Correlation Functions and Cumulants 
in Elliptic Flow Analysis

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Abstract

We consider various methods of flow analysis in heavy ion collisions and compare experimental data on corresponding observables to the predictions of our saturation model proposed earlier [1]. We demonstrate that, due to the nature of the standard flow analysis, azimuthal distribution of particles with respect to reaction plane determined from the second order harmonics should always be proportional to $\cos 2(\phi - \Psi_R)$ independent of the physical origin of particle correlations (flow or non-flow). The amplitude of this distribution is always physical and proportional to $v_2$. Two-particle correlations analysis is therefore a more reliable way of extracting the shape of physical azimuthal anisotropy. We demonstrate that two-particle correlation functions generated in our minijet model of particle production [1] are in good agreement with the data reported by PHENIX. We discuss the role of non-flow correlations in the cumulant flow analysis and demonstrate using a simple example that if the flow is weak, higher order cumulants analysis does not significantly reduce the contribution of non-flow correlations to elliptic flow observable $v_2$ in RHIC data.
1 Introduction

Differential elliptic flow in heavy ion collisions is defined as the coefficient of the second harmonic of particle distribution with respect to the reaction plane

\[ v_2(p_T) = \langle \cos 2(\phi_{p_T} - \Phi_R) \rangle \]  

(1)

where \( \phi_{p_T} \) is the azimuthal angle of the produced particle with transverse momentum \( p_T \), \( \Phi_R \) is the azimuthal angle of the reaction plane and the brackets denote statistical averaging over all particles with momentum \( p_T \) and over different events. According to emerging RHIC data differential elliptic flow is an increasing function of \( p_T \) for small \( p_T \), which stops increasing and saturates to a constant at \( p_T \gtrsim 1.5 - 2 \text{ GeV} \) [2, 3, 4]. While the hydrodynamic models are successful in describing the low-\( p_T \) increase of \( v_2(p_T) \) [5, 6], they disagree with the high-\( p_T \) data. Models incorporating jet quenching on top of hydrodynamic expansion also do not agree with the high-\( p_T \) data very well [7]. Describing the quark-gluon plasma (QGP) dynamics by covariant transport theory requires either extremely high initial gluon density or very large parton-parton scattering cross sections in order to fit the high-\( p_T \) elliptic flow data [8].

In our previous paper on the subject [1] we proposed a model of non-flow particle correlations in the initial stages of heavy ion collisions which described \( v_2(p_T) \) saturation data rather well. The model was based on particle production mechanism in the high energy regime when gluon and quark distribution functions of the colliding nuclei reach saturation [9]. As was suggested by McLerran and Venugopalan [10] the dominant gluon production mechanism in the early stages of the collision is given by the classical field of the nuclei. The field was found at the lowest order in \( \alpha_s \) in [11]. Extensive numerical simulations of the full solution have been performed in [12] while an analytical ansatz for the corresponding spectrum of the produced gluons was written down in [13]. At the high energies achieved by RHIC experiments the classical field alone can not account for particle production; thus quantum corrections become important. For an ansatz of gluon production cross section in AA including the nonlinear evolution of [14] see [15].

The essence of our model proposed in [1] is the following: to estimate the non-flow contribution to \( v_2(p_T) \) one has to calculate the single and double inclusive gluon production cross sections first in the framework of the simple McLerran-Venugopalan model and then include the nonlinear evolution effects [14] in them. Two produced gluons in the double inclusive cross section are of course azimuthally correlated with each other [1]. This correlation can contribute to \( v_2 \) after being averaged over all particle pairs, which is proportional to the total particle multiplicity squared. The latter was related to the single inclusive gluon production cross section similar to how it was done in [16]. Two comments are in order here. First of all double gluon production cross section of course can not be given by the classical field and is therefore not a classical quantity. Calculating it thus corresponds to the first (order \( \alpha_s \)) correction to McLerran-Venugopalan model. Secondly the correlations in the double inclusive gluon production cross section are not just back-to-back, as one would naively expect. One of the main advantages of saturation physics is that one does not have to assume that the momenta of the produced gluons are large in order to use small coupling and twist expansions like it is done in the collinear factorization approach. Saturation calculations include all twists and the coupling is kept small by the large saturation scale \( Q_s \) [10, 17, 18]. When the momenta of the produced two gluons are not extremely large the correlations are not only back-to-back since it
is not required anymore by transverse momentum conservation as some of the momentum can be carried away by other soft particles. A calculation of the lowest order double inclusive cross section was done in [19] showing not only back-to-back (\(\Delta \phi = \pi\)) but also collinear (\(\Delta \phi = 0\)) correlations.

Since the exact double inclusive gluon production cross section is not known we constructed a simple model of single- and double- gluon production [1] in the spirit of \(k_T\)-factorization approach used in [16]. The model successfully described the saturation of \(v_2(p_T)\) at high \(p_T\) as well as centrality dependence of \(v_2(B)\). The goal of this paper is to discuss compatibility of our model with other observables in the flow analysis.

There are other types of non-flow particle correlations in McLerran–Venugopalan model [20, 21] on top of the ones considered in [1]. Those are due to dependence of classical gluonic fields on nuclear overlap geometry and may contribute to \(v_2\) [20, 21]. (If taken separately these correlations can not reproduce the high-\(p_T\) behavior of \(v_2(p_T)\) given by RHIC data.) We will argue in Sect. 2 that these correlations are higher order in \(\alpha_s\) and are therefore parametrically suppressed.

In Sect. 3 we will demonstrate that even in the case of only non-flow correlations the distribution of particles with respect to experimentally determined reaction plane averaged over many events is proportional to \(\cos 2(\phi - \Psi_R)\) in agreement with STAR data [4]. (\(\phi\) and \(\Psi_R\) are azimuthal angles of the particle and reaction plane correspondingly.) This distribution is due to the fact that the reaction plane is also determined from the second harmonic of the multiplicity distribution. Therefore the \(\cos 2\phi\) shape of the distribution with respect to reaction plane does not reflect any physics. At the same time the amplitude of the distribution is given by physical correlations, that is by \(2v_2\) [1].

