MOND—theoretical aspects

Mordehai Milgrom

Department of Condensed Matter Physics, Weizmann Institute, Rehovot, Israel

I discuss open theoretical questions pertaining to the modified dynamics (MOND)—a proposed alternative to dark matter, which posits a breakdown of Newtonian dynamics in the limit of small accelerations. In particular, I point the reasons for thinking that MOND is an effective theory—perhaps, despite appearance, not even in conflict with GR. I then contrast the two interpretations of MOND as modified gravity and as modified inertia. I describe two mechanical models that are described by potential theories similar to (non-relativistic) MOND: a potential-flow model, and a membrane model. These might shed some light on a possible origin of MOND. The possible involvement of vacuum effects is also speculated on.

Key words: Cosmology, the dark matter problem, galaxy dynamics

1 Introduction—the modified dynamics

MOND is a modification of Newtonian dynamics in a form that obviates the need for dark matter, when applied to galactic systems. It does this by introducing a constant with the dimensions of an acceleration, $a_0$, and positing that standard Newtonian dynamics is a good approximation only for accelerations that are much larger than $a_0$. The exact behavior in the opposite limit is described by specific underlying theories like those described below. However, the basic point of MOND, from which follow most of the main predictions, can be encapsulated in the following approximate relation: a test particle at a distance $r$ from a mass $M$ is subject to the acceleration $a$ given by

$$a^2/a_0 \approx MGr^{-2},$$

when $a \ll a_0$, instead of the standard expression $a = MGr^{-2}$, which holds when $a \gg a_0$. Or, somewhat more generally, if $a_N$ is the Newtonian expression...
for the acceleration, then for $a_N \ll a_0$

$$a \approx (a_N a_0)^{1/2},$$

(2)

instead of $a \approx a_N$, which holds for $a_N \gg a_0$. The two expressions may be interpolated to give the heuristic relation

$$\mu(a/a_0)a \approx a_N,$$

(3)

where the interpolating function $\mu(x)$ satisfies $\mu(x) \approx 1$ when $x \gg 1$, and $\mu(x) \approx x$ when $x \ll 1$. This expression, while lacking from the formal point of view, is very transparent, and captures the essence of MOND. I shall describe below more presentable theories based on this basic relation, but these are still phenomenological theories into which the form of $\mu(x)$ has to be put in by hand. It will hopefully follow one day from a more basic underlying theory for MOND, which we still lack. Most of the implications of MOND do not depend strongly on the exact form of $\mu$. Much of the phenomenology pertinent to the mass discrepancy in galactic systems occurs in the deep-MOND regime ($a \ll a_0$), anyway, where we know that $\mu(x) \approx x$.

2 MOND phenomenology

The phenomenology dictated by MOND, and its application and testing in galactic systems: galaxies of all sorts, galaxy groups, clusters, and superclusters, is discussed in recent reviews; e.g. Milgrom (1998) and Sanders & McGaugh (2002). Here I only touch briefly on a few general aspects of the phenomenology.

It is important to recognize the message that the phenomenological success of MOND would carry. A sentiment is expressed occasionally that MOND—successful as it may be—is only a hypothesis that “saves the phenomena”; that MOND phenomenology might one day be understood within the dark-matter doctrine [e.g., Kaplinghat & Turner (2002), but see Milgrom (2002c)]. To be sure, this is still a far cry; and to appreciate how tall an order it would be, we note that the MOND idea, if it is taken just as the distillation of DM phenomenology, entails not one, but many independent laws that govern the mass discrepancy in galactic systems: the analog of Kepler laws in the solar system. Some of these laws involve $a_0$, and some do not. In those that do, $a_0$ appears in several independent roles if viewed just as a phenomenological parameter. (In the framework of MOND $a_0$ appears in two roles: the borderline acceleration between the MOND and the Newtonian regime, and a bench-mark acceleration deep in the MOND regime.)
Here are some of the laws:

1. Galaxy rotation curves are asymptotically flat.

2. The asymptotic rotational velocity of a galaxy is proportional to the fourth root of its total baryonic mass; the baryonic Tully-Fisher relation. The proportionality constant is $\left(Ga_0\right)^{1/4}$ (the role of $a_0$ as a deep-MOND parameter).

3. Galaxies with high central surface mass densities—corresponding to accelerations larger than $a_0$—should show no mass discrepancy in the inner parts. The discrepancy should appear only beyond a certain radius but always when the acceleration becomes comparable with $a_0$. (This is $a_0$ in its role as borderline acceleration.)

4. In LSB galaxies, which have low surface density everywhere, the mass discrepancy should start right from the center, and its magnitude is given by the inverse of the acceleration in units of $a_0$ ($a_0$ as a deep-MOND fiducial acceleration).

5. Self-gravitating, quasi-isothermal spheres cannot have mean accelerations much exceeding $a_0$ (or mean surface densities much exceeding $\sim a_0G^{-1}$)–with implications for round galactic systems, e.g. underlying the Fish law for elliptical galaxies ($a_0$ as a borderline acceleration).

Laws 1-4 are special cases of the sweeping law:

6. The distribution of the visible (baryonic) matter in every galaxy tightly determines the acceleration field of a galaxy and thus the distribution of the DM. The relation between the two mass distributions is given by eq. (3), where $a_0$ appears in both roles. This should hold despite the very different formation-evolution-interaction histories that the different galaxies have undergone.

Analogous laws hold for galactic systems other than galaxies. And they should be counted as separate laws in the framework of the DM picture. The fact that these systems look so different on the sky tells us that their baryonic component, at least, have undergone very different histories, so there is no reason, for example, why law 6 above would carry from one system type to another.

The above laws are phenomenologically independent in the sense that in the framework of the dark matter paradigm it is easy to conceive of baryon-plus-dark-matter galactic systems that obey any set of these laws, while breaking the others (? see a more detailed discussion with examples in) milkt. This is similar, for example, to the existence of different quantum phenomena; e.g., the black-body spectrum, the photoelectric effect, the hydrogen spectrum, superconductivity, etc., in which the Planck constant appears in different roles.
Without the quantum theory, these all seem unrelated. MOND is, likewise, a theory that unifies all the above laws of galaxy dynamics phenomenology.

