{type II:} & \frac{(1 - \rho) \omega}{N \rho + \sigma} \left( R \Omega^2 R + \Omega^2 \right) \end{cases}$$

(3.49)

It should be mentioned that the condition $\rho \neq 0$ appears to be important just here, since the relations (3.49) for type I algebra contain both scalar terms only under this restriction.

4 Differential Complexes of Invariant Forms.

As we argued before, among the algebras presented in Theorem 1 there exists the true algebra (or maybe the set of such algebras) of invariant differential forms on $GL_q(N)$. To make the connection with the differential calculi on quantum groups more clear we shall supply the homogeneous ACA’s listed in the Theorem 1 with a grade-1 nilpotent operator $d$ of external derivation. The definition of $d$ must respects the covariance properties (3.1) of Cartan 1-forms, i.e. $d$ must commute with the adjoint $GL_q(N)$-coaction on $\Omega$. Hence, the following general anzats is allowed:

$$\begin{cases} d \cdot \hat{\Omega} = x \hat{\Omega}^2 + y \omega \hat{\Omega} - z \hat{\Omega} \cdot d \\ d \cdot \omega = -\tau \omega \cdot d \end{cases}$$

(4.1)
Here $x$, $y$, $z$ and $t$ are some parameters to be fixed below. We stress that the last term in the right hand side of (4.1) defines the deformed version of Leibniz rules for differential forms. The ordinary Leibniz rules are restored under the limit $z = t = 1$. Note, that for the differential calculi on the quantum hyperplane the deformed version of the Leibniz rules have been considered in [24].

Now it is straightforward to obtain

**Theorem 2:** Under the restrictions of Theorem 1 there exists two distinct covariant differential complexes for type I algebras, defined by

$$\begin{align*}
\text{type IA}: & \quad \frac{d}{d} \cdot \hat{\Omega} = \hat{\Omega}^2 - \hat{\Omega} \cdot d, \\
& \quad \frac{d}{d} \cdot \omega = - \rho \omega \cdot d, \\
\text{type IB}: & \quad \frac{d}{d} \cdot \hat{\Omega} = \omega \hat{\Omega} - z \hat{\Omega} \cdot d, \\
& \quad \frac{d}{d} \cdot \omega = - \omega \cdot d.
\end{align*}$$

The differential complexes for type II and the nonstandard classical algebras are defined uniquely:

$$\begin{align*}
\text{type II}: & \quad \frac{d}{d} \cdot \hat{\Omega} = \hat{\Omega}^2 - \hat{\Omega} \cdot d, \\
& \quad \frac{d}{d} \cdot \omega = - \omega \cdot d, \\
\text{nonstandard classical case}: & \quad \frac{d}{d} \cdot \hat{\Omega} = \omega \hat{\Omega} - \hat{\Omega} \cdot d, \\
& \quad \frac{d}{d} \cdot \omega = - \omega \cdot d.
\end{align*}$$

Here all the inessential parameters are removed by $\omega$- and $\hat{\Omega}$-rescalings.

**Proof:** These restrictions are easily obtained by demanding $d^2 = 0$ and checking the compatibility of anzats (4.1) with the algebraic relations (3.34)-(3.36). We would like only to mention that the relation (3.37) plays an important role when elaborating the type I and II cases. Q.E.D.

Let us discuss which of the differential complexes listed in Theorem 2 can be treated as $q$-deformations of the complex of right-invariant forms on $GL(N)$. Comparing the formulae (3.34-3.36) and (4.2-4.4) with the conventional classical relations:

$$\begin{align*}
[\hat{\Omega}, \omega]_+ &= [\hat{\Omega}_1, \hat{\Omega}_2]_+ = 0, \\
\frac{d}{d} \cdot \hat{\Omega} &= \hat{\Omega}^2 - \hat{\Omega} \cdot d, \\
\frac{d}{d} \cdot \omega &= - \omega \cdot d,
\end{align*}$$

we conclude, that there are two different possibilities to deform the complex of $GL(N)$-invariant differential forms. The first is realized by the type IA differential complexes with the additional restriction on parameter $\lim_{q \rightarrow 1} \rho = 1$. Note, that in this case the Leibniz rules are deformed under quantization (for $\rho \neq 1$). The second possibility is realized by the type II differential complexes with $\lim_{q \rightarrow 1} \sigma = 0$. Here the Leibniz rules take their conventional form. We would like to mention that all the other types of differential complexes listed in Theorem 2 also may
be interested as an examples of 'exotical' differential complexes on \(GL(N)\) and \(GL_q(N)\), but this subject lies beyond the scope of the present paper.  

