On Soliton Content of
Self Dual Yang-Mills Equations

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ABSTRACT

Exploiting the formulation of the Self Dual Yang-Mills equations as
a Riemann-Hilbert factorization problem, we present a theory of pulling
back soliton hierarchies to the Self Dual Yang-Mills equations. We show
that for each map $\mathbb{C}^4 \rightarrow \mathbb{C}^\infty$ satisfying a simple system of linear equations
formulated below one can pull back the (generalized) Drinfeld-Sokolov
hierarchies to the Self Dual Yang-Mills equations. This indicates that
there is a class of solutions to the Self Dual Yang-Mills equations which
can be constructed using the soliton techniques like the $\tau$ function method.
In particular this class contains the solutions obtained via the symmetry
reductions of the Self Dual Yang-Mills equations. It also contains genuine
4 dimensional solutions. The method can be used to study the symmetry
reductions and as an example of that we get an equation exhibiting breaking
solitons, formulated by O. Bogoyavlenskii, as one of the $2 + 1$ dimensional
reductions of the Self Dual Yang-Mills equations.

1. Introduction.

The Self Dual Yang-Mills (SDYM) equations are an integrable system admit-
ting the zero-curvature representation involving a spectral parameter. Denoting the
spectral parameter by $\lambda$, where for simplicity we take $\lambda \in S^1$, we can write the zero
curvature condition as

\begin{equation}
\left[ \left( \frac{\partial}{\partial \bar{\tau}} + \lambda \frac{\partial}{\partial z} \right) - A_{\bar{\tau}} - \lambda A_z, \left( -\frac{\partial}{\partial \tau} + \lambda \frac{\partial}{\partial y} \right) + A_{\tau} - \lambda A_y \right] = 0.
\end{equation}

In the physical coordinates $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

\begin{equation}
y = x_1 + ix_2, \quad z = x_3 - ix_4
\end{equation}

and $\bar{y}$, $\bar{z}$ are complex conjugates.

It was R. Ward [1], who interpreted (1) as a differential consequence of an
assumption of holomorphic triviality over projective lines. The most succinct for-
mulation is via a Riemann-Hilbert factorization problem. Let $w_1 = y + \lambda \bar{\tau}$,
\[ w_2 = -\frac{\partial}{\partial y} + \lambda y, \partial_1 = \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial z}, \partial_2 = -\frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial z}. \] Then, Ward’s triviality condition reads:

\[ g(w_1, w_2, \lambda) = \phi_+(x, \lambda) \psi_+(x, \lambda), g \in SL(n, \mathbb{C}), \partial_1 g = \partial_2 g = 0 \quad \text{and} \quad \phi_+(\phi_-) \]

are matrices in \( SL(n, \mathbb{C}) \), holomorphic with respect to \( \lambda \) inside (outside) the unit circle. For the sake of our discussion we need a generalization of equation (3). Consider \( \mathcal{G} \) and \( \mathcal{H} \), both belonging to \( SL(n, \mathbb{C}) \) and both functions of \( \lambda \) and \( x \), satisfying

\[ \partial_1 \mathcal{G} = \Omega_1^+ \mathcal{G}, \partial_1 \mathcal{H} = \Omega_1^+ \mathcal{H}, \partial_2 \mathcal{G} = \Omega_2^- \mathcal{G}, \partial_2 \mathcal{H} = \Omega_2^+ \mathcal{H}, \]

where

\[ \Omega_i^+(x, \lambda) = \sum_{k=0}^{\infty} \omega_{ik}^+(x) \lambda^k, \Omega_i^-(x, \lambda) = \sum_{k=-\infty}^{1} \omega_{ik}^-(x) \lambda^k, i = 1, 2. \]

Then, the existence (and differentiability) of the Riemann-Hilbert factorization

\[ \mathcal{G} \mathcal{H}^{-1} = \phi_-^{-1} \phi_+ \]

implies that the analog of the soliton Baker-Akhiezer function \( \psi = \phi_- \mathcal{G} = \phi_+ \mathcal{H} \) satisfies

\[ \partial_1 \psi = (A_1 - \lambda A_2) \psi \]

\[ \partial_2 \psi = (-A_1 - \lambda A_2) \psi \]

where \( (A_y, A_y^-, A_z, A_z) \) is a self-dual connection. This statement is a simple consequence of equations (4) and (5) and Liouville theorem.