The strength of the case for MOND is further augmented by noting that, historically, these laws were not arrived at by distillation of existing data. They were, most of them, pure predictions of MOND. The only input in the construction of MOND was the hypothesized asymptotic velocity behavior in disc galaxies, to wit, laws 1 and 2 above taken as axioms, and assumed only for disc galaxies. And even these two were not established at the time to the extent they are today.

The second general point I want to make concerns a possibly very significant coincidence involving the actual value of $a_0$. The value that fits the data discussed above is about $10^{-8}\text{cm s}^{-2}$. This value of $a_0$ is of the order of some acceleration constants of cosmological significance. It is very nearly $a_{ex} \equiv c\dot{H}$, where $\dot{H} \equiv H_0/2\pi$ ($H_0$ is the Hubble constant) and, it is also of the order of $a_{cc} \equiv c(\Lambda/3)^{1/2}$, where $\Lambda$ is the emerging value of the cosmological constant (or “dark energy”).

Because the cosmological state of the universe changes, such a connection, if it is a lasting one, may imply that galaxy evolution does not occur in isolation, affected only by nearby objects, but is, in fact, responding constantly to changes in the state of the universe at large. For example, if the connection of $a_0$ with the Hubble constant always holds, the changing of the Hubble constant would imply that $a_0$ must change over cosmic times, and with it the appearance of galactic systems, whose dynamics $a_0$ controls. If, on the other hand, $a_0$ is a reflection of a true cosmological constant, then is might be a veritable constant.

### 3 MOND as an effective theory

The above proximity of $a_0$ to the cosmological acceleration scale, beyond its phenomenological significance, may hint at a deep connection between cosmology and local dynamics in systems that are very small on cosmological scales. Either cosmology somehow enters and affects local laws of physics, such as the law of inertia or the law of gravity, or a common agent affects both cosmology and local physics so as to leave on them the same imprint. This would mean that MOND—and perhaps more cherished notions, such as inertia—is a derived concept: an effective theory. An observed relation between seemingly unrelated constants appearing in a theory (in our case, $a_0$, the speed of light, and the radius of the horizon) may indicate that MOND is only an approximation of a theory at a deeper stratum, in which some of the constants do not really have any special role. For example, in experiments and observa-
tions confined to the vicinity of the earth’s surface, the free-fall acceleration, \( g \) attains the status of a “constant of nature”. It is numerically related to two other important “constants”: the escape speed \( c_e \) (objects thrown with a higher velocity never return) and the radius of the earth \( R_\oplus \). (This relation, \( g = \frac{c_e^2}{2R_\oplus} \), is practically the same as that between \( a_0 \), the speed of light, and the Hubble radius, in MOND.) But looking beyond the surface, and knowing about universal gravity, we know that all these “constants” actually derive from the mass and radius of the earth (hence the relation between the three). They are useful parameters when describing near surface phenomena, but quite useless in most other circumstances. In a similar vein, \( a_0 \) might turn out to be a derived constant, perhaps variable on cosmic time scales, perhaps even of no significance beyond the non-relativistic regime, where MOND has been applied so far. Its connection with the speed of light and the radius of the universe will, hopefully, follow naturally in the underlying theory that still eludes us.

Many instances of such effective theories are known. Even General Relativity is now thought to be an effective, low-energy approximation of a higher theory (e.g. a string-inspired theory); an idea that has been anticipated by Sakharov’s “induced gravity” idea.

4 Interpretations

Equations (1)-(3) have the form of a modification of the law of inertia, but since they are algebraic relations between the MOND and Newtonian accelerations they can simply be inverted to read \( a = \frac{F}{m} = a_N f(a_N/a_0) \), which seems to leave the second law intact, while modifying the Newtonian gravitational force \( ma_N \) to the MOND value \( ma \). Because gravitation is the sole force that governs galactic dynamics—the only corner where the mass discrepancy has been clearly observed—existing phenomenology does not distinguish well between the interpretations of MOND as modified gravity, and modified inertia. Although there are matter-of-principle differences between the two interpretations (see below) they pertain to observations that are not yet available. For now we must then investigate both options.

But what exactly is meant by “modifying gravity”, and “modifying inertia”? When dealing with pure gravity the distinction is not always clear. For example, the Brans-Dicke theory may be viewed as either. But when other interactions are involved, the distinction is clear. Obviously, modified inertia will enter the dynamics of systems even when gravity is negligible, unlike the case for modified gravity. Formally, the distinction might be made as follows. In a theory governed by an action principle we distinguish three part in the action: The pure gravitational part (for example, the Einstein-Hilbert action in GR),
the free action of the matter degrees of freedom (in GR it also encapsules their interaction with gravity), and the action of interactions between matter degrees of freedom (in GR they too engender sources for gravity). By “modifying gravity” I mean modifying the pure-gravity action; by “modifying inertia” I mean modifying the kinetic (free) matter actions.

To understand this definition remember that inertia is what endows the motion of physical objects (particles, fields, large bodies, etc.) with energy and momentum—a currency in the physical world. Motion itself is only of a descriptive value; inertia puts a cost on it. For each kind of object it tells us how much energy and momentum we have to invest, or take away, to change its state of motion by so much. This information is encapsuled in the kinetic action, which encodes the energy-momentum of the free degrees of freedom.

For example, take the standard, non-relativistic action for a system of particles interacting through gravity.

\[ S = S_\phi + S_k + S_{in} = -(8\pi G)^{-1} \int d^3r (\vec{\nabla} \phi)^2 + \sum_i (1/2) m_i \int dt v_i^2 - \int d^3r \rho(r)\phi(r). \] (4)

where \( \rho(r) = \sum_i m_i \delta(r - r_i) \), and \( m_i, r_i \) are the particle masses and positions. In GR, \( S_k \) and \( S_{in} \) are lumped together into the particle kinetic action \(- \int d\tau\)
(tau is the proper time of the particle).

Here, modifying gravity would mean modifying \( S_\phi \), while modifying inertia would entail changing \( S_k \).

