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RATIONAL SURFACES HAVING ONLY A FINITE NUMBER OF EXCEPTIONAL CURVES

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Abstract

We characterize the rational surfaces $X$ which have a finite number of $(-1)$-curves under the assumption that $-K_X$ is nef (i.e., the intersection number of $K_X$ with any effective divisor on $X$ is less than or equal to zero, where $K_X$ is a canonical divisor on $X$) and having self-intersection zero. A $(-1)$-curve is a smooth rational curve of self-intersection $-1$.

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1 Introduction

By a surface we mean here a compact complex analytic manifold of complex dimension two. According to the curves on surfaces, one can distinguish between three classes: there are those which have only a finite number of curves, i.e. the surfaces having algebraic dimension zero; those which have curves but not enough, i.e. the surfaces having algebraic dimension one; and those which are rich in curves, i.e. the projective ones. For more details see [5, Theorems 3.1, 4.1, 4.2, 4.3, 5.1], [6], and [12].

We restrict ourselves to curves which are smooth, rational and of self-intersection $-1$ (such curves are called $(-1)$-curves), and ask which surfaces have an infinite number of $(-1)$-curves. The answer is that only rational surfaces may have an infinite number of $(-1)$-curves (e.g., see [11]).

In this paper we give a characterization of rational surfaces which have a finite number of $(-1)$-curves under the assumption that an anti-canonical divisor of the surface is nef (see the definition in the next paragraph) and of self-intersection zero.

Let $X$ be a smooth projective rational surface. From now on we assume that $-K_X$ is nef (i.e., the intersection number of the divisor $K_X$ with any effective divisor on $X$ is less or equal to zero, where $K_X$ is a canonical divisor on $X$) and of self-intersection zero.

It is easy to see that $X$ is a blowing-up nine points (possibly infinitely near) of the complex projective plane.

According to the position of the nine points, $X$ may have a finite or an infinite number of $(-1)$-curves.

Masayoshi Nagata ([10], proposition 6a, p.282) proved that if the nine points are in general position, then $X$ has an infinite number of $(-1)$-curves (i.e., smooth rational curves of self-intersection $-1$).

Ulf Persson and Rick Miranda ([19]) studied the case when the nine points are the base points of a linear system of plane cubics without fixed components. In this case $X$ is an elliptic surface with a section. They classified all such surfaces which have a finite number of $(-1)$-curves and called them extremal jacobian elliptic rational surfaces. For each case, they gave the number of $(-1)$-curves.

We will use the following notations:

$\sim$ the linear equivalence of divisors on $X$;

$[D]$ the set of divisors $D'$ on $X$ such that $D' \sim D$;

$\text{Div}(X)$ the group of divisors on $X$;

$\text{NS}(X)$ the group quotient $\frac{\text{Div}(X)}{\sim}$ of $\text{Div}(X)$ by $\sim$ (the linear, algebraic and numerical equivalences are the same on $\text{Div}(X)$ since $X$ is a rational surface);

$D \cdot D'$ will denote the intersection number of the divisor $D$ with the divisor $D'$, in particular the self-intersection of $D$ is $D^2 = D \cdot D$;
the associated element to the divisor $D$ in the tensor product of the group $NS(X)$ with the field of rational numbers over the ring of integers.

Following [9], we will define a smooth projective rational surface having a finite number of $(-1)$-curves on it an extremal rational surface. Our main result gives a classification of extremal rational surfaces:

**Theorem 1.1** Let $X$ be a smooth projective rational surface having $-K_X$ nef and of self-intersection zero. Then the following are equivalent:

1. $X$ is extremal.

2. $X$ satisfy the two conditions below:

   a. the rank of the matrix $(C_i, C_j)_{i,j=1,\ldots,r}$ is equal to 8, where $\{C_i; i = 1,\ldots,r\}$ is the finite set of $(-2)$-curves on $X$; a $(-2)$-curve is a smooth rational curve of self-intersection $-2$.

   b. There exist $r$ strictly positive rational numbers $a_i$, $i = 1,\ldots,r$ such that $-K_X = \sum_{i=1}^{r} a_i C_i$.

This paper is organized as follows. In most cases $X$ denotes a smooth projective rational surface defined over the field of complex numbers $\mathbb{C}$ such that its anti-canonical class is nef and of self-intersection zero. In Sect. 2 we introduce some well-known facts about smooth rational surfaces. In Sect. 3 we prove that the existence of a family of $(-2)$-curves on $X$ such that its elements are linear dependents in the tensor product $NS(X) \otimes \mathbb{Q}$ of the Néron-Severi group $NS(X)$ of $X$ and the field of rational numbers $\mathbb{Q}$ over the ring of integers $\mathbb{Z}$ gives an elliptic structure to $X$ (see proposition 3.5, p.10); As a corollary, the number of $(-2)$-curves on $X$ is finite and bounded by the optimal integer 12 (see corollary 3.6, p.10). In Sect. 4 we give a useful criterion (see proposition 4.2, p.12) for the existence of an infinite number of $(-1)$-curves on the surface. In Sect. 5 we give the proof of theorem 1.1.