Now let us treat the \(SL_q(N)\)-case. The \(q\)-traceless generators \(\hat{\Omega}_{ij}\) can naturally be identified with the \((N^2 - 1)\)-dimensional basis of right-invariant 1-forms on \(SL_q(N)\). These generators form the closed algebra under external multiplication given in the Theorem 1 (see (3.34)), and, remarkably, the algebra of these generators doesn’t contain any random parameters. As the Theorem 2 states, the action of external derivative on these generators can be only defined like in the classical case: \(d \cdot \hat{\Omega} = \hat{\Omega}^2 - \hat{\Omega} \cdot d\) (see (4.2), or (4.4)). So, we conclude that the complex of \(SL(N)\)-invariant differential forms possess the unique \(q\)-deformation.

In the classical Lee-group theory the differential complex of invariant forms serves as the suitable basis in the whole de-Rham complex of all the differential forms on the group manifold. So, in order to get the full differential calculi on the linear quantum groups we have to supply the algebras obtained with the suitable cross-multiplication rules for \(T_{ij}\) and \(\Omega_{ij}\), and to define additionally the action of external derivative on \(T_{ij}\). Note, that in Woronowicz’s sheme [1] this questions are to be solved in the first place, when constructing the first order differential calculus. Not tempting to solve the problem in general we present here one example of such construction, and establish the correspondence between our homogeneous ACA’s and the existing examples of \(GL_q(N)\)-bicovariant differential calculi.

For the matrix group \(GL_q\) of a general rank \(N\) two versions of differential calculi have been considered. They were obtained first in the local coordinate representation, where the differential algebra is generated by the coordinate functions \(T_{ij}\), their differentials \(dT_{ij}\), and derivations \(D_{ij}\) (means \(\frac{\partial}{\partial T_{ij}}\)). We present here the full set of relations between such generators:

\[
\begin{align*}
\mathbf{R} \mathbf{T} \mathbf{T}' &= \mathbf{T} \mathbf{T}' \mathbf{R}, \\
\mathbf{R} \mathbf{d} \mathbf{T} \mathbf{d} \mathbf{T}' &= -\mathbf{d} \mathbf{T} \mathbf{d} \mathbf{T}' \mathbf{R}^{-1}, \\
\mathbf{R} \mathbf{d} \mathbf{T} \mathbf{T}' &= \mathbf{T} \mathbf{d} \mathbf{T}' \mathbf{R}^{-1}, \\
\mathbf{R} \mathbf{D} \mathbf{D}' &= \mathbf{D}' \mathbf{D} \mathbf{R}, \\
\mathbf{D} \mathbf{R} \mathbf{T} &= \mathbf{1} + \mathbf{T}' \mathbf{R}^{-1} \mathbf{D}' \\
\mathbf{D} \mathbf{R} \mathbf{d} \mathbf{T} &= \mathbf{d} \mathbf{T}' \mathbf{R}^{-1} \mathbf{D}'.
\end{align*}
\]

(4.7)

Here, as usual, \(\mathbf{D} = \mathbf{D} \otimes \mathbf{I}, \mathbf{D}' = \mathbf{I} \otimes \mathbf{D}, \mathbf{d} \mathbf{T} = \mathbf{d} \mathbf{T} \otimes \mathbf{I}, \mathbf{d} \mathbf{T}' = \mathbf{I} \otimes \mathbf{d} \mathbf{T}\). This algebra is checked to possess unique ordering for any quadratic and cubic monomials. The relations (4.7) were obtained in [10] and in \(R\)-matrix formulation in [11, 12]. The first two of relations (4.8) were appeared in [14, 15]. Note, that the algebra (4.7),(4.8) implies the commutativity of derivations \(D\) and external derivative \(d\). The defining relations for the second version of differential calculus can be obtained from (4.7), (4.8) by the symmetry transformation \(\mathbf{R} \leftrightarrow \mathbf{R}^{-1}\) of the type (3.33).