5 MOND as modified gravity

An implementation of MOND as a non-relativistic modified gravity was discussed by Bekenstein & Milgrom (1984), who replaced the standard, free, potential action \( S_\phi \) in eq. (4) by an action of the form

\[ S_\phi = -(8\pi G)^{-1} a_0^2 \int d^3r F[ (\vec{\nabla} \phi)^2 / a_0^2]. \] (5)

This gives, upon variation on \( \phi \), the equation

\[ \vec{\nabla} \cdot [\mu(\mid\vec{\nabla} \phi\mid)\vec{\nabla} \phi] = 4\pi G \rho(r), \] (6)

where \( \mu(z) \equiv dF(y)/dy|_{y=z^2} \). This theory, since it is derived from an action that has all the usual symmetries, satisfies all the standard conservation laws.
Its various implications have been discussed in Bekenstein & Milgrom (1984), Milgrom (1986), Milgrom (1997), and others.

An important point to note is that this theory gives the desired center-of-mass motion of composite systems: Stars, star clusters, etc. moving in a galaxy with a low center-of-mass acceleration are made of constituents whose internal accelerations are much higher than $a_0$. If we look at individual constituents we see bodies whose total accelerations are high and so whose overall motion is very nearly Newtonian. Yet, their motion should somehow combine to give a MOND motion for the center of mass. This is satisfied in the above theory as shown in Bekenstein & Milgrom (1984). A similar situation exists in GR. Imagine a tightly bound system of black holes, moving in the weak field of a galaxy, say. While the motions of the individual components are highly relativistic, and are governed by a non-linear theory, we know that these motions combine to give a simple Newtonian motion for the center of mass.

This field equation, generically, requires numerical solution, but it is straightforward to solve in cases of high symmetry (spherical, cylindrical, or planar symmetry), where the application of the Gauss law to eq. (6) gives the exact algebraic relation between the MOND ($\mathbf{g} = -\nabla \phi$) and Newtonian ($\mathbf{g}_N = -\nabla \phi_N$) acceleration fields:

$$\mu \left( \frac{\mathbf{g}}{a_0} \right) \mathbf{g} = \mathbf{g}_N,$$

which is identical to the heuristic MOND relation we started with. Note that in general, for configurations of lower symmetry, this algebraic relation does not hold (and, in general, $\mathbf{g}$ and $\mathbf{g}_N$ are not even parallel).

It is worth pointing out that in such a modified-gravity theory, the deep-MOND limit corresponds to a theory that is conformally invariant, as discussed in Milgrom (1997). Whether this has some fundamental bearings is not clear, but it does make MOND unique, and enables one to derive useful analytic results, such as an expression for the two-body force, and a virial relation, despite the obstacle of nonlinearity.

5.1 mechanistic models

Inasmuch as MOND is still in need of an underlying theory it may be useful to study mechanistic models or analogues that reproduce similar phenomenology. These may help elucidate the origin of the nonlinearity in (nonrelativistic) MOND, and perhaps the appearance of the same acceleration constant in both local dynamics and cosmology.
There is a large number of physical phenomena that are governed by an equation like eq. (6), each with its own form of the function $\mu(x)$, as detailed, e.g., in Milgrom (1997), or Milgrom (2002b). By choosing the right underlying physics a form of $\mu$ may perhaps be found that will correspond to MOND behavior.

It is well known, for example, that a stationary, potential flow is described by the Poisson equation: If the velocity field $\mathbf{u}(\mathbf{r})$ is derived from a potential, $\mathbf{u} = \nabla \phi$, then the continuity equation, which here determines the flow, reads $\nabla \cdot \nabla \phi = s(\mathbf{r})/\rho_0$, where $s(\mathbf{r})$ is the source density, and $\rho_0$ is the (constant) density of the fluid. When the fluid is compressible, but still irrotational, and barotropic [i.e. has an equation of state of the form $p = p(\rho)$] the stationary flow is described by the nonlinear Poisson equation. The Euler equation reduces to Bernoulli’s law

$$h(\rho) = -u^2/2 + \text{const.},$$

(8)

where $dh/d\rho \equiv \rho^{-1}dp/d\rho$. This tell us that $\rho$ is a function of $u = |\nabla \phi|$. Substituting this in the continuity equation gives

$$\nabla \cdot [\rho(|\nabla \phi|)\nabla \phi] = s(\mathbf{r}),$$

(9)

which has the same form as eq. (6) if we identify $\rho$ as $\mu$, and the source density $s$ with the normalized gravitational mass density $4\pi G \rho$. Note, however, that from the Bernoulli law, $d\rho/d|u| = -\rho|u|/c^2$, where $c^2 = dp/d\rho$ is the formal squared speed of sound. Thus, in the case of MOND, where $\mu$ is an increasing function of its argument, the model fluid has to have a negative compressibility $c^2 < 0$. A cosmological-constant-like equation of state, $p = -\rho$, with $c$ the speed of light gives $\rho(u) = \rho_0 \exp(u^2/2c^2)$, which is not what we need for MOND. The deep-MOND limit, $\mu(u) \approx u/a_0$, corresponds to $p = -(a_0^2/3)\rho^3$. To get the Newtonian limit at large values of $u$ the equation of state has to become incompressible at some finite density $\rho_0$, so that eq. (9) goes to the Poisson equation. (Note that $p = -c^2\rho$, which is the relation between the energy density and pressure of the vacuum is not an equation of state to be applied to local distortions of the vacuum, which cannot be described as a fluid, in general.)

The gravitational force is then the pressure+drag force on sources. For a small (test) static source $s$, at a position where the fluid speed is $\mathbf{u}$, the source imparts momentum to the flow at a rate $s\mathbf{u}$, and so is subject to a force $-s\nabla \phi$. The force between sources of the same sign is attractive, as befits gravity. The fluid density itself $\rho$ does not contribute to the sources of the potential equation, so it does not, itself, gravitate. Also note that, because $\rho = p = 0$ for $\mathbf{u} = 0$, the fluid behaves as if it has no existence without the
sources (masses) that induce velocities in it. This picture is still far from being directly applicable as an explanation of Newtonian gravity. For example, it is not clear how to obtain the barotropic equation of state that is needed to reproduce MOND. In particular, how does the infinite compressibility appear at a finite critical density, and what is the meaning of this density? Is this due to some phase transition? What happens at densities higher than this critical density? Are they accessible at all? Also, there seems to be a drag force on moving sources, which is undesirable. Note also that in the context of a time-dependent configuration the above equation of state is problematic as it implies waves carrying negative energy.