2. Generalized Drinfeld-Sokolov hierarchies.

A generalization of Drinfeld-Sokolov flows is presented fully in [2]. The canonical reference for the Drinfeld-Sokolov systems is [3]. Among the facts we shall need below are the following. Consider the Kac-Moody algebra \( A^{(1)}_{n-1}([4]) \). The canonical
The cyclic element $J_\lambda = \sum_{k=0}^{n-1} e_k$ has the property $J_\lambda^{kn} = \lambda^k$.

Let $\Delta = \Delta_- \cup \Delta_+$ be the set of all roots of $g = A_{n-1}^{(1)}$, and

$\pi = \{\alpha_0, \ldots, \alpha_{n-1}\}$ be the set of simple roots. $g$ admits the decomposition:

$$ g = g_- \oplus g_0 \oplus g_+ , $$

where

$$ g_\pm = \bigoplus_{\alpha \in \Delta_\pm} g_\alpha . $$

Recall that a subalgebra $p$ of $g$ is called a parabolic subalgebra if

$$ g_0 \oplus g_+ \subseteq p . $$

For any parabolic subalgebra $p$ there is a subset $\tilde{\pi}$ of $\pi$, such that if

$$ \tilde{\Delta}_+ = \left\{ \sum k_i \alpha_i \in \Delta_+ : \alpha_i \in \tilde{\pi} \right\} , $$

then $p = \left( \bigoplus_{\alpha \in \tilde{\Delta}_+} g_\alpha \right) \oplus g_0 \oplus g_+$. If, consequently, one defines $q = \bigoplus_{\alpha \in \Delta_+ \setminus \tilde{\Delta}_+} g_\alpha$,

then

$$ q = g \oplus p . $$

In our context, we assume that $\tilde{\pi}$ does not contain $\alpha_0$, that is $f_0$ does not belong to $p$. Thus $p$ differs from $g_0 \oplus g_+$ by some lower triangular matrices independent of $\lambda$. In fact this is just a lower triangular piece of a parabolic subalgebra of $sl(n, \mathbb{C})$. We fix now $p$ and $q$ and form the corresponding connected groups $G_p$ and $G_q$, both
subgroups of $G = \{ \text{space of maps: } S^1 \to SL(n, \mathbb{C}) \}$. The generalized Drinfeld-Sokolov flows can be defined through the factorization problem

$$
\sum_{e^{i\pi k} \in \mathbb{Z}} \int_{\mathbb{R}} t_k J^k_x \frac{g e^{i\pi k}}{k} \sum_{e^{i\pi k} \in \mathbb{Z}} t_k J^k_x = \phi^{-1}_-(t) \phi_+(t),
$$

(8)

where $\phi_- \in G_q, \quad \phi_+ \in G_p, \quad g \in G$. For all practical purposes we assume that all but a finite number of $t_k$’s vanish. The equations of (generalized) Drinfeld-Sokolov type arise as the compatibility conditions of

$$
\frac{\partial}{\partial t_k} \psi = U_k \psi, \quad k \in \mathbb{Z},
$$

(9)

where $\psi$ is the Baker-Akhiezer function

$$
\psi(t) = \phi_-(t) \sum_{e^{i\pi k} \in \mathbb{Z}} t_k J^k_x
$$

(10a)

and $U_k$ can be computed from (8) via

$$
U_k = (\phi_- J^k_x \phi_-^{-1})_p + (\phi_+ J^k_x \phi_+^{-1})_q
$$

(10b)