\footnote{For \(N = 2\) such an 'exotical' complexes have been considered in [16].}
The right-invariant 1-forms and vector fields are then constructed as

\[ \Omega = dT \cdot T^{-1}, \quad V = T \cdot D, \]  

(4.9)

and they possess the following algebra

\[ \begin{align*}
R \Omega RT &= T \Omega', \\
R V RT &= TV' + R T, \\
R \Omega R \Omega &= -\Omega R \Omega R^{-1}, \\
R V R V &= VR VR + VR - VR, \\
R \Omega R V &= VR \Omega R^{-1} + R \Omega.
\end{align*} \]  

(4.10, 4.11, 4.12, 4.13)

Here Eq’s.(4.11) are the commonly used commutation relations for \( GL_q(N) \)-invariant differential forms (see [12]-[15]). Comparing (4.11) with (3.49) we see that \( \Omega \)'s (4.9) realize the special case of type II external algebra with \( \sigma = -\kappa_q[N]q \). Eq’s.(4.12) is the well known commutation relations for \( GL_q(N) \)-invariant vector fields [3]-[5], but in a slightly different notations. To obtain this relations in the conventional form we have to pass to the new basis of generators \( Y = 1 - \lambda V \).

In this basis Eq’s.(4.13), (4.12) looks like

\[ \begin{align*}
R Y R Y &= Y R Y R, \\
R \Omega R Y &= Y R \Omega R^{-1}.
\end{align*} \]  

(4.14, 4.15)

Note, that our commutation relations of \( V \)'s with \( \Omega \)'s or \( Y \)'s (4.13), (4.14) are different from that presented in [13, 15] for invariant 1-forms and Lie derivatives.

The operator of external derivation in (4.7), (4.8) admits the following explicit representation

\[ d = Tr_q(\Omega V Y^{-1}) = Tr_q(dT D(1 - \lambda V)^{-1}), \]  

(4.16)

which surprisingly differs from expected formula \( Tr_q(dT D) = Tr_q(\Omega V) \). The operator (4.16) satisfy the nilpotence condition and the ordinary Leibniz rules. The form of relation (4.16) suggest us an idea of changing the definition (4.9) of invariant vector fields. Indeed, consider the new set of generators \( U_{ij} \) which is obtained from the old \( V \)'s by the nonlinear invertible transformation:

\[ U = \frac{V}{I - \lambda V}, \quad V = \frac{U}{I + \lambda U}. \]  

(4.17)

With a little algebra one can check that the commutation relations (4.10), (4.12), (4.13) can be concisely rewritten in terms of \( U \)'s

\[ \begin{align*}
R^{-1} U R^{-1} T &= T U' + R^{-1} T, \\
R^{-1} U R^{-1} U &= U R^{-1} U R^{-1} + R^{-1} U - U R^{-1}, \\
R \Omega R^{-1} U &= U R \Omega R + R \Omega.
\end{align*} \]  

(4.18, 4.19, 4.20)

Now, if we consider \( U_{ij} \) instead of \( V_{ij} \) as invariant vector fields on \( GL_q(N) \), then the formula for external derivative takes it standard form: \( d = Tr_q(\Omega U) \).
Finally, let us consider the simplest case of $GL_q(2)$-covariant differential calculus in more details. Note, that while the proof of Theorem I does not work for $N = 2$ the resulting formulae are applicable to this case as well. This can be directly checked by using only the general properties of R-matrix, namely Yang-Baxter equation, Hecke condition, and the $q$-trace formula. The fail of the general proof of Theorem I is due to the different structure of $Ad^{\otimes 2}$ decomposition in case $N = 2$ and is not crucial.