5.2 A membrane model for MOND

It is also well known that the shape of a membrane that corresponds to an extremal area solves a nonlinear equation of type (6), with a vanishing right hand side: If one describes the membrane as a hypersurface in an \((n+1)\)-dimension, Euclidean space with coordinates \(x^1, ..., x^n, \varphi\) taking the form \(\varphi(x^1, ..., x^n)\), the area (volume) of the membrane is given by

\[
A_M = \int d^n x [1 + (\vec{\nabla} \varphi)^2]^{1/2}, \tag{10}
\]

where the integral is over the (projected) volume in the \(x\) space, at the boundary of which \(\varphi\) is dictated. If the energy function of the membrane is proportional to the area, \(E_M = KA_M\) its minimization gives eq. (6) with \(\mu(z) \propto (1 + z^2)^{-1/2}\) (and \(\rho = 0\)). More generally we consider membrane energy functions that are still functionals of the membrane shape of the form

\[
E_M = (K/2) \int d^n x F[|\vec{\nabla} \varphi|^2]. \tag{11}
\]

Then think of our effective (non-relativistic) universe as a surface in an \((n+1)\)-dimensional Euclidean (or Minkowski) space such that at a point \((x, \varphi)\) on the surface, the \(\varphi\) coordinate is to be interpreted as the gravitation potential at \(x\). We now have to introduce gravitating masses as sources for the potential; in the membrane picture they will be some external agents that determine the shape of the membrane. We can do this, breaking the isotropy of the embedding Euclidian space, by assuming that there are \(\varphi\)-independent forces on the membrane acting in the \(\varphi\) direction. These are introduced by adding to the energy a term of the form \(a_0 \int d^n x \rho(x) \varphi(x)\). We can think of this term as resulting from a constant acceleration field of magnitude \(a_0\) acting in the \(\varphi\) direction, that couples linearly to some quantity \(\rho\) on the membrane with the dimensions of mass density. Note that the area energy function itself is just
\[ \int g^{1/2} d^nx, \text{ where } g \text{ is the determinant of the induced metric, and is covariant, and in particular isotropic in the embedding space. It is the force term that breaks the isotropy.} \]

We can define \( \phi = a_0 \varphi \), which then has the usual dimensions of a gravitational potential, and also define the gravitational constant as \( G = a_0^2/4\pi K \) and then write the combined energy (or action \( S = -E \)) as

\[ E_M = (a_0^2/8\pi G) \int d^nx F[(\vec{\nabla} \phi)^2/a_0^2] + \int d^nx \rho \phi. \tag{12} \]

The shape of the membrane \( \phi(x) \) that minimize the total energy is determined by eq. (6).

It can then be shown (\( ?, \text{ e.g.} \)), that the membrane produces forces on \( \rho \) in the lateral, \( x \) space. Take a (possibly finite) body made of mass distribution \( \rho \), and define the force on it as (minus) the gradient of the energy under rigid displacements of the body. The force is then writable as \( -\int d^nx \rho \vec{\nabla} \phi \), where \( \phi(x) \) is the shape of the membrane as determined by the total mass distribution, including that of the body itself. (Because of the non-linearity of the problem we cannot calculate the force using the potential determined by the masses other than the body.) These forces appear because it costs energy to rigidly displace the body, and they are interpreted as the gravitational forces on masses in the \( n \)-dimensional world in which we seem to live. This justifies the viewing of \( \phi \) as the gravitational potential.

Thus, a simple dynamical picture with non-interacting “masses” on the membrane that are subject to a uniform force field in the \( \varphi \) direction, yields a complex, effective picture of “gravitational” interactions between masses, mediated by the membrane. The functional form of the dependence of the membrane energy on its shape dictates the governing equation for the effective gravitation field.

Such membrane analogs of gravity are, of course, well known, from the simple demonstration of effective attraction between two masses placed on an horizontal, stretched, elastic surface, to the many recent discussions on the universe as a brane, in the high-energy literature (\( ?, \text{ see, e.g.,} \)) and references therein\( \text{[carter]} \). In such attempts, it has not yet been possible to derive the energy function from an underlying theory (e.g. string theory), and ad hoc energies are assumed to fill the bill. The emphasis here is on the potential application of the membrane to model MOND-like theories, and this is all in the choice of the energy functional for the membrane.

The theory obtained when the energy function is just the area is not what we need to model MOND. In the limit \( |\vec{\nabla} \varphi| \ll 1 \) (\(|\vec{\nabla} \phi| \ll a_0\)), we can write the area as \( \approx \int d^n x \ [1 + (\vec{\nabla} \varphi)^2/2] \), which gives the Poisson equation, not
the deep-MOND limit. (This is generic, and tends to happen when the energy density is finite at zero $\vec{\nabla} \phi$, as the next term in the expansion is, many times, the Poissonian $(\vec{\nabla} \phi)^2$.) In the opposite limit, $|(\vec{\nabla} \phi)^2| \gg 1$, the theory becomes singular. [In this limit $\mu(z) \to z^{-1}$, which is exactly the borderline between ellipticity and hyperbolicity of the field equation—the ellipticity condition being $d \ln [\mu(z)]/d \ln z > -1$.] Also, the theory does not permit concentrated masses: Applying Gauss theorem to the field equation gives

$$\int_{\sigma} \frac{\vec{\nabla} \phi \cdot ds}{[1 + (\vec{\nabla} \phi)^2]^{1/2}} = 8\pi GM/a_0,$$

where $M$ is the total mass in the volume whose surface is $\sigma$, from which follows that $M \leq a_0 S/8\pi G$, where $S$ is the total area of $\sigma$. So, there is a limit to the mass that can be put in a volume of a given surface area. Point masses, for example, are excluded: in three dimensions there is no solution for a radius below $(2GM/a_0)^{1/2}$ since $|\vec{\nabla} \phi| = \lambda/(1 - \lambda^2)^{1/2}$, where $\lambda = 2MG/a_0 r^2$.

However, this theory does include already an important feature of MOND: It has a critical, transition acceleration—which, in the units where the potential is a coordinate, is $a_0 = 1$—that separates two regions of rather different behavior of the gravitational field. It is likely that this critical value will also appear in the “cosmology” of the model—the behavior of the membrane as a whole. If indeed it does, it may hint at one possible way in which, in the real world, cosmology shares with local dynamics this critical constant. Note that although $a_0$ appears here as the constant $\phi$-acceleration—a role that smacks of cosmology—this, in itself, does not establish $a_0$ as an acceleration of cosmological significance. This will have to emerge from the dynamics of the membrane, which has to consider whether the membrane itself is coupled to the $a_0$ field, and what exactly carries inertia: the membrane, the masses $\rho$, or both. I leave the treatment of these questions to a future discussion.