2 Preliminaries

In this section we fix our notations and gather some well-known general facts concerning smooth projective rational surfaces.

2.1 Notation and First Properties

Let $X$ be a smooth projective rational surface defined over the field of complex numbers $\mathbb{C}$. Recall that the Néron-Severi group $NS(X)$ of $X$ is the group of divisor classes on $X$ modulo algebraic equivalence. The following facts are well-known:
• $NS(X)$ is finitely generated abelian group of rank $\rho(X)$ ($\rho(X)$ is called the Picard number of $X$);

• $NS(X)$ is torsion free abelian group;

• $\rho(X) = 10 - K_X^2$, where $K_X$ represent a canonical divisor on $X$;

• $NS(X)$ is equipped with a symmetric bilinear form induced from the intersection form defined on the set of divisors $Div(X)$ on $X$; this symmetric bilinear form will be called from now on the intersection form on $NS(X)$;

• $NS_\mathbb{Q}(X)$ is by definition the tensor product of $NS(X)$ and the field of rational numbers $\mathbb{Q}$ over the ring of integers $\mathbb{Z}$, i.e., $NS_\mathbb{Q}(X) = NS(X) \otimes \mathbb{Z} \mathbb{Q}$;

• $NS_\mathbb{Q}(X)$ is equipped with a symmetric bilinear form induced by the intersection form on $NS(X)$; this symmetric bilinear form will be called from now on the intersection form on $NS_\mathbb{Q}(X)$.

If $D$ is a divisor on $X$, we adopt the following notations:

• $[D]$ its equivalence class in $NS(X)$;

• $\overline{D}$ is the unique element associated to $[D]$ in $NS_\mathbb{Q}(X)$.

By definition:

• $K_X$ (resp. $-K_X$) denotes a canonical (resp. anti-canonical) divisor on the surface $X$;

• $[K_X]$ (resp. $[-K_X]$) is the canonical (resp. anti-canonical) class of $X$;

• $K_X^\perp$ is the orthogonal of $K_X$ in $NS_\mathbb{Q}(X)$, and by abuse of notation, the orthogonal of $[K_X]$ in $NS(X)$ is also denoted by $K_X^\perp$.

The intersection form defined on $Div(X)$ (resp. on $NS(X)$, resp. on $NS_\mathbb{Q}(X)$) is noted by $\cdot$. In particular for any two divisors $D$ and $D'$ on $X$, the equalities hold: $D.D' = \overline{D}.\overline{D'} = [D].[D']$; if $D = D'$, the intersection number of $D$ and $D'$ is called the self-intersection of the divisor $D$ and is denoted by $D^2$.

Similarly for every element $x$ of $NS_\mathbb{Q}(X)$, the rational number $x.x$ is denoted by $x^2$ and we say that it is the self-intersection of $x$.

The following well-known lemma is useful:

**Lemma 2.1** Let $X$ be a smooth projective rational surface such that the self-intersection of its canonical divisor $K_X$ is zero. Then the intersection form is negative semi-definite on $K_X^\perp$; and for every element $x \in K_X^\perp$, the following equivalence hold:
\[ x^2 = 0 \text{ if and only if } x = rK_X \text{ for some rational number } r. \]

We give here an elementary proof. Let \( x \) be an element of \( K_X^+ \), we can suppose that \( x \) is the divisor class of a divisor \( D \) on \( X \). If \( D^2 \) is strictly larger than zero, then by using the index theorem ([4]), \( K_X \) would be either of self-intersection strictly less that zero or numerically trivial. these are impossible since \( X \) is rational and \( K_X^+ = 0 \).

Let \( A \) be an ample divisor on \( X \), and consider the divisor \( E \) defined as \( E = A.K_X D - A.DK_X \). We have \( E^2 = 0 = A.E \).

Hence, by the index theorem ([4]), the divisor \( A.K_X D \) is numerically equivalent to \( A.DK_X \). Since \( X \) is rational and \(-K_X \) is an effective divisor, we obtain that \( D = \frac{A.D}{A.K_X}K_X \).

### 2.2 Riemann-Roch theorem for Rational Surfaces

Let \( X \) be a smooth projective surface, its canonical divisor is denoted by \( K_X \). If \( D \) is a divisor on \( X \), then:

- \( O_X(D) \) is the invertible sheaf associated to \( D \);

- for \( i = 0, 1, 2 \) : \( H^i(X, O_X(D)) \) is the \( i \)th cohomology group of the invertible sheaf \( O_X(D) \);

- for \( i = 0, 1, 2 \) : \( h^i(X, O_X(D)) \) is the dimension of \( H^i(X, O_X(D)) \).