Denote the components of matrix $\Omega$ as $\begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}$. Then from the covariant expressions (3.34-3.36) the following explicit ordering prescriptions can be extracted:

**Type I:** $\theta_2^2 = \theta_3^2 = 0$, $\theta_3\theta_2 = -\theta_2\theta_3$, 

\[
\begin{align*}
\theta_1^2 &= \frac{1}{q^{-2} + \rho} \left\{ q\lambda \rho \theta_2\theta_3 + (\rho - 1)\theta_1\theta_4 \right\}, \\
\theta_4^2 &= \frac{1}{q^{-2} + \rho} \left\{ q^{-2} \lambda \theta_2\theta_3 - q^{-2}(\rho - 1)\theta_1\theta_4 \right\}, \\
\theta_4\theta_1 &= -\frac{1}{q^{-2} + \rho} \left\{ (1 + q^{-2}\rho)\theta_1\theta_4 + q^{-1}\lambda(1 + \rho)\theta_2\theta_3 \right\}, \\
\theta_3\theta_1 &= \frac{1}{1 + \rho} \left\{ -\rho(1 + q^n)\theta_1\theta_3 + (\rho - q^n)\theta_3\theta_4 \right\}, \\
\theta_4\theta_3 &= \frac{1}{1 + \rho} \left\{ -(1 + q^n)\theta_3\theta_4 + (1 - q^n)\theta_1\theta_3 \right\}, \\
\theta_2\theta_1 &= \frac{1}{q^{-2} + q^n\rho} \left\{ -(1 + q^n)\rho\theta_1\theta_2 + (q^n\rho - 1)\theta_2\theta_4 \right\}, \\
\theta_4\theta_2 &= \frac{1}{q^{-2} + q^n\rho} \left\{ -(1 + q^n)\theta_2\theta_4 + (q^n - \rho)\theta_4\theta_2 \right\};
\end{align*}
\]

**Type II:** $\theta_2^2 = \theta_3^2 = 0$, $\theta_3\theta_2 = -\theta_2\theta_3$, 

\[
\begin{align*}
\theta_1^2 &= \frac{\mu}{1 - \mu} \theta_2\theta_3, \\
\theta_4^2 &= \frac{1 - q^{-2} - \mu}{1 - \mu} \theta_2\theta_3, \\
\theta_4\theta_1 &= -\theta_1\theta_4 + \lambda \frac{(q + q^{-1})\mu - q\theta_2\theta_3}{1 - \mu}, \\
\theta_3\theta_1 &= -\frac{1}{1 - \mu(1 + q^{-2})} \left\{ (1 - \mu)\theta_1\theta_3 + \frac{\mu}{q^n}\theta_3\theta_4 \right\}, \\
\theta_4\theta_3 &= -\frac{1}{1 - \mu(1 + q^{-2})} \left\{ \frac{1 - \mu}{q^n}\theta_3\theta_4 + (\mu - 1 + q^n)\theta_1\theta_3 \right\},
\end{align*}
\]

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\[ \theta_2 \theta_1 = -(1 - \mu) \theta_1 \theta_2 + \mu \theta_2 \theta_4, \quad (4.22) \]
\[ \theta_4 \theta_2 = -q^2(1 - \mu) \theta_2 \theta_4 + q^2(\mu - 1 + q^{-2}) \theta_1 \theta_2. \]

Here we use parameter \( \mu = \frac{\sigma + [N]q^k q^k}{\sigma + [N]} \mid_{N=2} \) instead of \( \sigma \) for sake of convenience. In this notation the case (4.11) corresponds to \( \mu = 0 \). An obvious restrictions \( \mu \neq \{1, (1 + q^{-2})^{-1}\} \) and \( \rho \neq \{-1, q^{-2}, q^{-4}\} \) arise when passing to the covariant relations to formulation in components (see remark 2 to Theorem 1).