We may liken such an appearance of $a_0$ in gravitational dynamics to the way the speed of light enters relativistic kinematics of bodies—e.g. in determining the lifetime of a moving muon—although $qua$ speed of light it has nothing to do with these kinematics. How does this happen? In Minkowski-type spaces, a space-time slope of 1 plays a critical role. (For historical reasons this slope has attained the dimensions of velocity and the value of $c$). On one hand it enters the kinematics of all particles; on the other, it is the constant slope on the world lines–null geodesics–of massless particles. Similarly, in the present context, an $x - \phi$ slope of 1 for the membrane is a transition value because of the form of the energy function of the membrane, so it enters local dynamics, and may enter cosmology. (For historical reasons having to do with how we measure the gravitational potential this slope has attained the dimensions of acceleration and the value of $a_0$.)
To write a membrane energy function that does give MOND we have to invoke again the special role of the $\varphi$ direction, as we did already when we assumed that the external field acts in the $\phi$ direction. We have to break the isotropy also in the energy function of the membrane.

For example, one of many energy functionals for a 3-dimensional membrane embedded in 4-dimensional Euclidean space that give MOND is

$$E_M = \left( \frac{K}{2} \right) \int d^3x (\vec{\nabla} \varphi)^2 \left[ \frac{(\vec{\nabla} \varphi)^2}{1 + (\vec{\nabla} \varphi)^2} \right]^{1/2}. \quad (14)$$

This can be given a geometrical meaning: If $\psi$ is the angle between the normal to the membrane and the (positive) $\varphi$ direction, then $\cos \psi = [1 + (\vec{\nabla} \varphi)^2]^{-1/2}$, and $\sin \psi = [(\vec{\nabla} \varphi)^2/(1 + (\vec{\nabla} \varphi)^2)]^{1/2}$. So, we can write the above energy functional as

$$E_M = \left( \frac{K}{2} \right) \int d^3x t g^2 \psi \sin \psi, \quad (15)$$

or

$$E_M = \left( \frac{K}{2} \right) \int dv t g^2 \sin^2 \psi, \quad (16)$$

where $dv = d^3x / \cos \psi$ is the volume element on the membrane.

We can also obtain an effective theory that lives in a curved $n$-dimensional space by starting with a foliation of the embedding space using coordinates $(\theta^1, ..., \theta^n, \varphi)$ in which the line element can be written as

$$ds^2 = d\varphi^2 + \varphi^2 g_{ij}(\vec{\theta}) d\theta^i d\theta^j. \quad (17)$$

If we now describe the membrane by $\tilde{\varphi}(\vec{\theta})$, The embedded metric on the membrane, in the coordinates $\vec{\theta}$, is $G_{ij} = \tilde{\varphi}_i \tilde{\varphi}_j + \varphi^2 g_{ij}$ (where $[]_i$ signifies the derivative with respect to $\theta_i$), whose determinant can be shown to be $G = g_{\tilde{\varphi}^2}(1 + g^{ij} \beta_i \beta_j)$, where $g$ is the determinant of $g_{ij}$, $g^{ij}$ is its inverse, $\beta \equiv \ln(\tilde{\varphi})$. So, the area element of the membrane is $dv = d^n\theta G^{1/2} = d^n\theta \ g^{1/2} \tilde{\varphi}^n \ (1 + g^{ij} \beta_i \beta_j)^{1/2}$. Note that the geometry of our universe is not that of the membrane: the effective metric is not $G_{ij}$, the induced metric on the membrane, but $g_{ij}$.

If $\psi$ is the angle between the membrane and the local, constant-$\varphi$ surface, then $\cos \psi = (1 + g^{ij} \beta_i \beta_j)^{-1/2}$. Again, membrane energy functionals that give MOND can be written that depend only on $\psi$. An example in three dimensions
is given by expression (16), which makes \( \phi \) a preferred direction. It is also a preferred direction as regards the external field, which is now assumed to lies in the \( \phi \) direction—i.e. is derived from a \( \phi \)-dependent potential \( V(\phi) \). Its interaction with \( \rho \), the density of masses on the membrane per unit \( d\Omega = d^3\theta \) (the covariant density being \( g^{-1/2}\rho \)), is described by
\[
E_I = \int d^3\theta \, \rho(\vec{\theta}) V[\tilde{\phi}(\vec{\theta})].
\]

This would give a MOND-like theory in the curved space having the metric \( g_{ij} \) in the coordinates \( \vec{\theta} \).

Why the energy function of the model, or real, membrane should take a form that reproduces MOND I do not know. Perhaps there is a clue in the fact that in this limit MOND becomes conformally invariant, as shown in Milgrom (1997). To see this in the present context, note that the form of the energy function in the deep-MOND limit (\( g^{ij}\beta_{,i}\beta_{,j} \ll 1 \)) is
\[
\int d^n\theta \, g^{1/2} \tilde{\phi}^n \, (g^{ij}\beta_{,i}\beta_{,j})^{3/2}.
\]
This is, clearly, invariant under replacement of the metric \( g_{ij}(\vec{\theta}) \) by \( \lambda(\vec{\theta}) g_{ij}(\vec{\theta}) \), for an arbitrary \( \lambda(\vec{\theta}) > 0 \) under which \( g^{ij} \rightarrow \lambda^{-1} g^{ij} \), and \( g \rightarrow \lambda^3 g \). Note that \( E_I \) is also conformally invariant, so that the field equation for \( \phi \) is invariant as well.

I have limited myself to the static case here, and will discuss the dynamics of the system elsewhere.

6  MOND as modified inertia

Newtonian inertia itself has not been immune to changes over the years. Special Relativity entails a familiar modification, replacing the single-particle kinetic action in eq. (4) by
\[
-mc^2 \int dt \, [1 - (v/c)^2]^{1/2}.
\]
This gives an equation of motion
\[
F = md(\gamma v)/dt = m\gamma [a + \gamma^2 v \cdot a] / c^2,
\]
where \( \gamma \) is the Lorentz factor.