where the subscripts $p$ and $q$ denote the projections on $\mathbb{Z}$ and $\mathbb{Z}$. The theory depends in an essential way on the sign of $k$. For $k > 0$ we have a well known statement [3] that $U_k$ is a differential polynomial (with respect to the first flow $t_1 = x$) in the entries of $U_1$. As a result the corresponding compatibility condition is an evolution equation. This is not so for $k < 0$. To illustrate that we quote a few results from [2]. For $n = 2$, $\pi = \{ \alpha_1 \}, \ k = 1, 3$

$$
U_1 = J_\lambda + q,
$$

$$
q = \begin{pmatrix} u & 0 \\ Du - u^2 & -u \end{pmatrix},
$$

and the compatibility of the first and the third flow gives the potential $KdV$ equation $u_t = \frac{1}{4} u_{xxx} + \frac{7}{2} (u_x)^2$. On the other hand for $k = 1, -1$ one obtains the stationary Bogoyavlenskii equation [5], [2], $u_{xxx}t = 4(u_{xx} + u_x \cdot u_t + 2u_x u_{xt})$. When $\pi = 0$, one gets the modified $KdV$ hierarchy. Including the ”negative” flows in this case is well known to yield the Toda equations.

Now suppose $t_k$'s are functions of the variables appearing in the SDYM equations, namely $(y, \overline{y}, z, \overline{z})$. The question we want to address is that of admissible choices of a functional dependence of generalized Drinfeld-Sokolov flows for which the factorization problem (8) goes into the factorization problem (5). In other words we are looking for a map $\phi : \mathbb{C}^4 \rightarrow \mathbb{C}^\infty, (y, \overline{y}, z, \overline{z}) \rightarrow \mathfrak{t} = \{t_k\}_{k \in \mathbb{Z}}$ which pulls back (9) to (6a) and (6b). To this end the functions

$$G = e^{\sum_{k \in \mathbb{Z}} t_k J^k_\lambda} \quad \text{and} \quad H = e^{\sum_{k \in \mathbb{Z}} t_k J^k_\lambda} \quad g, \quad g \in G$$

are required to satisfy equation (4). Using the fact that $J^k_\lambda = \lambda^k$, we obtain

(11a) $$\frac{\partial}{\partial y} t_k + \frac{\partial}{\partial z} t_{k-n} = 0 ,$$

(11b) $$-\frac{\partial}{\partial z} t_k + \frac{\partial}{\partial y} t_{k-n} = 0 ,$$

for

$$k > n \quad \text{or} \quad k < 0 .$$

Equations (11a) and (11b) can be viewed as differential recurrence relations. For each choice of a parametrization $\phi$ of Drinfeld-Sokolov flows subject to (11a) and (11b) one gets a class of solutions to the SDYM equations. By cross differentiation of (11a), (11b) we get that all $t_k$'s, $k \neq 0 \mod n$, satisfy the Laplace equation:

(12) $$\frac{\partial^2 t_k}{\partial y \partial y} + \frac{\partial^2 t_k}{\partial z \partial z} = 0$$

It is in fact quite easy to solve equations (11a), (11b) in the class of analytic functions in $(y, \overline{y}, z, \overline{z})$. We have

Theorem Given $2(n-1)$ functions $t_k = g_k$, $-(n-1) \leq k \leq -1$ or $1 \leq k \leq n-1$, analytic in $(y, \overline{y}, z, \overline{z})$ and satisfying (12), the general solution to (11a), (11b) is

(13a) $$t_{k+n} = \mathcal{D}^j g_k + \sum_{m=0}^{t-1} \mathcal{D}^m \psi_m , \quad 1 \leq k \leq n - 1$$
(13b) \[ t_k = \mathcal{E}^{t} g_k + \sum_{m=0}^{\ell-1} \mathcal{E}^{m} \psi_m, \quad -(n-1) \leq k \leq -1, \]

for \( \ell \geq 1 \). By definition

(13c) \[ (Df)(y, \overline{y}, z, \overline{z}) \equiv \int_{0}^{y} (\partial_y f)(y, \overline{y}, z, \overline{z}) dy - \int_{0}^{\overline{y}} \partial_{\overline{y}} f(y, \overline{y}, z, \overline{z}) d\overline{y} \]

(13d) \[ (Ef)(y, \overline{y}, z, \overline{z}) = \int_{0}^{y} \partial_{y} f(y, \overline{y}, z, \overline{z}) dy - \int_{0}^{\overline{y}} \partial_{\overline{y}} f(y, \overline{y}, z, \overline{z}) d\overline{y}. \]

The functions \( \psi_m(\overline{\psi}_m) \) are analytic functions of \( y, z(\overline{y}, \overline{z}), \) otherwise arbitrary.