Let us compare these results with that presented for \( GL_{p,q}(2) \) case in Ref. [16]. First, we note that by assumption the left-invariant 1-forms in [16] admit the decomposition \( \Omega = T^{-1} \cdot d T \), and the external derivative satisfies the undeformed version of Leibniz rules. Hence, the formula \( d \cdot \Omega = -\Omega \cdot d \) is postulated. Moreover, the relation \( d \Omega \sim \omega, \Omega \) is also implied. Hence, the differential calculi obtained in [16] must be of the second type. Indeed the relations (8.5) of [16] can be transformed into the form (4.22) if we note that due to the conditions (6.25) [16] for parameter \( N \) (see (8.1-3) [16]) the following quadratic relation is satisfied:
\[ (1 + r^2) \left( 2 + N \left( 1 + r - (1 + r^2)s \right) \right) = (1 + r + rN)^2. \quad (4.23) \]

Here \( r = pq \) (see Eq. (8.5) [16]) is the only combination of deformation parameters that enters the external algebra of invariant forms. This is not surprisingly since \( GL_{p,q}(2) \) \( R \)-matrix, when suitably normalized, satisfies Hecke relation
\[ R^2 = 1 + (r^\frac{1}{2} - r^{-\frac{1}{2}})R. \quad (4.24) \]

Hence, we expect that parameter \( r^\frac{1}{2} \) of [16] corresponds to ours \( q^{-1} \) (the inverse power here is due to the substitution \( q \leftrightarrow q^{-1} \) that should be done to pass from the right-invariant forms of Eq. (4.22) to the left-invariant ones).

Variable \( s \) of (4.23) parametrizes the different external algebras in [16] and it should be corresponded to ours \( \mu \). Actually, using (4.23) it is straightforward to check that Eqs. (8.5) of [16] are equivalent to (4.22) with the following substitutions to be made:
\[ r \leftrightarrow q^{-2}, \quad \frac{r^{-1} - 1 - rN}{r + r^{-1}} \leftrightarrow \mu. \]

Summarizing all the above, we conclude that our type II differential complexes \( q \) from one hand generalize for arbitrary \( N \) the \( N = 2 \) the formulae given in [16] and from another hand include the bicovariant calculi considered in [10]-[15].

## 5 Conclusion

Here we make some comments on constructing the differential calculi for type II complexes (3.34), (3.36), (4.4), and discuss briefly the problem of \( SL_q(N) \)-reduction of the \( GL_q(N) \)- differential calculi.
Since all the type II differential complexes are isomorphic (see first remark to the Theorem 1), we expect that \( \sigma = -\kappa_q [N] \) differential calculus (4.7),(4.8) can be transformed to the case of any \( \sigma \). In order to realize this transformation we consider the new set of generators \( \{ T^i_j^{(g)} \} \) of the algebra \( \text{Fun}(GL_q(N)) \)

\[
T^i_j^{(g)} = g(z) T_{ij} ,
\]

where \( z = \det_q(T) \) and \( g(z) \) is an arbitrary function of \( z \). It is clear that \( \det_q(T^{(g)}) = z g(z)^N \), and therefore the choice \( g(z) = z^{-1/N} \) leads to the \( SL_q(N) \)-case of [13, 15]. Using commutation relations (4.7) and (4.10) one can deduce

\[
z dT = q^2 dT z \ , \quad \lambda q^N dT = [T, \omega] \Rightarrow \lambda q^N dg(z) = \omega(g(q^2 z) - g(z)) .
\]

Now we introduce new Cartan 1-forms \( \Omega^g = dT^{(g)}(T^{(g)})^{-1} \) which relate to the old ones via the following formulae

\[
\Omega^g = g dT T^{-1} g^{-1} + dg \cdot g^{-1} = \Omega G(z) + \omega \frac{G(z) - 1}{\lambda q^N} .
\]

Here \( G(z) = g(q^2 z) g^{-1}(z) \). Note, that eqs.(4.7) and (4.11) give the following formulas

\[
\lambda q^N d\Omega = \{ \Omega, \omega \} = \lambda q^N \Omega^2
\]