**Definition 2.2** A divisor \( D \) on a smooth projective surface \( X \) is effective if \( n_i \geq 0 \) for all \( i = 1, \ldots, r \), where \( r \) is a non-negative integer and where \( D = n_1C_1 + \ldots + n_rC_r \) (the \( C_i \), \( i = 1, \ldots, r \) are reduced irreducible curves on \( X \)).

An element of \( NS(X) \) is effective if it is the class of an effective divisor on \( X \).

A proof of the next lemma is given in [4, proposition 7.7, p. 157]

**Lemma 2.3** An element \( [D] \) of \( NS(X) \) is effective if and only if the integer \( h^0(X, O_X(D)) \) is not equal to zero.

It is easy to see that:

**Lemma 2.4** If \( D \) is an effective divisor on a smooth projective rational surface \( X \), then the integer \( h^2(X, O_X(D)) \) is equal to zero.

**Definition 2.5** Let \( D \) be a divisor on a smooth projective surface \( X \). The Euler-Poincaré characteristic \( \chi(D) \) of \( D \) is the following integer:

\[
\chi(D) = h^0(X, O_X(D)) - h^1(X, O_X(D)) + h^2(X, O_X(D)),
\]

where \( O_X(D) \) is the invertible sheaf associated to \( D \). This integer is also called the Euler-Poincaré characteristic of the sheaf \( O_X(D) \) (resp. of the class \( [D] \) of \( D \) in the Néron-Severi group \( NS(X) \) of \( X \)) associated to the divisor \( D \).
The Riemann-Roch theorem is:

**Theorem 2.6** Let $D$ be a divisor on a smooth projective rational surface $X$, the Euler-Poincaré characteristic $\chi(D)$ of the divisor $D$ is given by:

$$\chi(D) = 1 + \frac{1}{2}(D^2 - K_X.D)$$

**Definition 2.7** A smooth projective rational surface is anti-canonical if the anti-canonical class of this surface is effective.

It is easy to see:

**Lemma 2.8** Every smooth rational surface for which the self-intersection of its canonical class is larger or equal to zero is anti-canonical.

The adjunction formula is:

**Lemma 2.9** Let $Y$ be a smooth projective surface. For every irreducible curve $C$ on $Y$, the equality hold:

$$C^2 + C.K_Y = 2p_a(C) - 2,$$

where $K_Y$ is a canonical divisor on $Y$ and $p_a(C)$ is the arithmetic genus of the curve $C$.

**Definition 2.10** Let $X$ be a smooth projective surface.

- A divisor $D$ on $X$ is nef if and only if $D.C \geq 0$ for every irreducible curve $C$ on $X$.
- An element of $NS(X)$ is nef if it is the class of a nef divisor on $X$.

Using the Riemann-Roch theorem and the adjunction formula, one can obtain:

**Lemma 2.11** Let $X$ be a smooth projective rational surface such that its anti-canonical class is nef. The self-intersection of a reduced irreducible curve $C$ on $X$ of self-intersection non-positive is either $-2$ or $-1$; and the following hold:

- $C^2 = -2$ if and only if $C$ is a smooth rational curve of self-intersection $-2$.
- $C^2 = -1$ if and only if $C$ is a smooth rational curve of self-intersection $-1$.

If $T$ is a reduced irreducible curve on the smooth rational surface $X$ such that $T$ is not orthogonal to canonical divisor on $X$, then the following properties are valid:

**Lemma 2.12** With the same notations as above.
\(i-\) \(NS_{\mathbb{Q}}(X) = \mathbb{Q}T \oplus K_X^1\), in particular if \(T\) is a smooth rational curve of self-intersection \(-1\) on \(X\), then we have: \(NS(X) = \mathbb{Z}[T] \oplus K_X^1\).

\(ii-\) the dimension of the \(\mathbb{Q}\)-vector space \(K_X^1\) is \(9 - K_X^2\).

Proof. \([i-]\) Let \(x \in NS_{\mathbb{Q}}(X)\):

- if \(x \in \mathbb{Q}T \cap K_X^1\), we have:
  \[
  \begin{cases}
    x.K_X = 0; \\
    x = \alpha T,
  \end{cases}
  \text{ for some } \alpha \in \mathbb{Q}.
  
  Hence \(0 = \alpha \overline{T}.K_X^1\) and consequently \(\alpha = 0\).

  It follows that \(\mathbb{Q}T \cap K_X^1 = \{0\}\);

- on the other hand, we have:
  \[
  x - \frac{K_X.x}{K_X.T}T \in K_X^1,
  \]
  since
  \[
  \frac{K_X}{K_X.T}(x - \frac{K_X.x}{K_X.T}T) = K_X.x - \frac{K_X.x}{K_X.T}K_X.T = K_X.x - K_X.x = 0.
  
  In conclusion, \(NS_{\mathbb{Q}}(X) = \mathbb{Q}[T] \oplus K_X^1\).