In Sect. 4 we show that two-particle correlation functions in our model are consistent with the data reported by PHENIX [22]. To see that, one has to relax the large rapidity interval condition employed for simplicity in [1], which does not apply to the PHENIX detector and to flow analysis in general.

We conclude in Sect. 5 by discussing higher cumulant flow analysis. Higher cumulants analysis was proposed in [23] as a way to reduce the contribution of non-flow effects to \(v_2\). We consider the case when all of standard (2nd cumulant) \(v_2\) is due to non-flow two-particle correlations, similar to [1]. We then calculate the higher order cumulants in this model and extract \(v_2\) from them. While parametrically the conclusion of [23] still holds, the numerical values of non-flow \(v_2\) extracted from the fourth and higher cumulants for RHIC are not significantly smaller than \(v_2\) from the two-particle correlation function in agreement with the results of recent STAR analysis [3].

## 2 Different Types of Non-Flow Correlations

The two-particle multiplicity distribution is given in our model [1] by

\[
\frac{dN}{d^2k_1 dy_1 d^2k_2 dy_2} = \frac{dN}{d^2k_1 dy_1 d^2k_2 dy_2} + \frac{dN_{corr}}{d^2k_1 dy_1 d^2k_2 dy_2} \tag{2}
\]

We would like to point out that the true reaction plane angle \(\Phi_R\) used in Eq. (1) is, in principle, different from the reaction plane angle \(\Psi_R\) determined from the flow analysis.
with the first term in Eq. (2) given by a product of two single particle distributions due to classical gluon fields (disconnected piece) [11, 12, 13] and the second term corresponding to the correlations estimated in [1] (connected piece). Let us make some parametric estimates in the framework of classical particle production where the coupling is small $\alpha_s(Q_s) \ll 1$ and together with atomic number $A \gg 1$ it forms a resummation parameter $\alpha_s^2 A^{1/3} \sim 1$ [17]. First we note that at the leading order single particle distribution is

$$\frac{dN}{d^2 k \, dy} \sim \frac{S_A^A}{\alpha_s}$$

(3)

with $S_A^A = \pi R^2$ the cross sectional area of the nucleus. The gluons produced by classical fields have typical transverse momentum of the order $k_T \sim Q_s \gg 1/R$ and thus do not “know” of the nuclear overlap geometry. This way the distribution of these gluon modes in Eq. (3) is azimuthally symmetric. Therefore the leading contribution of the first term in Eq. (2) does not contribute to $v_2$. However, the soft momentum tail of the single gluon distribution with $k_T \sim 1/R$ does depend on geometry of the overlap [20]. If one assumes that large $Q_s$ is sufficient to keep $\alpha_s$ small even for these soft modes one can perform a calculation of these geometric effects similar to how it was done in [20, 21]. One would obtain the following corrected version of Eq. (3) [20]

$$\frac{dN}{d^2 k \, dy} \sim \frac{S_A^A}{\alpha_s} \left[ 1 + o \left( \frac{1}{Q_s^2 S_A^A} \right) \right].$$

(4)

As $Q_s^2 \sim \alpha_s^2 A^{1/3} \sim 1$ the correction in Eq. (4) is of order $1/\alpha_s$. To calculate the contribution of this correction to elliptic flow observable one can either use Eq. (4) directly in Eq. (1) or use the definition of $v_2$ through two-particle correlations [24, 25]

$$\langle v_2 \rangle = \sqrt{\langle e^{2i(\phi_1-\phi_2)} \rangle}.$$  

(5)

In the end one obtains the contribution to $v_2$ of geometric corrections to classical fields [20]

$$v_2^{\text{geom}} \sim \frac{1}{S_A^A}.$$  

(6)

In contrast the second term in Eq. (2) is parametrically of order [1]

$$\frac{dN_{\text{corr}}}{d^2 k_1 \, dy_1 \, d^2 k_2 \, dy_2} \sim S_A^A.$$  

(7)

This can be understood by combining two classical gluon fields and letting them interact. This corresponds to squaring Eq. (3), dropping one power of $S_A^A$ (gluons have to be at the same impact parameter to interact) and multiplying by $\alpha_s^2$ (interaction). Using Eqs. (5) and (3) the contribution of correlations in Eq. (7) to $v_2$ can be estimated as [1]

$$v_2^{\text{non-flow}} \sim \frac{\alpha_s}{\sqrt{S_A^A}}.$$  

(8)

To see which contribution to $v_2$ is dominant we observe that $S_A^A \sim A^{2/3} \sim 1/\alpha_s^4$ since $\alpha_s^2 A^{1/3} \sim 1$. Plugging this into Eqs. (6) and (8) gives $v_2^{\text{geom}} \sim \alpha_s^4$ and $v_2^{\text{non-flow}} \sim \alpha_s^3$. Therefore $v_2^{\text{non-flow}} \gg$
$v_2^{\text{geom}}$ when the coupling is small indicating that two-particle correlations are parametrically more important for $v_2$ than geometrical dependence of classical fields. In the actual RHIC data the saturation scale was estimated to be $Q_s^2 \approx 2 \text{ GeV}^2$ [16] corresponding to $\alpha_s(Q_s) \approx 0.3$ which means that geometrical effects of [20, 21] may still be numerically important, though probably only in the soft transverse momentum region of $k_T \sim 1/R$.