Using these equations and relations (4.7) and (4.11) we obtain the set of \( GL_q(N) \)-differential calculi parametrized by the function \( G(z) \):

\[
RT^{(g)} T^{(g)} = T^{(g)} T^{(g)} R ,
\]

\[
R \Omega^g R \Omega^g + \Omega^g R \Omega^g R^{-1} = RU^g R + U^g ,
\]

\[
T^{(g)} \Omega^g = R \Omega^g RT^{(g)} G(z) + \Omega^g T^{(g)} (G(z) - 1) + \omega^{(g)} (1 - R^2 G(z)) T^{(g)} \left( [N]_q + \frac{\lambda q^N G(z)}{\alpha = 1} \right)^{-1} ,
\]

where

\[
U^g = (\Omega^g)^2 (1 - G(z)) + \omega \Omega^g \frac{G(q^2 z) - G(z)}{\lambda q^N} ,
\]

\[
\omega^{(g)} = Tr_{q^2} \Omega^g = \omega \left( G(z) + \frac{[N]_q G(z) - 1}{\lambda q^N} \right) .
\]

Let us consider the case when the function \( G(z) \) is a constant. For example for \( g(z) = z^\alpha \) we have \( G(z) = q^{2\alpha} \) and Eqs.(5.5) give us the one-parametric set of differential calculi which can be naturally corresponded to the set of type II
differential complexes (3.49):

$$\mathbf{R} T^{(a)} T^{(a)}' = T^{(a)} T^{(a)}' \mathbf{R},$$

$$\mathbf{R} \Omega^{(a)} \mathbf{R} \Omega^{(a)} + \Omega^{(a)} \mathbf{R} \Omega^{(a)} \mathbf{R}^{-1} = \mu_a \left( \mathbf{R} (\Omega^{(a)})^2 \mathbf{R} + (\Omega^{(a)})^2 \right),$$

$$T^{(a)} \Omega^{(a)}' = \mathbf{R} \Omega^{(a)} \mathbf{R} T^{(a)} (1 - \mu_a) - \Omega^{(a)} T^{(a)} \mu_a - \left( \frac{\mu_a}{\xi(\mu_a)} \right) \mathbf{R} (1 + \frac{\mu_a}{\lambda}) \omega^{(a)} T^{(a)},$$

where \( \mu_a = 1 - q^{2a} \), \( \xi(\mu_a) = q^N (1 - \mu_a) - \frac{\mu_a}{\lambda} [N]_q \) and

$$\omega^{(a)} = \text{Tr}_q \Omega^{(a)} = q^{-N} \xi(\mu_a) \omega.$$ (5.9)

Now let us explore the possibilities of \( SL_q(N) \)-reduction of this calculi. First, if we put \( \alpha = -1/N \) (as it was done in Refs. [13, 15]), then we have in the commutation relations (5.8) unavoidable additional 1-form generator \( \omega^{-1/2} \) and, thus, the number of Cartan’s 1-forms is \( N^2 \) but not \( N^2 - 1 \) as in the undeformed case of \( SL(N) \). Second, one could try to put \( \omega^{(a)} \) to zero choosing parameters \( \alpha \) and \( q \) as

$$q^{-\alpha} \xi(\mu_a) = q^{\alpha + N} + [\alpha][N]_q = 0.$$ (5.10)

In particular this equation is fulfilled for the \( q \) being a root of unity: \( q^{2N} = q^{2^2} = q^{2^2} \), which doesn’t contradict with the condition \( \alpha = -1/N \). However, for the case of (5.10) we have in third equation of (5.8) \( 0 \) ambiguity, which resolves as

$$q^{N-\alpha} \omega^{(a)} \left( q^{\alpha + N} + [\alpha][N]_q \right)^{-1} = \omega,$$ (5.11)

and we can not put it to zero having in mind that \( \lambda q^N dT^{(a)} = [T^{(a)}, \omega] \) and \( [\det_q T^{(a)}, \omega] \neq 0 \). Therefore, the differential calculi (5.8) doesn’t admit the correct \( SL_q(N) \)-reduction even for the special values of the quantization parameter \( q \).