\([ii-]\) is a consequence of \([i-]\) and the fact that \(\rho(X) = 10 - K_X^2\).

2.3 Some linear Algebra Results

We give here some linear algebra results that we will use in section.

**Definition 2.13** Let \(n\) be a non-negative integer. Consider two \(n\)-tuple \(Z(t) = (z_1(t), \ldots, z_n(t))\) and \(Z'(t) = (z'_1(t), \ldots, z'_n(t))\) of real numbers (resp., \(Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}\) and \(Z'(t) = \begin{pmatrix} z'_1(t) \\ \vdots \\ z'_n(t) \end{pmatrix}\)) of real numbers.

We write \(Z(t) \leq Z'(t)\) if and only if the inequality \(z_i(t) \leq z'_i(t)\) hold for all \(i = 1, \ldots, n\).

**Definition 2.14** Let \(n\) be a non-negative integer, A family \((Z(t))_{t \in I}\) of \(n\)-tuple \(Z(t) = (z_1(t), \ldots, z_n(t))\) (where \(I\) is non empty set) with coefficients in \(\mathbb{Q}\), or in \(\mathbb{R}\), is bounded if there exists some scalar \(b\) such that for every \(i\) in \(\{1, \ldots, n\}\) and for every \(t\) in \(I\), the inequality hold: \(|z_i(t)| \leq b\), where \(|x|\) denotes the absolute value of the scalar \(x\).

The next lemma is elementary and the proof is omitted.

**Lemma 2.15** Let \(A = (a_{i,j})\) be an invertible matrix of order \(n\) with coefficients in \(\mathbb{Q}\), or in \(\mathbb{R}\). For every family \((Z(t))_{t \in I}\) (where \(I\) is a non empty set), the following assertions are equivalents:

1. the family \((Z(t))_{t \in I}\) is bounded.
2. the family \((AZ(t))_{t \in I}\) is bounded.

**Corollary 2.16** Let \(Z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix}\) be a family of real numbers. and let \(A = (a_{i,j})\) be a square matrix of order \(n\) with coefficients in \(\mathbb{Q}\), or in \(\mathbb{R}\), having the following shape:

\[
A = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}
\]

(where \(D\) is an invertible matrix of order \(n - 1\)).

1. If the family \((Z(t))_{t \in I}\) is bounded, then \((AZ(t))_{t \in I}\) is bounded too.

2. The converse is true if there exist some scalars \(d_1, \ldots, d_n\) such that:

   a. \(d_n \neq 0\);
   
   b. for every \(t\) in \(I\), the following equality hold:

   \[
z_1(t)^2 + \cdots + z_{n-1}(t)^2 - \sum_{1 \leq i < j \leq n-1} a_{i,j} z_i(t)z_j(t) = d_1z_1(t) + \cdots + d_{n-1}z_{n-1}(t) + d_nz_n(t).
   \]

**Proof.** Let \(Y(t) = AZ(t)\). Then

\[
Y(t) = \begin{pmatrix} DZ'(t) \\ 0 \end{pmatrix}
\]

If we write for every \(t \in I\), \(Y(t)\) as \(Y(t) = \begin{pmatrix} Y'(t) \\ y_n(t) \end{pmatrix}\) and \(Z(t)\) as \(Z(t) = \begin{pmatrix} Z'(t) \\ z_n(t) \end{pmatrix}\), we have for every \(t \in I\), \(Y'(t) = DZ'(t)\) and \(y_n(t) = 0\).

The fact that \((Z(t))_{t \in I}\) is bounded shows that \((Z'(t))_{t \in I}\) is bounded too; Since \(D\) is invertible, the lemma 2.15 shows that \((DZ'(t))_{t \in I}\) is bounded, that is \((Y'(t))_{t \in I}\) is bounded, it follows then \((Y(t))_{t \in I}\) is bounded (since \(y_n(t) = 0\) for all \(t \in I\).

Conversely, if \((Y(t))_{t \in I}\) is bounded, then \((DZ'(t))_{t \in I}\) is bounded too, and by using the lemma 2.15, \((Z'(t))_{t \in I}\) is bounded.

The hypothesis a, and b confirm that the number \(z_n(t)\) is a function of \(z_1(t), \ldots, z_{n-1}(t)\). Hence the family \((z_n(t))_{t \in I}\) is bounded. In conclusion, \((Z(t))_{t \in I}\) is bounded.

3 **Criterion for a Rational Surface to Be Elliptic**

Let \(X\) be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero. Recall that a \((-2)\)-curve on \(X\) is a smooth rational curve of self-intersection \(-2\).