**Examples** Let \( t_1 = \alpha y + \beta \overline{y} + \gamma z + \delta \overline{z} + t_1^0, \quad t_k = 0 \quad 2 \leq k \leq n - 1. \) Then \( t_{1+n} = \alpha z - \gamma \overline{y} + \psi_1(y, z), \quad t_{1+2n} = \overline{z} \partial_y \psi_1 - \overline{y} \partial_z \psi_1 + \psi_2(y, z) \) and so forth. We can terminate this sequence by choosing \( \psi_1 = \psi_2 = 0 . \) As a special case we might take \( n = 2, \quad t_1 = \overline{y} + y, \quad t_3 = \overline{z}, \) which leads to a symmetry reduction of the SDYM to the (potential) KdV equation discussed in [6], see also [7]. Another simple choice of \( \phi \) obtained when on takes \( t_1 = \overline{y} + y, \quad t_3 = \overline{z}, \quad t_1 = z \) gives in the case of the maximal parabolic \( \hat{\pi} = \{ \alpha_1 \} \) a solution of the SDYM which is simultaneously parametrized by the (potential) KdV equation and the stationary case of the Bogoyavlenskii equation [3]. Such solutions will in general depend on three independent variables thus giving a three dimensional solution of the SDYM. For \( \hat{\pi} = \emptyset, \) one obtains the modified KdV and the Sine-Gordon equation respectively. The choice of \( t_1 = \overline{y}, \quad t_{-1} = y \) gives also rise to the stationary case of the Bogoyavlenskii equation.

The above theorem allows one to choose the ”times” \( t_k \) to have nonlinear dependence on \( (y, \overline{y}, z, \overline{z}) \). We have seen above an example of \( \phi \) whose rank is 3. It is equally easy to give an example of \( \phi \) with maximal rank of 4. As an example one can take

\[
\begin{align*}
t_{-1} &= y^2 + t_{-1}^0 \\
t_1 &= \overline{y} \overline{z} + zy + t_1^0 \\
t_{n+1} &= z \overline{z} - y \overline{y} + t_{n+1}^0 \\
t_{2n+1} &= -\overline{y} \overline{z} + t_{2n+1}^0
\end{align*}
\]

all other times being zero. Thus by employing more then two flows from the soliton hierarchy one can in principle construct four dimensional solutions of SDYM. It
remains an interesting open question as to what class of solutions one obtains this way.

**Remark** One can develop a similar theory to the one presented in this letter starting with the AKNS systems and the AKNS hierarchy. It will be published elsewhere.

4. **More Examples**

   (i) rational solutions to the KdV;

   (ii) the Bogoyavlenskii equation

   It seems that the most interesting cases to consider are classes of solutions to generalized Drinfeld-Sokolov hierarchy which are in some sense finite dimensional. The class of rational solutions, soliton solutions, or even more generally finite gap solutions are good examples of those. To illustrate this point let us consider the rational solutions to the KdV equation vanishing at infinity. By a result of Segal and Wilson [8; section 7] they are obtained by factorizing

   \[
g_d(\lambda) = \begin{pmatrix}
    \lambda^{-(d+1)/2} & 0 \\
    0 & \lambda^{d+1}
\end{pmatrix}, \quad d \in \mathbb{Z}_+, \text{ d is odd}
\]

   or

   \[
g_d(\lambda) = \begin{pmatrix}
    \lambda^{d/2} & 0 \\
    0 & \lambda^{-d/2}
\end{pmatrix}, \quad d \in \mathbb{Z}_+, \text{ d is even}
\]