And, physics is replete with instances of modified, acquired, or effective inertia. Electrons and holes in solids can sometimes be described as having a greatly modified mass tensor. Mass renormalization and the Higgs mechanism, modify particle masses and/or endow them with mass: an effective, approximate description that encapsulates the effects of interactions of the particles, with vacuum fields in the former instance, and with the Higgs field in the latter. The effects of a fluid on a body embedded in it may sometimes be described
as a contribution to the mass tensor of the body, because its motion induces motion in the fluid which carries energy and momentum. So, modified inertia might also well lie in the basis of MOND.

Consider non-relativistic modifications of inertia that incorporate the basic principle of MOND. We seek to modify the particle kinetic action $S_k$ in eq. (4) into an action of the form $S_k[R, a_0]$, which is a functional of the particle trajectory, $R$ [symbolically representing some trajectory $r(t)$], and depends also on the single constant, $a_0$. The potential part of the action remains the standard one. The modified kinetic action should satisfy the following asymptotic requirements: 1. In the formal limit $a_0 \to 0$, corresponding to all acceleration measures in the system being much larger than the actual value of $a_0$, the action should attain its standard Newtonian form (this is similar to obtaining the classical limit of quantum mechanics by taking the formal limit $\hbar \to 0$). 2. To retain MOND phenomenology, according to which, in the deep MOND limit, $G$ and $a_0$ appear only through their product $Ga_0$, we should have in the limit $a_0 \to \infty$, $S_k \propto a_0^{-1}$. This can be seen by rescaling $\phi$ into $\phi/G$ in eq. (4) (and dividing the action by $G$).

The equation of motion is then of the form

$$A(R, t, a_0) = -(\vec{\nabla} \phi)[r(t)],$$

where the generalized acceleration $A$ is a functional of the trajectory, and a function of the time $t$, and $\vec{\nabla} \phi$ is to be calculated at the momentary position $r(t)$.

The theory should also satisfy the more subtle requirement of the correct center-of-mass motion discuss in the previous section.

General properties of such theories are discussed in detail in Milgrom (1994). Here I summarize, very succinctly, some of the main conclusions.

If the action enjoys the usual symmetries: translational, rotational, and Galilei invariance, then, to satisfy the two limits in $a_0$ the action must be non-local. This means that the action cannot be written as $\int L \, dt$, where $L$ is a function of a finite number of derivatives of $r(t)$. This might look like a disadvantage, but, in fact, it is a blessing. A local action for MOND would have had to be a higher-derivative theory, and, as such, it would have suffered from the several severe problems that beset such theories. A non-local theory need not suffer from these. A non-local action is also a more natural candidate for an effective theory.

While nonlocal theories tend to be rather unwieldy, they do lend themselves to a straightforward treatment of the important issue of rotation curves. This
is done via a virial relation that physical, bound trajectories can be shown to satisfy:

\[ 2S_k[R, a_0] - a_0 \frac{\partial S_k}{\partial a_0} = \langle r \cdot \nabla \phi \rangle, \]

where \( \phi \) is the (unmodified) potential in which the particle is moving, \( \langle \rangle \) marks the time average over the trajectory, and \( S_k \) is the value of the action calculated for the particular trajectory (\( S_k \) is normalized to have dimensions of velocity square). In the Newtonian case this reduces to the usual virial relation. Applying this relation to circular orbits in an axi-symmetric potential, and noting that, on dimensional grounds, we must have \( S_k(r, v, a_0) = v^2 \nu(v^2/a_0) \) (where \( r \) and \( v \) are the orbital radius and velocity), we end up with the expression for the velocity curve

\[ \frac{(v^2/r)}{\mu(v^2/a_0)} = d\phi/dr, \]

where \( \mu(x) = \nu(x)[2 + d \ln \nu(x)/d \ln x] \). Thus the algebraic relation that was first used in MOND as a naive application of eq. (3), and which all existing rotation-curve analyzes use, is exact in modified-inertia MOND. In modified gravity this expression is only a good approximation.

I recently noticed the following scaling property of deep-MOND solutions in modified inertia: As explained above, the single-particle kinetic action in the limit \( a_0 \to \infty \) has to be of the form

\[ S_k(R, a_0) \approx s(R)/a_0, \]

where \( s \) is a functional of the trajectory alone.

In this limit, the virial relation takes the form \( s(R) = a_0 \langle r \cdot \nabla \phi \rangle/3 \), where \( s \) has the dimensions of acceleration-times-velocity-squared.

It also follows that the equation of motion in an external potential field becomes

\[ A(R, t, a_0) = Q(R, t)/a_0 = -(\nabla^2 \phi)[r(t)], \]

\( Q \) has dimensions of squared acceleration, and hence has the following scaling property: if we scale the trajectory \( R \) given by \( r(t) \) to \( R^* \) given by \( r^*(t) = \lambda r(t/\zeta) \), then

\[ Q(R^*, t) = \lambda^2 \zeta^{-4} Q(R, t/\zeta). \]
The action itself scales as: $S_k(R^*, \lambda \zeta^{-2} a_0) = \lambda^2 \zeta^{-2} S_k(R, a_0)$. If the potential field itself is homogeneous in $r$; i.e., it satisfies $\phi(\lambda r) = \lambda^{1-\alpha} \phi(r)$ we get a scaling property for the solutions: If $r(t)$ is a solution of the equation of motion, then so are the whole one-parameter family $r_\zeta(t) = \zeta^{4/(2+\alpha)} r(t/\zeta)$.

For example, the asymptotic gravitational field of a bounded mass ($\phi \propto r^{-1}$) has $\alpha = 2$, so if $r(t)$ is a solution, so is $r_\zeta(t) = \zeta r(t/\zeta)$ for all $\zeta$. The velocity on these trajectories is $v_\zeta(t) = d\zeta r_\zeta/dt = v(t/\zeta)$, and so does not change with the dilation of the orbit. This is a generalization of the notion of asymptotic flatness of rotation curves from circular orbits to arbitrary ones: every orbit is a member of a family of self-similar ones, with different sizes, but the same velocity at the corresponding phase of the orbit.