In this section we prove that if \({C_1, \ldots, C_k}\) is a set of \((-2)\)-curves on \(X\) such that the vectors \(\{\overline{C_1}, \ldots, \overline{C_k}\}\) are linearly dependent in \(NS_{\mathbb{Q}}(X)\) (where \(k\) is a non-negative integer), then the surface \(X\) is elliptic. As a corollary, we prove that the number of \((-2)\)-curves on \(X\) is bounded.
by 12; and this bound is optimal.

Recall some definitions:

Let \( f : X \rightarrow B \) be a relatively minimal elliptic surface \( X \) over a smooth curve \( B \) of genus \( g \). By this we always mean the following:

\( X \) is a smooth projective surface defined over \( \mathbb{C} \) equipped with an elliptic relatively minimal fibration \( f : X \rightarrow B \). That is, \( f \) is a surjective morphism satisfying the following conditions:

1. Almost all the fibres are elliptic curves;
2. Each fibre does not contain a \((-1)\)-curve as a component.

If \( D = n_1C_1 + \cdots + n_pC_p \) is the decomposition of the divisor \( D \) in reduced irreducible components \( C_1, \ldots, C_p \), then it will be called multiple of multiplicity \( m \) if and only if the greatest common divisor of the integers \( n_1, \ldots, n_p \) is equal to \( m \) and \( m \geq 2 \).

It follows that if \( D \) is multiple of multiplicity \( m \), then \( D = mD' \) where \( m \) is the multiplicity of \( D \) and \( D' \) is a divisor which is not multiple.

We denote by:

- \( \chi \) the Euler characteristic of the surface \( X \) \((\chi = \chi(O_X))\);
- \( F \) a general fiber of \( f \);
- \( F_b \) the fiber of \( f \) over the point \( b \in B \);
- \( F_{b_1}, \ldots, F_{b_r} \) the multiple fibers of \( f \) of multiplicity \( m_1, \ldots, m_r \) respectively.

Recall that the Néron-Severi group \( NS(X) \) of \( X \) is the group of divisor classes on \( X \) modulo algebraic equivalence. If \( C \) is a divisor on \( X \), we denote by \([C]\) its class in \( NS(X)\) and by \( \overline{C} \) the element associated to \([C]\) in \( NS_{\mathbb{Q}}(X) \), where \( NS_{\mathbb{Q}}(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{Q} \).

We have ([1, corollaire 12.3, p. 162]):

\[
K_X = (2g - 2 + \chi + r - \sum_{i=1}^{r} \frac{1}{m_i})F. \tag{1}
\]

We need the following lemma:

**Lemma 3.1** Let \( f : X \rightarrow B \) be an elliptic fibration of a smooth projective surface \( X \) over a smooth curve \( B \). Then the topological character of a reducible fibre is greater than or equal to the number of irreducible components of this fiber. With equality if and only if this fiber is of type \( I_n \) (where \( n \) is greater than or equal to 2).

Arnaud Beauville ([3, the lemma of the page 345]), showed that there exist some inequalities between the genus \( g \) of the curve \( B \) and the irregularity \( q \) of the surface \( X \) \((q = h^1(X, O_X), O_X \)
is the structure sheaf of $X$) mainly:

$$g \leq q \leq g + 1,$$

In particular if $X$ is rational, then $q = 0$ and the genus $g$ of the curve $B$ is necessarily zero. Thus $B$ is isomorphic to $\mathbb{P}^1$. This fact motivates the following definition:

**Definition 3.2** An elliptic rational surface is a relatively minimal elliptic rational surface over $\mathbb{P}^1$.

The following result is straightforward:

**Lemma 3.3** Let $X$ be an elliptic rational surface.

The anti-canonical class of $X$ is nef.

The following result is straightforward:

**Lemma 3.4** Let $X$ be an elliptic rational surface.

The anti-canonical class of $X$ is nef.

This is our main result in this section:

**Proposition 3.5** Let $X$ be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero.

If there exists a family of $(-2)$-curves on $X$ such that its elements are linearly dependents in $NS_{\mathbb{Q}}(X)$, then the surface $X$ is elliptic.

**Proof.** Let $\{C_1, \ldots, C_k\}$ be a family of $(-2)$-curves on $X$ such that its elements are linearly dependents in $NS_{\mathbb{Q}}(X)$ ($k$ is a non-negative integer). We can suppose that there exist some non-negative integers $a_1, a_2, \ldots, a_k$ such that:

$$\Sigma_{i=1}^{r} a_i [C_i] = \Sigma_{j=r+1}^{k} a_j [C_j],$$

where $r$ is an integer satisfying $1 \leq r \leq k - 1$.

Define $C$ and $C'$ as:

$$C = \Sigma_{i=1}^{r} a_i C_i \text{ and } C' = \Sigma_{j=r+1}^{k} a_j C_j.$$

We have: $0 \geq C^2 = C.C' \geq 0$, then $C^2 = 0 = C'.^2$ and $C$ is linearly equivalent to $C'$.

The complete linear system $[C]$ is without fixed components, without base points and by using Stein factorisation gives an elliptic fibration $f : X \dasharrow \mathbb{P}^1$ such that $f$ is relatively minimal.