Finally we should mention that HBT correlations are also present in the model and should in principle be added to the right hand side of Eq. (2). Parametrically HBT correlations are of the order

$$\frac{dN^{HBT}}{d^2k_1 dy_1 d^2k_2 dy_2} \sim \frac{S_A^2}{\alpha_s^2} \sim \frac{1}{\alpha_s^{10}}$$

and are much larger than any other correlations mentioned above being comparable to the uncorrelated piece. (The estimate of Eq. (9) could be obtained by squaring Eq. (3). To derive the HBT correlation coefficient one has to consider interference of the particle production amplitudes contributing to the first term in Eq. (2) [26].) However these correlations are important only when $|\vec{k}_1 - \vec{k}_2| \lesssim 1/R$ and could be excluded from experimental flow analysis by imposing corresponding cuts on particle momenta. The contribution of HBT correlations to the flow observables was studied in detail in [27].

We should also point out that non-perturbative non-flow correlations due to jet fragmentation, hadronization and resonance decay are not included in our model and could be somewhat important numerically in the data. We assume that their effect is subleading.

### 3 Azimuthal Correlations: Reaction Plane Analysis

#### 3.1 Simple Example: Autocorrelations

Let us consider a simple example of scattering events in which $N$ independent particles are produced each having a homogeneous azimuthal probability distribution $dn/d\phi \equiv (1/N) dN/d\phi = 1/2\pi$. (We set the normalization to 1 for simplicity.) Let us define the reaction plane in each event using all $N$ particles according to the standard flow analysis [28]. After choosing weights to be equal to one the reaction plane angle $\Psi_R$ for $v_2$ analysis is determined by [28]

$$\tan 2\Psi_R = \frac{\sum_{i=1}^N \sin 2\phi_i}{\sum_{j=1}^N \cos 2\phi_j}$$

where out of two roots between 0 and $2\pi$ one has to take the one with same signs for $\cos 2\Psi_R$ and $\sum_{j=1}^N \cos 2\phi_j$. Here the “reaction plane angle” $\Psi_R$ of course has nothing to do with the real reaction plane since all particles are assumed to be produced independent of geometry and of each other. Our goal is to calculate azimuthal distribution of one of the particles with respect to reaction plane determined by Eq. (10). Of course the correlation of each of the particles with the reaction plane in Eq. (10) is due to our inclusion of this particle in the definition of the reaction plane and is not physical. It is, nevertheless, instructive to see what the azimuthal shape of these autocorrelations would be.

The distribution of particles with respect to the reaction plane is defined as

$$\frac{dn}{d\phi_1 d\Psi_R} = \int_0^{2\pi} d\phi_2 \ldots d\phi_N \frac{dn}{d\phi_1 d\phi_2 \ldots d\phi_N} \delta\left(\tan 2\Psi_R - \frac{\sum_{i=1}^N \sin 2\phi_i}{\sum_{j=1}^N \cos 2\phi_j}\right)$$
\[
\frac{1}{\cos^2 2\Psi} \theta \left( \cos 2\Psi_R \sum_{k=1}^{N} \cos 2\phi_k \right) \tag{11}
\]

with the distribution function for \( N \) independent particles

\[
\frac{dn}{d\phi_1 \, d\phi_2 \ldots d\phi_N} = \frac{dn}{d\phi_1} \frac{dn}{d\phi_2} \ldots \frac{dn}{d\phi_N} = \frac{1}{(2\pi)^N}. \tag{12}
\]

For simplicity we will put \( \Psi_R = 0 \) in what follows. Representing \( \delta \) - and \( \theta \) - functions as integrals we rewrite Eq. (11) as

\[
\left. \frac{dn}{d\phi_1 \, d\Psi_R} \right|_{\Psi_R=0} = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \frac{1}{2\pi i} \frac{1}{\eta - i\epsilon} \int_0^{2\pi} d\phi_2 \ldots d\phi_N \sum_{j=1}^{N} \cos 2\phi_j \times e^{-i\xi \sin 2\phi_1 + i\eta \cos 2\phi_1} \left( \cos 2\phi_1 \left[ J_0(\sqrt{\xi^2 + \eta^2}) \right]^{N-1} \right)
\]

\[
+ (N-2) \frac{i\eta}{\sqrt{\xi^2 + \eta^2}} J_1(\sqrt{\xi^2 + \eta^2}) \left[ J_0(\sqrt{\xi^2 + \eta^2}) \right]^{N-2}, \tag{13}
\]

after integration over angles the expression becomes

\[
\left. \frac{dn}{d\phi_1 \, d\Psi_R} \right|_{\Psi_R=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \frac{1}{2\pi i} \frac{1}{\eta - i\epsilon} \exp \left( -N(\xi^2 + \eta^2) \right) \frac{N}{4} \left( \frac{N}{2} \right)^2 \tag{14}
\]

where the first term in the curly brackets of Eq. (14) corresponds to \( j = 1 \) term in the sum of Eq. (13), while the second term includes the rest of the sum. To evaluate Eq. (14) in the \( N \to \infty \) limit let us note that since \( J_0(\sqrt{\xi^2 + \eta^2}) < 1 \) for all real non-zero \( \sqrt{\xi^2 + \eta^2} \) and \( J_0(0) = 1 \) the integrals in Eq. (14) are dominated by small values of \( \xi \) and \( \eta \). Expanding the Bessel function we write

\[
\left[ J_0(\sqrt{\xi^2 + \eta^2}) \right]^{N-1} \bigg|_{N \to \infty} \approx \exp \left( -N(\xi^2 + \eta^2) \frac{4}{4} \right). \tag{15}
\]

The second term in the brackets of Eq. (14) becomes

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \frac{N}{2} \exp \left( -N(\xi^2 + \eta^2) \frac{4}{4} \right) = \frac{1}{(2\pi)^2} \tag{16}
\]

where we put the exponent of Eq. (14) to one in our large-\( N \) approximation. Due to Eq. (15) the integral in Eq. (14) is dominated by \( \xi \sim \eta \sim 1/\sqrt{N} \). Therefore expansion of exponent in Eq. (14) beyond 0th order gives \( o(1/N) \) subleading corrections.