If $\phi$ is an harmonic-oscillator field (not necessarily spherical) for which $\alpha = -1$, $r_\zeta(t) = \zeta^4 r(t/\zeta)$ is always a solution in the deep-MOND regime, if $r(t)$ is. We see that in this case the orbital period scales as the orbit size to the $1/4$ power. This example is relevant, for example, for constant-density spheres, for which the potential is harmonic.

Note that for the general homogeneous (power-law) potential, the acceleration on the trajectory $r_\zeta$ is $a_\zeta = d^2 r_\zeta/dt^2 = \zeta^{2\alpha/(2+\alpha)} a(t/\zeta)$, while, from the scaling of the potential field, $(\vec{\nabla} \phi)(r_\zeta) = \zeta^{-4\alpha/(2+\alpha)} (\vec{\nabla} \phi)(r)$. So we see that, within the family, $a_\zeta(r_\zeta) \propto (\vec{\nabla} \phi)(r_\zeta)]^{1/2}$. This proportionality evokes the basic, deep-MOND relation eq. (2). But note that it holds only when comparing corresponding points on scaled trajectories in the same family. It is certainly not true in general that the acceleration is proportional to the square root of the Newtonian acceleration.

This brings to mind another important difference between the two interpretations of MOND: In (non-relativistic) modified gravity the gravitational field is modified, but in this modified field all bodies at the same position undergo the same acceleration. In modified inertia the acceleration depends not only on position, but also on the trajectory. In the case of SR the acceleration depends on the velocity as well, but in more general theories it might depend on other properties of the orbit: as explained above it most probably is a functional of the whole orbit. Because translational invariance is retained, there is, of course, still a generalized momentum whose rate of change is a function of position only ($m_\gamma v$ in SR) but this rate is not the acceleration. This larger freedom in modified inertia comes about because we implement the modification via a modification of the action as a functional of the trajectory; namely, a function of an infinite number of variables; so, different trajectories might suffer different modifications. In modifying gravity we modify one function of the three coordinates (the gravitational potential). This is an obvious point, but is worth making because in interpreting data we tend to equate observed accelerations with the gravitational field because of our wont with Newtonian
inertia. While this is still true in modified gravity it is not so in modified inertia.

We can exemplify this point by considering the claimed anomaly in the motions of the Pioneer 10 and 11 spacecraft. Analysis of their motion have shown an unexplained effect (Anderson) that can be interpreted as being due to an anomalous, constant acceleration towards the sun of about $8 \cdot 10^{-8}$ cm s$^{-2}$, which is of the order of $a_0$. This might well be due to some systematic error, and not to new physics. This suspicion is strengthened by the fact that an addition of a constant acceleration of the above magnitude to the solar gravitational field is inconsistent with the observed planetary motions (e.g. it gives a much too large rate of planetary perihelion precession). MOND could naturally explain such an anomalous acceleration: We are dealing here with the strongly Newtonian limit of MOND, for which we would have to know the behavior of the extrapolating function $\mu(x)$ at $x >> 1$, where $\mu \approx 1$. We cannot learn about this from galaxy dynamics, so we just parameterize $\mu$ in this region by $\mu \approx 1 - \xi x^{-n}$. (This is not the most general form; e.g. $\mu$ may approach 1 non analytically in $x^{-1}$, for example as $1 - \exp(-\xi x)$.) Be that as it may, if $n = 1$ we get just the desired effect in MOND: the acceleration in the field of the sun becomes $M_\odot Gr^{-2} + \xi a_0$ in the sun’s direction. In a modified gravity interpretation this would conflict with the observed planetary motions, which, as stated above, are not known to undergo such anomalous acceleration; but, in the modified-inertia approach it is not necessarily so. It may well be that the modification enters the Pioneers motion, which corresponds to unbound, hyperbolic motions, and the motion of bound, and quasi-circular trajectories in a different way. For example, the effective $\mu$ functions that correspond to these two motions might have different asymptotic powers $n$.

7 vacuum effects and MOND inertia

Because MOND revolves around acceleration, which is so much in the heart of inertia, one is directed, with the above imagery in mind, to consider that inertia itself, not just MOND, is a derived concept reflecting the interactions of bodies with some agent in the background. The idea, which is as old as Newton’s second law, is the basic premise of the Mach’s principle. The great sense that this idea makes has lead many to attempt its implementation. The agent responsible for inertia had been taken to be the totality of matter in the Universe.

Arguably, an even better candidate for the inertia-producing agent, which I have been considering since the early 1990s, in the hope of understanding MOND’s origin, is the vacuum. The vacuum is known to be implicated in producing or modifying inertia; for example, through mass renormalization
effects, and through its contribution to the free Maxwell action in the form of the Euler-Heisenberg action (Itzykson & Zuber, 1980). Another type of vacuum contributions to inertia have been discussed by Jaekel & Reynaud (1993). But, it remains moot whether the vacuum can be fully responsible for inertia.

The vacuum is thought to be Lorentz invariant, and so indifferent to motion with constant speed. But acceleration is another matter. As shown by Unruh in the 1970s, an accelerated body is alive to its acceleration with respect to the vacuum, since it finds itself immersed in a telltale radiation, a transmogrification of the vacuum that reflects the accelerated motion. For an observer on a constant-acceleration ($a$) trajectory this radiation is thermal, with $T = \alpha a$, where $\alpha \equiv \hbar/2\pi kc$. The effect has been also calculated approximately for highly relativistic circular motions; the spectrum is then not exactly thermal. The effect is non-local; i.e., depends on the full trajectory.

Unruh’s result shows that the vacuum can serve as an inertial frame. But this is only the first step. The remaining big question is how exactly the vacuum might endow bodies with inertia. At any rate, what we want is the full MOND law of inertia, with the transition occurring at accelerations of order $a_0$ that is related to cosmology. We then have to examine the vacuum in the context of cosmology. How it affects, and is being affected by, cosmology. One possible way in which cosmology might enter is through the Gibbons-Hawking effect, whereby even inertial observers in an expanding universe find themselves embedded in a palpable radiation field that is an incarnation of the vacuum. The problem has been solved for de Sitter Universe, which is characterized by a single constant: the cosmological constant, $\Lambda$, which is also the square of the (time independent) Hubble constant. In this case the spectrum is also thermal with a temperature $T = \alpha c(\Lambda/3)^{1/2}$.