An immediate consequence is:

**Corollary 3.6** Let $X$ be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero.

If the number of $(-2)$-curves on $X$ is greater than or equal to ten, then $X$ is elliptic.
Nagata [10, lemme 6.1, p. 292] proved that if \( X \) is the blowing-up nine points of the projective plane, then the set of \((-2)\)-curves is finite. Here we give another proof of his result in the more general context of smooth projective rational surfaces \( X \) such \( K_X^2 = 0 \), where \( K_X \) is a canonical divisor of the surface. This result is a consequence of the proposition above.

**Corollary 3.7** Let \( X \) be a smooth projective rational surface such that the self-intersection of its canonical divisor is zero. Then the number of \((-2)\)-curves is less than or equal to 12, and this bound is optimal.

**Proof.** Consider a family \( \{ C_1, C_2, \ldots, C_k \} \) of \((-2)\)-curves on \( X \), where \( k \) is an integer. Two disjoint possibilities may occur:

If the vectors \( C_1, C_2, \ldots, C_k \) are linearly independants in \( NS(Q)(X) \) which is of dimension 10, then \( k \leq 9 \).

If not, \( X \) admits a relatively minimal elliptic fibration \( f : X \to \mathbb{P}^1 \). Recall the equality (see [2] lemme VI.4 page 95):

\[
\chi_{\text{top}}(X) = \chi_{\text{top}}(B)\chi_{\text{top}}(F) + \sum_{b \in B}(\chi_{\text{top}}(F))
\]

Here, we have:

- \( B = \mathbb{P}^1 \);
- \( \chi_{\text{top}}(X) \) is the topological character of \( X \);
- \( \chi_{\text{top}}(F) \) is the topological character of \( F \);
- \( F_b \) is the fiber of \( f \) over the point \( b \in B \);
- \( \chi_{\text{top}}(F_b) \) is the topological character of \( F_b \) for \( b \in B \).

If \( \chi \) is the Euler characteristic of the surface \( X \), the Noether formula is:

\[
12\chi = K_X^2 + \chi_{\text{top}}(X) \quad (\chi = \chi(O_X)).
\]

Since \( K_X^2 = 0 \) and \( \chi_{\text{top}}(F) = 0 \) (since \( F \) is an elliptic curve) we have:

\[
12\chi = \sum_{b \in B}\chi_{\text{top}}(F_b).
\]

We denote by \( F_1, F_2, \ldots, F_t \) the reducible fibers of \( f \). We have for every \( i = 1, 2, \ldots, t \):

\[
l_i \leq \chi_{\text{top}}(F_i),
\]

where \( l_i \) is the number of irreducible components of \( F_i \).
Which gives
\[ \sum_{i=1}^{l} l_i \leq 12\chi. \]

Hence the number of \((-2)\)-curves is less or equal than \(12\chi = 12\) (since \(X\) is rational).

The number of \((-2)\)-curves of the extremal jacobian elliptic rational surface \(X_{3333}\) ([9, Tableau 5.3, page 549]) is 12, which ends the proof.

4 Some Rational Surfaces Having an Infinite Number of \((-1)\)-Curves

In what follows a \((-1)\)-divisor \(D\) on a smooth projective surface \(Z\) is a divisor on \(Z\) such that \(D^2 = -1 = D.K_Z\), where \(K_Z\) is a canonical divisor on \(Z\). We need the following result:

**Proposition 4.1** Let \(X\) be a smooth projective rational surface such that \(K_X^2 = 0\). Suppose that the intersection form is negative definite on the space spanned over the field of rational numbers by the \(\overline{C_i}, i = 1, \ldots, l\), where the \(C_i\) are \((-2)\)-curves on \(X\).

Then there exists an infinite number of \((-1)\)-divisors such that each \((-1)\)-divisor intersect positively each \(C_i\), \(i = 1, \ldots, l\).

**Proof.** The hypothesis “the intersection form is negative definite on the space spanned by the \(\overline{C_i}, i = 1, \ldots, l\)” shows that the vectors \(\{\overline{C_i}, i = 1, \ldots, l\} \) are linearly independents in the orthogonal of the canonical divisor of \(X\). In particular, \(l\) is less than or equal to 8. For a large integer \(n\) (e.g., \(n \geq E_0.C_j\) for all \(j = 1, \ldots, l\), where \(E_0\) is a fixed \((-1)\)-curve), we consider the unique divisor \(\Delta_n\) whose support is in the set of \((-2)\)-curves \(C_i, i = 1, \ldots, l\) and defined by:
\[ \Delta_n.C_j = |d|(n - E_0.C_j) \]
for each \(j = 1, \ldots, l\), where \(d\) is the determinant of the matrix \((C_i.C_j)_{1 \leq i, j \leq l}\) (\(|d|\) is the absolute value of \(d\)).