   The manifold of rational solutions \( \mathcal{M} \) is a union of disconnected pieces \( \mathcal{M}_d \) each labeled by a positive integer \( d = \dim \mathcal{M}_d \). As a result, whenever one performs the factorization of

   \[
e^{\sum t_k J_k^+=} g_d(\lambda)e^{-\sum t_k J_k^-},
\]

   \( \phi_- \) will depend only on the first \( d \) flows \( t_1, t_3, \cdots t_{2d-1} \). We can however easily choose \( t_1, t_3, \cdots t_{2d-1} \) to have, say, polynomial dependence on \( (y, \bar{y}, z, \bar{z}) \) subject to (13a). Thus we will obtain a large class of (complex) rational solutions to the SDYM equations labeled by \( d \). It would be interesting to contrast these with instanton solutions.

   To get some insight into the type of solutions we are getting we will consider two simplest cases corresponding to \( d = 1 \) and \( d = 2 \). In the first case

   \[
g_1(\lambda) = \begin{pmatrix}
    \lambda^{-1} & 0 \\
    0 & \lambda
\end{pmatrix},
\]

   7
and the factorization (8) can easily be performed. The matrix \( \phi_- \) depends now on only one variable, namely \( t_1 \). A straightforward computation gives

\[
\phi_- = \begin{pmatrix}
1 & \frac{1}{t_1} \\
0 & 1
\end{pmatrix},
\]

Let us make a choice of parametrization for \( t_1, \cdots \) satisfying (11a) and (11b). For convenience we set \( t_1(y, \overline{y}, z, \overline{z}) = \chi \). It is quite easy to compute now the corresponding self-dual connection. To this end we use the formulas (6a) and (6b), and \( \phi_- \) given above. The chain rule implies that

\[
(15a) \quad A_y = \begin{pmatrix}
0 & 0 \\
-\frac{\partial \chi}{\partial \overline{y}} & 0
\end{pmatrix},
\]

\[
(15b) \quad A_\overline{y} = \begin{pmatrix}
-\frac{\partial \chi}{\chi} & \frac{\partial \chi}{\chi} + \frac{\partial \chi}{\overline{y}} \\
\frac{\chi^2}{\overline{y}} & \frac{\partial \chi}{\chi}
\end{pmatrix},
\]

\[
(15c) \quad A_z = \begin{pmatrix}
0 & 0 \\
\frac{\partial \chi}{\partial \overline{z}} & 0
\end{pmatrix},
\]

\[
(15d) \quad A_\overline{z} = \begin{pmatrix}
-\frac{\partial \chi}{\chi} & -\frac{\partial \chi}{\chi} + \frac{\partial \chi}{\overline{z}} \\
0 & \frac{\partial \chi}{\chi}
\end{pmatrix}.
\]

One can directly check that indeed we have obtained a self-dual connection. We would like to point out that (15a)-(15d) give the self-dual connections for any \( \chi \) satisfying the Laplace equation (12). Moreover, as the next example shows, the resulting connections depend in general on all 4 variables \( y, \overline{y}, z, \overline{z} \) and thus the method proposed above is capable of producing some 4 dimensional solutions to the SDYM equations. Indeed, let us choose

\[
t_1 = \chi = \overline{y} \overline{z} + yz, \\
t_3 = z \overline{z} - y \overline{y}, \\
t_5 = -\overline{y} \overline{z}.
\]

The formulas (15a)-(15d) simplify now to

\[
A_y = \begin{pmatrix}
0 & 0 \\
-\overline{y} & 0
\end{pmatrix},
\]

8
\[ A_y = \begin{pmatrix} \frac{z}{y-z^2} & \frac{y+z^2 y}{y-z^2} & 0 \\ 0 & 0 & \frac{z}{y-z^2} \end{pmatrix}, \]

\[ A_z = \begin{pmatrix} \frac{1}{y} \\ 0 \\ \frac{1}{y} \end{pmatrix}, \]

\[ A_{\tau} = \begin{pmatrix} \frac{-z}{y-z^2} & \frac{-y+z^2 y}{y-z^2} & 0 \\ 0 & 0 & \frac{z}{y-z^2} \end{pmatrix}. \]