The first term in Eq. (14) gives

\[
\frac{1}{2\pi} \cos 2\phi_1 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, \frac{1}{2\pi i} \frac{1}{\eta - i\epsilon} \exp \left( -N(\xi^2 + \eta^2) \frac{4}{4} \right) = \frac{1}{(2\pi)^2} \cos 2\phi_1 \sqrt{\frac{\pi}{N}} \tag{17}
\]
where we have used
\[ \frac{1}{\eta - i\epsilon} = \text{P.V.} \frac{1}{\eta} + \pi i \delta(\eta). \]

Combining Eqs. (16) and (17) and inserting non-zero \( \Psi_R \) we end up with

\[ \frac{dn}{d\phi \, d\Psi_R} = \frac{1}{(2\pi)^2} \left(1 + \sqrt{\frac{\pi}{N}} \cos 2(\phi - \Psi_R)\right) \tag{18} \]

up to \( o(1/N) \) corrections.

We can draw the following general conclusion from Eq. (18). If a particle is included in the reaction plane angle definition from the \( n \)-th harmonic it is correlated to the reaction plane with the shape of the correlations’ distribution in the large multiplicity limit given by \( \cos n(\phi - \Psi_R) \). Note that (auto)correlations leading to \( \cos 2\phi \) distribution of Eq. (18) are non-flow correlations by their definition.

### 3.2 Correlations with Respect to Reaction Plane

Now we are going to generalize the simple model of the previous subsection to the case of some weak pairwise physical correlations between the particles, similar to the correlations considered in [1]. Namely let us consider the case where the particle number 1, which correlations to the reaction plane will be studied, is excluded from reaction plane definition but may have some physical correlations with the particles 2, \ldots, \( N \) contributing to reaction plane determination. We thus remove autocorrelation of particle number 1 with reaction plane in accordance with the standard method of flow analysis [28]. However, particle 1 can now be physically correlated to, say, particle 2. The latter is included in reaction plane definition and is therefore autocorrelated to the reaction plane with \( \cos 2\phi \) distribution of Eq. (18). Our claim here is that it is this \( \cos 2\phi \) autocorrelations distribution, and not the physical correlations, which will be dominant in determining the azimuthal shape of the correlations of particle 1 to the reaction plane. In what follows we will keep in mind the cuts commonly employed in flow analysis, which were recently used by STAR [4]. In [4] only particles with \( p_T < 2 \text{ GeV} \) were included in the reaction plane definition (corresponding to our particles 2, \ldots, \( N \)) and correlations of particles with \( p_T > 2 \text{ GeV} \) (corresponding to our particle 1) to this reaction plane were studied.

We start by defining a correlated two-particle distribution given for example by minijet correlation discussed in [1] and normalized according to

\[ \frac{dn_{\text{corr}}}{d\phi_1 \, d\phi_2} = \frac{1}{N(N-1)} \frac{dN_{\text{corr}}}{d\phi_1 \, d\phi_2}. \tag{19} \]

The distribution of \( N \) particles now becomes

\[ \frac{dn}{d\phi_1 \, d\phi_2 \ldots d\phi_N} = \frac{dn}{d\phi_1 \, d\phi_2} \ldots \frac{dn}{d\phi_N} + (N-1) \frac{dn_{\text{corr}}}{d\phi_1 \, d\phi_2 \, d\phi_3} \ldots \frac{dn}{d\phi_N} + \ldots \tag{20} \]

where the factor of \( N - 1 \) accounts for all possible pairwise correlations of particle 1 with particles 2, \ldots, \( N \). In Eq. (20) we omit possible pairwise correlation of particles 2, \ldots, \( N \) with each other which are of the same order as the correlations shown in Eq. (20) but do not
contribute to the azimuthal asymmetry of the distribution of particle 1 with respect to the reaction plane, giving only an additive small constant. The correlations of more than one pair of particles are neglected in Eq. (20) since they are suppressed by higher powers of $\alpha_s [1]$. 

To determine the distribution of particle 1 with respect to reaction plane we have to substitute distribution of Eq. (20) into Eq. (11). Evaluation of the first term in the resulting expression could be easily done repeating the steps which led to the first term in Eq. (18). One thus obtains

$$
\left. \frac{dn}{d\phi_1 d\Psi_R} \right|_{\Psi_R=0} \approx \frac{1}{(2\pi)^2} + (N - 1) \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{d\eta}{2\pi i} \frac{1}{\eta - i\epsilon} \int_{0}^{2\pi} \frac{d\phi_2 \ldots d\phi_N}{(2\pi)^{N-2}} \frac{dn_{corr}}{d\phi_1 d\phi_2} \times \sum_{j=2}^{N} \cos 2\phi_j e^{-i\xi \sum_{i=2}^{N} \sin 2\phi_i + i\eta \sum_{k=2}^{N} \cos 2\phi_k}.
$$

(21)

We are interested only in $j = 2$ term in Eq. (21) since it comes with $\cos 2\phi_2$ and can give non-trivial correlations with the reaction plane. In all the other terms in the sum of cosines the integration over $\phi_2$ would give an additive constant small compared to the first term on the right hand side of Eq. (21). Assuming that $dn_{corr}/d\phi_1 d\phi_2$ is an even function of $\phi_1 - \phi_2$ only (which is, probably, true for all perturbative spin-independent pairwise correlations and is certainly true for correlations considered in [1]) we write

$$
\int_{0}^{2\pi} \frac{d\phi_2}{d\phi_1} \frac{dn_{corr}}{d\phi_1 d\phi_2} \cos 2\phi_2 = \cos 2\phi_1 \int_{0}^{2\pi} \frac{d\phi_2}{d\phi_1} \frac{dn_{corr}}{d\phi_1 d\phi_2} \cos (2\phi_1 - \phi_2)
$$

$$
= \cos 2\phi_1 2\pi v_2(1) v_2(2) \frac{dn}{d\phi_1} \frac{dn}{d\phi_2} = \cos 2\phi_1 \frac{1}{2\pi} v_2(1) v_2(2),
$$