The divisor \(E_n\) defined for large \(n\) by:
\[ E_n = E_0 + \Delta_n - (2E_0.\Delta_n + \Delta_n^2)(-K_X) \]
is a \((-1)\)-divisor intersecting positively each \(C_i, i = 1, \ldots, l\). And for \(E_n \neq E_m\) for \(n \neq m\).

**Proposition 4.2** Let \(X\) be a smooth projective rational surface such that the anti-canonical class is nef and of self-intersection zero.

If the intersection form is negative definite on the space spanned by \(\overline{C_i}\), where the \(C_i\) constitute a connected component of the set of all \((-2)\)-curves on \(X\), then the number of \((-1)\)-curves on \(X\) is infinite.
Proof. By hypothesis we can find a connected component \( \{C_1, \ldots, C_l\} \) of the family of all \((-2)\)-curves on \( X \) such that the intersection form is negative definite on the space spanned by the \( C_i, i = 1, \ldots, l \). An application of the proposition above yields to the existence of an infinite family \( E_n \) of \((-1)\)-divisors such that \( E_n.C_i \geq 0 \). By construction the element \( E_n \) is equal to

\[
E_n = E_0 + \Delta_n - (2E_0.\Delta_n + \Delta_n^2)(-K_X)
\]

where the support of \( \Delta_n \) is in the set \( C_i, i = 1, \ldots, l \) and \( E_0 \) is a fixed \((-1)\)-curve on \( X \).

For each \((-2)\)-curve \( C \) different from \( C_i, i = 1, \ldots, l \), we have \( C.C_i = 0 \) for all \( i = 1, \ldots, l \); then \( C.E_n = C.E_0 \geq 0 \). Consequently \( E_n \) intersect every \((-2)\)-curve on \( X \) positively. Hence by [7, Theorem, p.3], \( E_n \) is irreducible.

5 Proof of Theorem 1.1

Let \( s \) be the rank of the matrix \((C_i, C_j)_{i,j=1,\ldots,r}\), where \( \{C_i; i = 1, \ldots, r\} \) is the finite set of \((-2)\)-curves on \( X \).

Suppose that \( X \) is extremal, we claim that \( s = 8 \).

Proof of the claim:

If \( s = 0 \), i.e., \( X \) has no \((-2)\)-curves, then the nine points are in general position and therefore \( X \) has an infinite number of \((-1)\)-curves.

If \( s \in \{1, \ldots, 7\} \), then we can find a divisor \( D \) such that the following conditions hold:

1. \( D^2 < 0 \),
2. \( D.K_X = 0 \),
3. \( D.C_i = 0 \), for each \( i = 1, \ldots, r \).

Then fix a \((-1)\)-curve \( E_0 \), and for each integer \( n \), consider the \((-1)\)-divisor \( E_n \) defined as follows:

\[
E_n = E_0 + nD - (nE_0.D + \frac{n^2}{2} D^2)(-K_X).
\]

By [7, Theorem, p.3] \( E_n \) is a \((-1)\)-curve. Two distinct integers \( n \) and \( m \) give two distinct \((-1)\)-curves \( E_n \) and \( E_m \); so \( X \) has an infinite number of \((-1)\)-curves.

Hence \( s = 8 \) is a necessarily condition in order that \( X \) to be extremal.

In what follows \( s \) is equal to 8. We will distinguish between two cases:

1. **First Case:** If the vector space over the field of rational numbers spanned by the set of \((-2)\)-curves is not equal to the orthogonal of the canonical divisor, then according to the proposition 4.2 ( page 12), \( X \) will have an infinite number of \((-1)\)-curves.

2. **Second Case:** If the vector space over the field of rational numbers spanned by the set of \((-2)\)-curves is equal to the orthogonal of the canonical divisor, then we will distinguish between the case where \( X \) is elliptic or not:
**First subcase:** If $X$ is not elliptic, then according to proposition 3.5 (10), the $(2)$-curves are linearly independents over the field of rational numbers; so the number of $(2)$-curves on $X$ is nine. Two cases come:

- if the set of $(2)$-curves is connected, then $-K_X$ is a linear combination of strictly positive integers of all $(2)$-curves.
- If the set of $(2)$-curves is not connected, then $-K_X$ is not a linear combination of strictly positive rational numbers of all $(2)$-curves. Applying proposition 4.2 (p. 12) to any connected component of the set of $(2)$-curves for which the intersection form is negative definite gives the fact that $X$ contains an infinite number of $(1)$-curves.

**Second subcase:** If $X$ is elliptic, then $-K_X$ is a linear combination of strictly positive rational numbers of all $(2)$-curves.