In the case of \( d = 2 \)

\[ g_2(\lambda) = \begin{pmatrix} \frac{1}{\lambda} \\ 0 \\ \lambda^{-1} \end{pmatrix}, \]

\[ \phi_- = \begin{pmatrix} 1 + \frac{f_1}{(t_1-3t_3)} & -\frac{f_1^2}{3} \frac{1}{(t_1-3t_3)^2} & \end{pmatrix} \lambda^{-1} - \frac{3}{3} \frac{f_1^2}{(t_1-3t_3)^2} \lambda^{-1} \].

We now set \( t_1(y, \overline{y}, z, \overline{z}) = \chi_1 \) and \( t_3(y, \overline{y}, z, \overline{z}) = \chi_3 \), both assumed to satisfy (11a) and (11b), and proceed as above. The final result is [9]:

\[ A_y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]

\[ A_{\overline{y}} = \begin{pmatrix} -3(3\chi_3 + 2\chi_1) \frac{\partial \chi_1}{\partial \overline{y}} - 3\chi_1 \frac{\partial \chi_1}{\partial y} + (\chi_1^2 - 3\chi_1^2 \chi_3) \frac{\partial \chi_1}{\partial \overline{y}} & (18\chi_3 + 3\chi_1^2) \frac{\partial \chi_1}{\partial \overline{y}} - 9\chi_3 \frac{\partial \chi_1}{\partial y} + (9\chi_1^2 - 6\chi_3^2 \chi_3 + \chi_1) \frac{\partial \chi_1}{\partial \overline{y}} \end{pmatrix}, \]

\[ A_z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]

\[ A_{\tau} = \begin{pmatrix} -3(3\chi_3 - 2\chi_1) \frac{\partial \chi_1}{\partial \overline{y}} + 3\chi_1 \frac{\partial \chi_1}{\partial y} + (\chi_1^2 - 3\chi_1^2 \chi_3) \frac{\partial \chi_1}{\partial \overline{y}} & (18\chi_3 + 3\chi_1^2) \frac{\partial \chi_1}{\partial \overline{y}} - 9\chi_3 \frac{\partial \chi_1}{\partial y} + (9\chi_1^2 - 6\chi_3^2 \chi_3 + \chi_1) \frac{\partial \chi_1}{\partial \overline{y}} \end{pmatrix}. \]

Thus we have found this time a self-dual connection parametrized by two scalar functions \( \chi_1 \) and \( \chi_2 \), which need only satisfy the Laplace equation (12) and the conditions (11a) and (11b).

Another interesting subject would be to consider systems of equations defined by the factorization

\[ \epsilon \sum_k \partial_k \lambda^k g(w; \lambda) e^{-\sum_k \partial_k \lambda^k} = \phi + (t) \phi_+ (t), \quad \partial_1 g = \partial_2 g = 0. \]

The whole theory developed in this paper applies to this factorization without any major changes.
Since not much is known about these systems (see [10] for some information) we restrict ourselves to a particularly interesting example which is the (full) Bogoyavlenskii equation that corresponds to choosing $n = 2$ and

$$ t_1 = \mathcal{F}, \quad t_{-1} = y. $$

For the maximal parabolic case and $n = 2$ we obtain [5]

$$ u_{yz} = u_{yy} u_y - 4(u_{yy} + u_{yy} u_y + 2u_{yy} u_{y}). \tag{17} $$

To derive this equation one needs to compute the zero curvature condition for $U_1$ and $U_{-1}$ using (10a) and (10b).

The proof that this equation is indeed a symmetry reduction of the SDYM is in [10].

Finally, we would like to point out that the method proposed in this paper relies heavily on the factorization problem (5). The examples of solutions to the SDYM we have given were obtained by explicitly performing such a factorization. However this is not the only way. The other way would have been to apply the representation theoretic method [11] in which the factorizations like the one we are using are recovered from the appropriate $\tau$ function.

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