(22)

where we have used the definition of flow observable from Eq. (5) [24, 25] and substituted $dn/d\phi = 1/2\pi$. $v_2(1)$ and $v_2(2)$ are elliptic flow variables for the particles 1 and 2 correspondingly. Let us now consider a specific example of flow analysis when the particle 1 represents all particles with fixed value of transverse momentum $p_T$ while particle 2 represents particles in a broad transverse momentum range which go into reaction plane definition (e.g. all particles with $p_T < 2$ GeV as in STAR analysis [4]). Then $v_2(1)$ would correspond to differential elliptic flow $v_2(p_T, B)$, while $v_2(2)$ would be the averaged elliptic flow $v_2(B)$, where $B$ is the impact parameter of the collision. With the help of Eq. (22), Eq. (21) becomes

$$
\left. \frac{dn}{d\phi_1 d\Psi_R} \right|_{\Psi_R=0} \approx \frac{1}{(2\pi)^2} + \cos 2\phi_1 \frac{1}{2\pi} (N - 1) v_2(p_T, B) v_2(B)
$$

$$
\times \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \frac{d\eta}{2\pi i} \frac{1}{\eta - i\epsilon} \left[ J_0(\sqrt{\xi^2 + \eta^2}) \right]^{N-2},
$$

(23)

which can be rewritten in the large-$N$ approximation as

$$
\left. \frac{dn}{d\phi_1 d\Psi_R} \right|_{\Psi_R=0} \approx \frac{1}{(2\pi)^2} + \cos 2\phi_1 \frac{1}{2\pi} N v_2(p_T, B) v_2(B)
$$
Performing the integrations in Eq. (24) similar to how it was done in obtaining Eq. (17) and reintroducing non-zero $\Psi_R$ we write

$$\frac{d\eta}{d\phi_{p_T} \ d\Psi_R} \approx \frac{1}{(2\pi)^2} \left[ 1 + \sqrt{\pi N} v_2(p_T, B) \cos 2(\phi_{p_T} - \Psi_R) \right].$$

Eq. (25) represents the main point of this Section: the shape of the correlations of particles with respect to the reaction plane determined by the standard elliptic flow analysis is proportional to $\cos 2(\phi_{p_T} - \Psi_R)$ independent of the physical nature of these correlations. Therefore particle distribution with respect to reaction plane carries no information about the azimuthal shape of physical correlations.

Nevertheless, the coefficient in front of the cosine in Eq. (25) is given by the strength of physical correlations. To understand the nature of the coefficient in Eq. (25) let us calculate reaction plane resolution in our model. If we divide all $N - 1$ particles defining the reaction plane into two subgroups of (roughly) $N/2$ particles each and determine the reaction plane angles in each subgroup independently ($\Psi_1$ and $\Psi_2$), the reaction plane resolution would be given by [28]

$$\Delta = \sqrt{2 \langle \cos 2(\Psi_1 - \Psi_2) \rangle}$$

where the averaging is taken over many same–multiplicity events. To determine the strength of the reaction plane correlations in our model we note that the sub-event planes are correlated in it only due to pairwise correlations between the particles. One particle from plane $\Psi_1$ is correlated to plane $\Psi_2$ via the distribution in Eq. (25). The same particle is correlated to its own plane $\Psi_1$ via autocorrelations of Eq. (18). To determine correlations between the two planes introduced by this one particle we have to multiply the two distributions and average over all angles and momenta of the particle $\phi$. The total correlations would be obtained by taking this correlation due to one particle to the $N/2$ power to account for all $N/2$ particles in the sub-event plane definition. Expanding the resulting expression up to the lowest order in correlations we end up with

$$\frac{d\eta}{d\Psi_1 \ d\Psi_2} \propto 1 + \frac{\pi}{4} N v_2(B)^2 \cos 2(\Psi_1 - \Psi_2).$$

Using Eqs. (26) and (27) we can determine the reaction plane resolution as

$$\Delta = \frac{\sqrt{\pi N}}{2} v_2(B),$$

which is in agreement with [28, 29]. With the help of Eq. (28), Eq. (25) can be rewritten as

$$\frac{d\eta}{d\phi_{p_T} \ d\Psi_R} \approx \frac{1}{(2\pi)^2} \left[ 1 + 2 v_2(p_T, B) \Delta \cos 2(\phi_{p_T} - \Psi_R) \right].$$

Now we can see that, independent of the nature of correlations contributing to $v_2$, particle distribution with respect to reaction plane has a $\cos 2\phi$ shape with the amplitude given by
differential elliptic flow observable \( v_2(p_T, B) \) times the event plane resolution. In a recent paper by STAR [4] the data on particle distribution with respect to reaction plane was reported, which was fitted rather well with the ansatz of Eq. (29) [4]. Eq. (29) tells us that since our model describes the data on \( v_2(p_T, B) \) rather well [1] it also agrees with the recent data on correlations with respect to reaction plane reported by STAR [4].