In conclusion if $X$ is extremal, then it satisfies the two conditions below:

a. the rank of the matrix $(C_i, C_j)_{i,j=1,\ldots,r}$ is equal to 8, where $\{C_i ; i = 1, \ldots, r\}$ is the finite set of $(2)$-curves on $X$;

b. There exist $r$ strictly positive rational numbers $a_i, i = 1, \ldots, r$ such that $-K_X = \sum_{i=1}^{r} a_i C_i$.

Conversely, the fact that the rank of the matrix $(C_i, C_j)_{i,j=1,\ldots,r}$ is equal to eight enables us to suppose for example that the vectors $\overline{C}_1, \ldots, \overline{C}_8$ are linearly independents in $NS_{\Omega}(X)$ and the intersection form (induced by the intersection form on $K_X^1$) is negative definite on the vector space over the field of rational numbers spanned by the vectors $\overline{C}_1, \ldots, \overline{C}_7$.

Since $-K_X$ is not trivial and of self-intersection zero, the vectors $\overline{C}_1, \ldots, \overline{C}_8$ and $-K_X$ are linearly independents in $NS_{\Omega}(X)$, in particular

- the elements $[C_1], \ldots, [C_8]$ and $[-K_X]$ are linearly independents in $NS(X)$;
- the vector space $< \overline{C}_1, \ldots, \overline{C}_8, -K_X >$ is equal to $K_X^1$.

It is easy to prove the following lemma:

**Lemma 5.1** Let $E$ be a $(1)$-curve on $X$. Then the equality $-K_X = \sum_{i=1}^{r} a_i \overline{C}_i$ gives rise the inequalities:

$$0 \leq E.C_i \leq \frac{1}{a_i}$$

for every $i = 1, \ldots, r$.

Let $E_0$ be a fixed $(1)$-curve on $X$ and consider a $(1)$-curve $E$ on $X$. There exist some rational numbers $a_1(E), \ldots, a_9(E)$ such that:

$$E = E_0 + a_1(E)\overline{C}_1 + \cdots + a_9(E)\overline{C}_8 + a_9(E)(-K_X).$$
We claim that:
the numbers \( \alpha_1(E), \ldots, \alpha_9(E) \) may be taken such that their denominators are bounded.
In fact, let \( M \) be the lattice of \( NS(X) \) defined by the free family
\( \{[C_1], \ldots, [C_8], [-K_X]\} \), i.e., \( M = \mathbb{Z}C_1 \oplus \ldots \oplus \mathbb{Z}C_8 \oplus \mathbb{Z}(-K_X) \). The set \( \{E - E_0 / E \text{ est une courbe } (-1) \text{ sur } X \} \) is a subset of \( NS(X) \) and \( K_X^\vee \); if \( M^+ \) denotes \( \{x \in NS(X) / \text{ pour un certain entier naturel non nul } n, nx \in M\} \), then \( M^+ = NS(X) \cap K_X^\vee \). Since \( \frac{NS(X)}{M} \) is a finitely generated group, the group \( (\frac{NS(X)}{M})_t \) whose elements are the torsion elements of \( \frac{NS(X)}{M} \) is finite ([8, Theorem 8.5, p. 46]). Since \( \frac{NS(X)}{M} = \frac{M^+}{M} \), there exists \( N \in \mathbb{N}, N \neq 0, \forall x \in M^+ N x \in M \); in particular there exists some non-negative integer \( N \) such that for every \( (-1) \)-curve \( E N(E - E_0) \in \mathbb{Z}C_1 \oplus \ldots \oplus \mathbb{Z}C_8 \oplus \mathbb{Z}(-K_X) \).

According to the lemma 5.1 (p. 14) we have for all \( i = 1, 2, \ldots, 9 \):

\[
-E_0.C_i \leq \sum_{j=1}^{j=8} C_i.C_j \alpha_j(E) \leq \frac{1}{a_i} - E_0.C_i.
\]

In matrix form:

\[
\begin{pmatrix}
-E_0.C_1 \\
\vdots \\
-E_0.C_8 \\
0
\end{pmatrix} \leq \begin{pmatrix}
D & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_8 \\
\alpha_9
\end{pmatrix} \leq \begin{pmatrix}
\frac{1}{a_1} - E_0.C_1 \\
\vdots \\
\frac{1}{a_8} - E_0.C_8 \\
0
\end{pmatrix}.
\]

where \( D = (C_i.C_j)_{1 \leq i, j \leq 8} \).

The equality \( E^2 = -1 \) gives

\[
\alpha_1(E)^2 + \cdots + \alpha_8(E)^2 - \sum_{1 \leq i < j \leq 8} C_i.C_j \alpha_i(E) \alpha_j(E) = E_0.C_1 \alpha_1(E) + \cdots + E_0.C_8 \alpha_8(E) + \alpha_9(E).
\]

Since \( D \) is a negative definite square matrix of order eight, it is invertible and an application of corollary 2.16 (p. 8) gives the fact that \( (\alpha_1(E), \ldots, \alpha_8(E), \alpha_9(E))_E \) is bounded.

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