Now, how one may hopes to construct the consistent bicovariant differential calculus on \( SL_q(N) \)? The nice way of making the reduction from \( GL_q(N) \)-case doesn’t work for type II differential calculi (5.8). May be the cross-multiplication presented in (5.8) is not the unique possibility of construsting the differential calculi starting from the type II complexes. May be the difficulties will be overcome if we use the type IA complexes instead of the type II ones. But here for \( \rho \neq 1 \) we meet the serious problems when constructing the local coordinate representation of the type \( \Omega = dT \cdot T^{-1} \). So, only the type IA differential complex with \( \rho = 1 \) seems to be good candidate for construction of consistent differential calculus on \( GL_q(N) \) with it’s possible reduction to \( SL_q(N) \). We hope to revert to these problems in further publications.
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A Appendix

Here we present the ‘ordered’ expressions for quadratic combinations \((\tilde{\Omega} \circ \tilde{\Omega}^2)_{21}\), \((\tilde{\Omega}^2 \circ \tilde{\Omega})_{21}\) (see \((3.42)\)), and collect some formulas which were used in derivation of these expressions.

Consider the sequence \(x_i\) defined iteratively

\[
x_0 = q^N + q^{-N}, \quad x_1 = [N]_q + \lambda q^N, \quad x_{i+2} = x_i + \lambda x_{i+1}.
\]  

(A.1)

Define

\[
y_i = (-1)^{i+1} \frac{x_{1-i} x_1 - x_{-i} x_2}{|x|}, \quad y_{i,k} = (-1)^{i+k} \frac{x_{1-i-k} x_{1-k} - x_{-i-k} x_{2-k}}{|x|},
\]  

(A.2)

where \(|x| \equiv [N + 2]_q [N - 2]_q\). It is straightforward to show that \(y_{i,k} = y_i\) for any \(i\) and \(k\), and \(y_i\) are calculated by the following simple iteration:

\[
y_0 = 1, \quad y_1 = \lambda, \quad y_{i+2} = y_i + \lambda y_{i+1}.
\]  

(A.3)

When simplifying the final expressions for \((\tilde{\Omega} \circ \tilde{\Omega}^2)_{21}\) and \((\tilde{\Omega}^2 \circ \tilde{\Omega})_{21}\) we use the following properties of the matrix functions \(x(R), \bar{x}(R)\):

\[
R^k x(R) = x_k 1 + x_{k+1} R, \quad R^k \bar{x}(R) = \frac{(-1)^k}{|x|} (-x_{2-k} 1 + x_{1-k} R),
\]

\[
R^k x_i \bar{x}(R) = (-1)^{i+k} \left( y_{-i-k} 1 - x_{1-i} R^{1-i} \bar{x}(R) \right),
\]

\[
R^k \bar{x}(R) = \left( x_{2-k} 1 + x_{1-k} R \right) R^{-k} \bar{x}(R).
\]
together with (A.1)-(A.3). The result is

\[
(\hat{\Omega}^2 \circ \hat{\Omega})_{21} = E^{-1}(R) \left[ \epsilon(R) \left\{ 1 + \frac{x(R)}{|x|} \delta(R) \right\} (\hat{\Omega}^2 \circ \hat{\Omega})_{12} \right.
\]

\[+ R \left\{ \epsilon(R) \left( \lambda - \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} \right) \right. + \frac{\tau_{x}}{|x|} \left( [N]_{q} + R \delta(R) \right) \left\} (\hat{\Omega}^2 \circ \hat{\Omega})_{12} \right.
\]

\[- \tau R^2 \epsilon(R) \left\{ \pi(R) + \frac{x(R)}{|x|} R^{-2} \right\} \omega(\hat{\Omega} \circ \hat{\Omega})_{12}
\]

\[+ \left\{ \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} R^2 (1 + R^2 \pi(R)) + \frac{\tau_{x}}{|x|} R \left( [N]_{q} + \lambda qN (1 + R^2) + (1 + \lambda^2) R \delta(R) \right) \right\} (\hat{\Omega}^2)_{1}
\]

\[- \left\{ \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} R^2 (1 + R^2 \pi(R)) + \frac{\tau_{x}}{|x|} \left( qN (1 + R^2) + \lambda R \delta(R) \right) \right\} (\hat{\Omega}^2)_{2}
\]