4 Azimuthal Correlations: Correlation Functions

We saw in the previous section that the reaction plane analysis cannot determine the physical shape of azimuthal particle correlations. In this section we consider another method of azimuthal correlations analysis – the correlation function method. We believe that this method correctly determines the shape of azimuthal correlations. The two-particle azimuthal correlation function is defined by [22]

\[
C(\Delta \phi) = \frac{dN_{\text{real}}/d\Delta \phi}{dN_{\text{mixed}}/d\Delta \phi} \frac{N_{\text{mixed}}}{N_{\text{real}}},
\]

where \( dN_{\text{real}}/d\Delta \phi \) is the number of particle pairs observed in the same event with a given azimuthal opening angle \( \Delta \phi \) and \( dN_{\text{mixed}}/d\Delta \phi \) is the number of particle pairs selected from two different events with the same azimuthal opening angle. \( N_{\text{real}} \) and \( N_{\text{mixed}} \) are total numbers of pairs in the same and in different events correspondingly. Only events with the same multiplicity are selected for the analysis of \( C(\Delta \phi) \). In this section we will calculate the contribution of two-particle correlations in our minijet model [1] to the azimuthal correlation function.

The number of particle pairs produced in the same event with a given azimuthal opening angle \( \Delta \phi \) can be obtained by integrating Eq. (2)

\[
\frac{dN_{\text{real}}}{d\Delta \phi} = 2\pi \int dk_1^1 dk_2^1 dy_1^1 dy_2^1 \frac{dN}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1},
\]

while the total number of pairs is

\[
N_{\text{real}} = \int d^2 k_1^1 d^2 k_2^1 dy_1^1 dy_2^1 \frac{dN}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1}.
\]

Particle pair where particles are selected from two different events obviously have no physical correlations. Therefore their azimuthal angle distribution is trivial and is given by

\[
\frac{1}{N_{\text{mixed}}} \frac{dN_{\text{mixed}}}{d\Delta \phi} = \frac{1}{2\pi}.
\]

Combining Eqs. (31), (32), (33) and (2) with Eq. (30) yields

\[
C(\Delta \phi) = \frac{(2\pi)^2 \int dk_1^1 dk_2^1 dy_1^1 dy_2^1 \left( \frac{dN}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1} + \frac{dN_{\text{corr}}}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1} \right)}{\int d^2 k_1^1 d^2 k_2^1 dy_1^1 dy_2^1 \left( \frac{dN}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1} + \frac{dN_{\text{corr}}}{d^2 k_1^1 dy_1^1 d^2 k_2^1 dy_2^1} \right)}.
\]

To calculate the correlation function we, therefore, need to know single- and double- inclusive particle distributions.
We calculated the two-particle multiplicity distribution of Eq. (2) in our previous work [1] assuming that there is a large rapidity interval \(|y_1 - y_2| \sim 1/\alpha_s \gg 1\) between the correlated particles. It turns out that this approximation preserves all important ingredients of the model as long as we are not concerned with the rapidity dependence of elliptic flow and with the azimuthal angular distributions. At the same time it considerably simplifies calculations allowing for a simple interpretation of the final result. However, the large rapidity interval assumption does not hold in the actual flow analyses. In order to properly describe data one has to relax the \(|y_1 - y_2| \gg 1\) condition. Below we intend to compare our model’s predictions to the correlation function data reported by PHENIX [22]. The (pseudo)rapidity acceptance of the PHENIX detector is limited to the interval of 0.7 units in the central rapidity region and the \(|y_1 - y_2| \gg 1\) condition certainly does not hold there.

Single particle distribution was written in our model [1] as

\[
\frac{dN}{d^2 k_1 dy_1} = 2\alpha_s C_F S_\perp \int \frac{d^2 q_1}{q_1^2} \frac{dx G_A}{dq_1^2} \frac{dx G_A}{d(k_1 - q_1)^2}.
\]

Relaxing \(|y_1 - y_2| \gg 1\) condition to \(|y_1 - y_2| \sim 1\) yields a more general expression for the two-particle distribution

\[
\frac{dN_{corr}}{d^2 k_1 dy_1 d^2 k_2 dy_2} = \frac{N_c \alpha_s^2}{\pi^2 C_F S_\perp} \int \frac{d^2 q_1}{q_1^2} \int \frac{d^2 q_2}{q_2^2} \delta^2(q_1 + q_2 - k_1 - k_2)
\]

\[
\times \frac{dx G_A}{dq_1^2} \frac{dx G_A}{dq_2^2} A(q_1, q_2, k_1, k_2, y_1 - y_2),
\]

where \(A(q_1, q_2, k_1, k_2, y_1 - y_2)\) is the two-to-four particles amplitude in the quasi-multi-Regge kinematics (\(|y_1 - y_2| \sim 1\)). \(A\) was found in an impressive calculation performed in [19]. We refer the reader to the reference [19] for an explicit expression for \(A(q_1, q_2, k_1, k_2, y_1 - y_2)\). In the leading logarithmic approximation \(|y_1 - y_2| \sim 1/\alpha_s \gg 1\) this amplitude reduces to

\[
A(q_1, q_2, k_1, k_2, y_1 - y_2) = \frac{q_1^2 q_2^2}{k_1^2 k_2^2}.
\]

This expression for \(A\) was used in our model before [1] yielding only back-to-back correlations between the particles. It was argued in [19] that inclusion of the next-to-leading logarithmic corrections is essential for understanding the angular distribution of the correlation function. In particular, on top of the back-to-back correlations, they introduce correlations at small angles \(\Delta \phi\) between the particles in a pair [19].