In the context of MOND it is interesting to know what sort of radiation an observer sees, who is accelerated in a non-trivial universe: if the Unruh temperature is related to inertia, then it might be revealing to learn how this temperature is affected by cosmology. This can be gotten for the case of a constant-acceleration observer in a de Sitter Universe. For this case the radiation is thermal with a temperature $T = \alpha (a^2 + c^2 \Lambda/3)^{1/2}$ (Deser & Levin, 1997). Inertia, which is related to the departure of the trajectory from that of an inertial observer, who in de Sitter space sees a temperature $\alpha c(\Lambda/3)^{1/2}$, might be proportional to the temperature difference

$$\Delta T = \alpha [(a^2 + c^2 \Lambda/3)^{1/2} - c(\Lambda/3)^{1/2}], \quad (26)$$

and this behaves exactly as MOND inertia should: it is proportional to $a$ for $a >> a_0 \equiv 2c(\Lambda/3)^{1/2}$, and to $a^2/a_0$ for $a << a_0$; and, we reproduce the connection of $a_0$ with cosmology. Of course, this sort of argument pertains
to linear, constant-acceleration motion, while more general trajectories will probably behave differently. But the emergence of an expression à-la MOND in this connection with the vacuum is intriguing.

For inertia to result somehow from the resistance of the vacuum, an accelerated observer should be able to tell from vacuum effects alone, not only the magnitude of its acceleration, as in the Unruh effect, but also all sorts of other properties of its orbit, as enter the generalized acceleration. The Unruh radiation is supposedly isotropic so its angular distribution for a point observer does not carry directional information. If, however, we consider a finite-size observer it is possible that different parts of the observer are seeing different Unruh radiation, from which difference more general properties of the trajectory may be read. Consider, for example, two points, infinitesimally nearby, each moving on an hyperbolic (constant acceleration) trajectory along the $z$ axis: $z_i(t) = (t^2 + g_i^{-2})^{1/2}$, $i = 1, 2$. The proper distance between the point remains constant (i.e., they move as a rigid body) and equals $|g_2^{-1} - g_1^{-1}|$. More generally, the different points of a rigid body, translationally accelerated at a constant acceleration, move on parallel hyperbolic trajectories, but each with its own acceleration parameter according to its position along the direction of the acceleration. Each point then has its own Unruh temperature, and the direction of the body’s acceleration can be read off the temperature distribution.

8 Relativistic theories

A relativistic extension of MOND, which we still do not have, is needed for conceptual completion of the MOND idea. It is also needed because we already have observed relativistic phenomena that show mass discrepancies, and we must ascertain that there too the culprit is not dark matter but modified dynamics.

There are no local black holes whose surface acceleration is in the MOND regime; i.e., for which $MG/r_s^2 < a_0$. This would require the Schwarzschild radius to satisfy $r_s > c^2/2a_0$, which, by the cosmological coincidence, is larger than the Hubble distance. The only system that is strongly general relativistic and in the MOND regime is the Universe at large. This, however means that we would need a relativistic extension of MOND to describe cosmology. In fact, as I have indicated, MOND itself may derive from cosmology, so it is possible that the question of the origin of MOND will have to be tackled as part and parcel of that of MOND cosmology. And, because the cosmological expansion is strongly coupled with the process of structure formation this too will have to await a modified relativistic dynamics for its treatment.
Several relativistic theories incorporating the MOND principle have been discussed in the literature, but none is wholly satisfactory (\textit{?}, see, e.g.) and references therein\textit{[bm,pcg,stratified].}

There have also been attempts to supplement MOND with extra assumptions that will enable the study of structure formation, so as to get some glimpse of structure formation in MOND. For these see Milgrom (1989), Sanders (2001), Stachniewicz & Kutschera (2002), and Nusser (2002).

Gravitational light deflection, and lensing, is another phenomenon that requires modified relativistic dynamics. It is tempting to take as a first approximation the deflection law of post-Newtonian General Relativity with a potential that is the non-relativistic MOND potential (\textit{?}, see e.g. analyses by)\textit{[qin,mortlock].} This, however, is in no way guaranteed. In GR this is only a post-Newtonian approximation, and perhaps it would turn out to be a post-Newtonian approximation of MOND (i.e. an approximation of MOND in the almost Newtonian, $a >> a_0$ regime). But, there is no reason to assume that it is correct in the deep-MOND regime. Even in the framework of this assumption one needs to exercise care. For example, the thin-lens hypothesis, by which it is a good approximation to assume that all deflecting masses are projected on the same plane perpendicular to the line of sight, breaks down in MOND. For example, $n$ masses, $M$, arranged along the line of sight (at inter-mass distances larger that the impact parameter) bend light by a factor $n^{1/2}$ more than a single mass $nM$.

No less important is another MOND effect without analog in Newtonian dynamics: the external-field effect (EFE) of MOND, by which, if a system is embedded in an external field whose acceleration dominates over the internal acceleration, it is the former that controls the dynamics. In particular, in the Bekenstein & Milgrom (1984) formulation of MOND as modified gravity, if a mass (say a galaxy) is embedded in an external field (due, say, to large scale structure) the dynamics becomes quasi-Newtonian beyond radii where the internal acceleration falls below the external one (Milgrom, 1986). Furthermore, the MOND, effective gravitational field is non-spherical due to the EFE, even if the mass (of the galaxy) is spherical. The field is not even elliptical, but—assuming that the external field is constant—becomes elliptical asymptotically with an asymptotic axes ratio of $2^{-1/2}$. Hoekstra et al. (2002), for example, ignore the EFE, altogether, when they assess the performance of MOND in light of their galaxy, weak-lensing analysis. One of their complaints against MOND is, in fact, that their analysis gives a “halo cutoff radius” beyond which a faster, Newtonian-like fall-off in the signal is seen, just as expected from the EFE due, e.g., to acceleration of galaxies by LSS. This, I find, happens in the Hoekstra et al. analysis where the galaxy’s intrinsic acceleration is $\sim 4 \times 10^{-10} h \ cm \ sec^{-2}$, which is of the order of the LSS acceleration of galaxies.
References

Hoekstra, H., Yee, H., & Gladders, M. 2002, this volume
Itzykson, C., & Zuber, J.B., Quantum Field Theory (McGraw-Hill, 1980)
Jaekel, M.T., & Reynaud, S., 1993, J. de Physique 3, 1093