\[+ \left\{ \tau \pi(R) \epsilon(R) \left( \lambda R - \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} \right) - \frac{\tau_{x}}{|x|} R \epsilon(R) + \lambda qN \frac{\tau_{x}}{|x|} R^{-1} F(R) \right\} \omega(\hat{\Omega})_{12}
\]

\[+ \left\{ \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} + \frac{\tau_{x}}{|x|} \right\} \epsilon(R) + \lambda qN \frac{\tau_{x}}{|x|} R^{-1} F(R) \right\} \omega(\hat{\Omega})_{12} \right)
\]

\[(\hat{\Omega} \circ \hat{\Omega})_{21} = E^{-1}(R) \left[ \left\{ \epsilon(R) \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} - \frac{\tau_{x}}{|x|} \left( \lambda qN R^3 - x_{-1} R + R^2 \delta(R) \right) \right\} (\hat{\Omega} \circ \hat{\Omega})_{12}
\]

\[+ \left\{ R^2 \epsilon(R) \left( 1 - \pi(R) \delta(R) + \tau \sigma \frac{x_{2} + \lambda x_{3}}{|x|} \right) + \tau \sigma (R^4 + R^6) \delta(R) \pi^2(R) - \left( \frac{\tau_{x}}{|x|} R^2 \right)^2 \right\} (\hat{\Omega}^2 \circ \hat{\Omega})_{12}
\]

\[- \tau R^2 \left( \pi(R) + \frac{x(R)}{|x|} R^{-2} \right) \left( \epsilon(R) + \tau \sigma R^2 \pi(R) \right) \omega(\hat{\Omega} \circ \hat{\Omega})_{12}
\]

\[+ \left\{ \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} R^3 G(R) + \tau \sigma R^2 \left( \frac{\tau_{x}}{|x|} + \frac{x(R)}{|x|} \pi(R) \delta(R) \right) - \left( \frac{\tau_{x}}{|x|} \right)^2 (1 + \lambda^2) R^2 \right\} (\hat{\Omega}^3)_{1}
\]

\[- \left\{ \frac{\delta x_{1} + qN x_{2} - \lambda}{|x|} R^3 G(R) + \tau \sigma R \left( \frac{\tau_{x}}{|x|} + \frac{x(R)}{|x|} \pi(R) \delta(R) \right) - \lambda R \left( \frac{\tau_{x}}{|x|} R^2 \right) \right\} (\hat{\Omega}^3)_{2}
\]

\[+ \left\{ \frac{\tau_{x}}{|x|} R^2 \left( \epsilon(R) (-x_2 + (1 + \lambda^2 - x_3 \pi(R) R) \delta(R)) + \tau \sigma (1 + \lambda^2) \right)
\]

\[+ \tau \pi(R) \left( \epsilon(R) \left( \tau \sigma \frac{x(R)}{|x|} + R^2 \right) - \lambda \tau \sigma R^3 \pi(R) \right) \right\} \omega(\hat{\Omega}^2)_{1}
\]

\[+ \left\{ \frac{\tau_{x}}{|x|} \left( \epsilon(R) (x_1 + (x_2 R \pi(R) - \lambda) \delta(R)) - \lambda \tau \sigma R \right)
\]

\[+ \tau^2 \sigma \pi(R) \left( \frac{x(R)}{|x|} \epsilon(R) + R^2 \pi(R) \right) \right\} \omega(\hat{\Omega}^2)_{2} \right]
\]

where

\[
\delta(R) = \delta + qN R^2 \quad \epsilon(R) = 1 + R^2 \pi(R) \delta(R) \quad F(R) = 1 + R^{-4} \frac{\pi(R)}{|x|} \delta(R) \quad G(R) = \epsilon(R) + \tau \sigma R^2 \pi(R)
\]

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\[ E(\mathbf{R}) = \left( 1 + \frac{\delta x_0 + q^N x_{-2}}{|x|} \right) \epsilon(\mathbf{R}) + \frac{\tau \sigma}{|x|} \left( q^{-N} + q^N \mathbf{R}^2 \right). \]  

Note, that these expressions are significantly simplified under restriction \( \tau \sigma = 0 \).

References