The unintegrated gluon distribution due to the quasi-classical non-Abelian Weiszäcker-Williams (WW) field \(A^{WW}(z)\) of the nucleus is given by [11, 17]

\[
\frac{dx G_A(x, q^2)}{dq^2} = \frac{2}{(2\pi)^2} \int d^2 z e^{-izq} \int d^2 b \text{Tr} \langle A^{WW}(0) A^{WW}(z) \rangle
\]

\[
= \frac{2}{\pi(2\pi)^2} \int d^2 z e^{-izq} S_\perp C_F \frac{S_\perp}{\alpha_s z^2} (1 - e^{-\frac{1}{4}z^2 q_s^2}),
\]

(38)
Figure 1: Two-particle azimuthal angle correlation function given by our model compared to PHENIX √s = 130 GeV data for different cuts in the transverse momentum and centralities. We used the following values of parameters: Λ = 0.15 GeV, A = 197, α_s = 0.3.

where $b$ is the gluon’s impact parameter (which we can trivially integrate over in a cylindrical nucleus case considered here) and $Q_s^2$ is a certain scale at which nonlinear nature of the gluon field becomes evident. In [1] we argued that one has to include quantum evolution effects in quasi-classical formulae in order to describe some important features of the particle spectrum. In particular we took into account the fact that evolved gluon distribution scales as a function of $p_T/Q_s$ only, where $Q_s$ is a saturation scale which emerges from the solution to the nonlinear evolution equation [14]. We argued in [1] that evolution effects can be mimicked by writing the argument of the Glauber exponent of Eq. (38) in the form

$$-rac{1}{4}z^2 Q_s^{cl^2}(z) = -\left(\xi^2 Q_s^2/4q^2\right) \frac{\ln Q_s/\xi}{\ln(Q_s/\Lambda)}$$

where $\xi = qz$ and $\Lambda$ is the infrared cutoff. Thus, after integration over angles in Eq. (38) we obtain

$$\frac{dx G_A(x, q^2)}{dq^2} = \frac{1}{\pi^2} \frac{S_1 C_F}{\alpha_s} \int_0^{Q_s/\Lambda} \frac{d\xi}{\xi} J_0(\xi) \frac{S_1 C_F}{\alpha_s} \left(1 - e^{-(\xi^2 Q_s^2/4q^2)} \frac{\ln Q_s/\xi}{\ln(Q_s/\Lambda)}\right)$$

Formulae (34), (35), (36), and (40), together with the expression for $A$ given in [19] comprise all necessary theoretical information for calculation of the correlation function $C(\Delta \phi)$. In order to compare our correlation function to PHENIX data [22] we have to integrate in Eq. (34) over $y_1$ and $y_2$ in the rapidity interval from $-0.35$ to $+0.35$ (PHENIX detector acceptance) and over $k_1$ and $k_2$ in the transverse momentum intervals specified by the data analysis of [22]. The two-particle distribution of Eq. (36) with $A$ from [19] has a collinear singularity at
small opening angles $\Delta \phi$ due to the contribution of $1 \to 2$ gluon splitting [19]. It is possible that the corresponding increase of the correlation function would be reduced after inclusion of higher order perturbative corrections [30]. The small angle particle correlations also receive contributions from HBT correlations, resonance decays and hadronization functions. While some of these correlations are excluded by appropriate cuts in the experimental flow analysis, some might still remain. To model the effect of these phenomena on the correlation function in our model we have imposed a lower cutoff on the invariant mass of the two particles produced in Eq. (36), with the value of the cut obtained from comparison to the data [22]. The best fit was obtained by requiring that $(k_1 + k_2)^2 = 2 (k_1 \cdot k_2 - k_1 k_2 \cosh(\Delta y) - k_1 \cdot k_2) > 0.07 \text{ GeV}^2$.

Results of our calculations are presented in Fig. 1. The values of parameters that were used are specified in the caption of Fig. 1. Integrations over $q_{1,2}$ in Eq. (36) were cut off in the infrared limit of $q_{1,2} = 0$ by $2\Lambda$. The value of the saturation scale giving the best fit for $0.3 < p_T < 2.5 \text{ GeV}$ data is slightly lower than the saturation scale giving the best fit for $0.5 < p_T < 2.5 \text{ GeV}$ data ($Q_s = 0.8 \text{ GeV}$ and $Q_s = 1 \text{ GeV}$ correspondingly). This slight discrepancy is probably due to the fact that non-perturbative (soft) effects become more important for lower $p_T$ data ($0.3 < p_T < 2.5 \text{ GeV}$) which could be effectively mimicked by reducing the saturation scale. The fall-off of our correlation function at the very small angles is an artifact of the applied invariant mass cut. More research is needed to completely quantitatively understand the correlation function at small opening angles $\Delta \phi$. Overall we observe a good agreement of our model with experimental data reported by PHENIX [22].

5 Higher Order Cumulants

A new method of elliptic flow analysis was recently proposed in [23]. It was suggested to measure flow using not only two-particle correlations of Eq. (5) [24, 25] but also four-, six- and other higher order correlation functions. The flow observable can be extracted from cumulants formed out of these correlation functions. For instance the fourth order cumulant for elliptic flow is defined as [23]

$$c_2\{4\} \equiv \langle e^{2i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle \equiv \langle e^{2i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \rangle - 2 \langle e^{2i(\phi_1 - \phi_4)} \rangle$$

and in case of only flow correlations is equal to [23]

$$c_2\{4\} = -v_2^4.$$  

The advantage of the higher order cumulant analysis is in the fact that, as argued in [23], if flow is larger than non-flow correlations, the contribution of the latter to $v_2$ extracted from higher order cumulants is suppressed by powers of particle multiplicity. For instance, if $M$ particles are emitted in a collision it can be easily shown that the contribution of non-flow correlations to $v_2$ from Eq. (5) scales as $1/\sqrt{M}$. At the same time a similar analysis shows that non-flow $v_2$ extracted from the fourth order cumulant of Eq. (42) scales as $1/M^{3/4}$, i.e., it is suppressed by an extra factor of $1/M^{1/4}$. Corresponding data analysis based on Eqs. (41) and (42) has been carried out at STAR [3].

In saturation particle production models one should use the number of participants $N_{\text{part}} \sim S_A Q_s^2$ instead of $M$ [16, 1]. For $\sqrt{s} = 130 \text{ GeV}$ at RHIC the number of participants for the
most central collisions is \( N_{\text{part}}(B = 0) = 344 \) [16] which gives a suppression factor \( 1/N_{\text{part}}^{1/4} \approx 0.23 \). This factor increases with decreasing multiplicity and, correspondingly, centrality of the collision. We are going to give a somewhat more careful estimate of what exactly the suppression factor is in the framework of our model of non-flow correlations.

In principle to estimate the contribution of saturation physics to the fourth order cumulant from Eq. (41) one has to calculate the four particle inclusive cross section in the quasi-classical approximation of [10, 11, 12, 13, 17]. This task seems to be extremely difficult. Constructing a model similar to what was done for double inclusive cross section is dangerous since at this high order the model inspired by factorization approaches may not be valid at all. What we are going to do here is to neglect this four particle correlations and estimate the contribution of two-particle correlations of Eq. (2) to the fourth order cumulant in Eq. (41).

Substituting the distribution of Eq. (2) into Eq. (41) we obtain

\[
c_2^2 \{4\} = 2 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_3} \frac{dN_{\text{corr}}}{d\phi_2} \frac{dN_{\text{corr}}}{d\phi_4} e^{2i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} \times \left( \frac{1}{\int \frac{dN}{d\phi_1} \frac{dN}{d\phi_2} + \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2}} + 6 \int \frac{dN}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_3} \frac{dN_{\text{corr}}}{d\phi_4} + 3 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2} \frac{dN_{\text{corr}}}{d\phi_3} \frac{dN_{\text{corr}}}{d\phi_4} - \frac{1}{\left( \int \frac{dN}{d\phi_1} \frac{dN}{d\phi_2} + \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2} \right)^2} \right),
\]

where the integral sign denotes integration over all azimuthal angles to follow. The numerator is the same in both terms in Eq. (41). It gives the prefactor in Eq. (43). However the denominators are not exactly the same in the two terms in Eq. (41), and this is reflected by the difference in the parentheses of Eq. (43). The denominator of the second term is equal to the number of pairs squared, while the denominator of the first term is equal to the number of quadruplets of particles.

Expanding the expression in parentheses of Eq. (43) to the lowest non-trivial order in \( dN_{\text{corr}}/d\phi_1 d\phi_2 \) yields

\[
c_2^2 \{4\} \approx -8 \left[ \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_3} e^{2i(\phi_1 - \phi_3)} \right]^2 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2} = -8 c_2^2 \{2\}^2 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2} = -8 v_2^2 \{2\}^4 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2}
\]

where we follow the notation of [23] for elliptic flow variable \( v_2^2 \{2\} \) extracted from the second order cumulant \( c_2^2 \{2\} \) (see Eq. (5)). Using Eq. (43) we can deduce from Eq. (44) the following expression for the elliptic flow extracted from the fourth order cumulant

\[
v_2^\{4\} = v_2^\{2\} \left[ \frac{8 \int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2}}{\int \frac{dN}{d\phi_1} \frac{dN}{d\phi_2}} \right]^\frac{1}{4}. \quad (45)
\]

To put a lower bound on the suppression factor in Eq. (45) we first note that since cosine is always less or equal to one

\[
\int \frac{dN_{\text{corr}}}{d\phi_1} \frac{dN_{\text{corr}}}{d\phi_2} \geq v_2^\{2\}^\{2\}. \quad (46)
\]
According to the $\sqrt{s} = 130$ GeV data reported by STAR the elliptic flow for the most central collisions is $v_2 = 1.87 \pm 0.25\%$ which after insertion into Eqs. (46) and (45) gives the suppression factor

$$\frac{v_2^4}{v_2^2} \geq 0.23,$$ (47)

where the precise agreement with the number cited before is coincidental. Using the full expression in Eq. (43) instead of Eq. (45) does not significantly change the result.

The suppression factor of Eq. (45) can be estimated in the framework of our minijet model [1]. The result is

$$\frac{v_2^4}{v_2^2} \approx 0.5 - 0.75,$$ (48)

where the discrepancy is due to infrared cutoff sensitivity of our model. An exact calculation in the saturation framework of [10, 11, 13, 17, 15] should give a more precise estimate. Both numbers in Eqs. (47) and (48) are consistent with the STAR data presented in Fig. 13 of [3]. There the ratio of the elliptic flow extracted from the fourth and the second order cumulants is in the range of 0.25 − 0.5 for the most central events, where the discrepancy is due to different methods of calculating the fourth order cumulant [3].

Using the same two-particle correlations model we can estimate the ratio of the elliptic flow variable extracted from the 6th-order cumulant to the one extracted from the 2nd-order cumulant to be

$$\frac{v_2^6}{v_2^2} = \left[ 108 \left( \frac{\int dN_{corr} d\phi_1 d\phi_2}{\int dN d\phi_1 d\phi_2} \right)^2 \right]^{1/6} \approx 0.5 - 0.75$$ (49)

where we give a numerical estimate of the ratio in our model for the most central collisions.

Fig. 13 of [3] shows that the ratio of $v_2^4/v_2^2$ increases approaching unity with decreasing centrality down to 60% centrality. This is in qualitative agreement with our Eq. (45), which implies that $v_2^4/v_2^2 \sim 1/N_{part}^{\frac{1}{4}} \sim 1/(S_A Q_s^2)^{\frac{1}{4}}$ so that the ratio increases with decreasing centrality. At the same time if the difference between $v_2^2$ and $v_2^4$ reported by STAR was due to elimination of non-flow effects as expected in [3] it should have been increasing with decreasing centrality which is true only for the most peripheral data. Despite the qualitative agreement of our model with the data of [3] we do not attempt to fit the STAR fourth order cumulant data at the moment since for a consistent fit one needs to include the contributions of the 4-particle correlations in Eq. (41).

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References


