Quantum Gravity on a Circle and the
Diffeomorphism Invariance of the Schrödinger Equation

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Abstract

We study a model for quantum gravity on a circle in which the notion of a classical metric tensor is replaced by a quantum metric with an inhomogeneous transformation law under diffeomorphisms. This transformation law corresponds to the co-adjoint action of the Virasoro algebra, and resembles that of the connection in Yang–Mills theory. The transformation property is motivated by the diffeomorphism invariance of the one dimensional Schrödinger equation. The quantum distance measured by the metric corresponds to the phase of a quantum mechanical wavefunction. The dynamics of the quantum gravity theory are specified by postulating a Riemann metric on the space $Q$ of quantum metrics and taking the kinetic energy operator to be the resulting laplacian on the configuration space $Q/\text{Diff}_0(S^1)$. The resulting metric on the configuration space is analyzed and found to have singularities. The second-quantized Schrödinger equation is derived, some exact solutions are found, and a generic wavefunction behavior near one of the metric singularities is described. Finally some further directions are indicated, including an analogue of the Yamabe problem of differential geometry.
1. Introduction

Understanding gauge theories such as Yang–Mills theory and General Relativity at the quantum level is a problem of fundamental significance in theoretical physics. It is useful to study these problems in simpler lower dimensional contexts [1], [2], [3]. Yang–Mills theory on a cylinder has been solved by canonical methods in Ref. [4]. It would be interesting also to study the canonical quantization of lower dimensional theories of gravity, in analogy with that of Yang–Mills theory. The analogy between classical Yang–Mills theory and General Relativity has been very fruitful for both fields. See e.g., [5]. However, this analogy is imperfect because the dynamical variable (metric) of General Relativity is a tensor while that of Yang–Mills theory (connection) has an inhomogeneous transformation law. We will study a canonical theory of gravity in one–dimension, in which the dynamical variable of gravity has an inhomogeneous transformation law, analogous to that of the connection. Since our theory is only invariant under the diffeomorphism group of the circle (not of the two dimensional space–time) it is to be viewed as a partially gauge fixed theory. This is analogous to the canonical formalism of Yang–Mills theory in the temporal gauge $A_0 = 0$.

We will now argue that an inhomogeneous transformation law for the metric is natural, if we regard the devices that measure distances to be quantum mechanical. In a purely classical theory, the measurement of length and time can be accomplished using rods and clocks, leading to the picture of the metric as a tensor [6]. However, in a quantum theory of matter, it would no longer make sense to use classical objects such as rods to measure distance: the position of the endpoints cannot be measured with infinite accuracy. Some quantum device must be used instead. One could use an interference experiment to measure the ‘phase delay’ of the wavefunction between two points. It would not be possible to define the length of a curve this way, since we cannot assign a particular path to the wave; yet, it should be possible to define a distance between two points this way.
The distance between two points then would be an average over all possible paths that connect two points rather than the length of the minimal path. The classical concept of distance should then be the eikonal approximation of this quantity. It is not yet possible to implement this idea in general, but we will be able to do so in the case of one-dimensional geometry.

In one dimensional Riemannian geometry (on a circle), the metric $ds^2 = g(x)dx^2$, is determined by a positive function $g : S^1 \to R$. The distance between two points is $W(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{g(x)}dx$ and is independent of the choice of the co-ordinate $x$. This can be thought of as the solution to the eikonal form of the geodesic equation,

$$
\left( \frac{\partial W(x_1, x_2)}{\partial x_2} \right)^2 - g(x_2) = 0
$$

(1)

with the boundary condition $W(x_1, x_1) = 0$. If $W$ transforms as a scalar and the metric $g$ transforms as a second rank covariant tensor, this equation is invariant under change of co-ordinates. If we identify $g(x) = E - V(x)$, $E$ being the total energy and $V$ the potential energy, this is the Hamilton–Jacobi equation of a non-relativistic point particle of mass $m = 1/2$, and the distance function $W(x_1, x_2)$ is essentially Hamilton’s characteristic function, restricted to energy $E$. By making this connection we establish the Jacobi (or Maupertuis) principle of classical mechanics [7]: trajectories of a point particle of energy $E$ moving in potential $V(x)$ are geodesics of the conformally flat metric $ds^2 = (E - V(x))dx^2$. This fact is true in higher dimensions as well, but the identification of Hamilton’s characteristic function with the distance function must be modified. The point is that there is a correspondence between conformally flat Riemannian geometry and the dynamics of point particles moving in potentials in flat space. In one dimension all metrics are in fact flat so this correspondence can be made for all one dimensional metrics.

A natural way, then, to discover a quantum generalization of one dimensional Riemannian geometry is to base the quantum geometry on a similar correspondence with the
dynamics of non-relativistic quantum particles in flat space, these being described by the Schrödinger equation:

\[
\frac{\partial^2 \psi}{\partial x^2} + \hat{q} \psi = 0.
\] (2)

where we should identify \( \hat{q}(x) = \frac{1}{\hbar^2}(E - V(x)) \), and we will find it convenient for later arguments to define \( q(x) \) by \( \hat{q} = -\frac{12\pi}{\hbar^2}q + \frac{1}{4} \) and \( \frac{b}{12\pi} = \hbar^2 \). This differential equation with a periodic coefficient \( q \) is called the Hill's equation in the classical literature  [8].

The quantum metric, in analogy to the classical situation, is \( \hat{q} \). This metric defines a quantum distance which may be identified with the change of phase of the wavefunction \( \psi \). In fact if we rewrite \( \psi(x = x_2) \) with boundary condition \( \psi(x_1) = 1 \) as \( \psi(x_1, x_2) = e^{i\frac{\hbar \pi}{W(x_1, x_2)}} \), we find that in the classical limit, \( \hbar \to 0 \), (2) reproduces the classical distance equation (1) with the appropriate boundary condition. There is some ambiguity in the choice of \( \psi \) owing to the fact that the quantum distance formula is second order, and the classical distance equation is first order. We are free to choose a second boundary condition on \( \psi \) without affecting the classical limit. We will see later that a convenient choice is to take \( \psi(x_1, x_1 + 2\pi) = e^{i\text{arccos}(\frac{\hbar}{2}\text{tr}M_q)} \), where \( M_q \) is the monodromy matrix of the Hill's operator.

As mentioned, the classical Hamilton-Jacobi equation is invariant under diffeomorphisms of the circle. To interpret the Schrödinger equation (2) as a quantum distance formula it too must exhibit this symmetry. This equation appears, at first, not to be invariant under a change of co-ordinates. However, if \( \psi \) transforms as a half-density

\[
\psi \mapsto \phi \circ \psi, \quad \phi \circ \psi(x) = \psi(\phi(x))[\phi'(x)]^{-\frac{1}{2}}
\] (3)

and \( q \) transforms as follows,

\[
q \mapsto \phi \circ q, \quad \phi \circ q(x) = q(\phi(x))\phi'^2(x) - \frac{b}{2\pi} \sigma\phi(x)
\] (4)
the equation is invariant under diffeomorphisms [9]. Here,
\[
\sigma_\phi(x) = \frac{1}{6} \left[ \frac{1}{2} \frac{\phi'''}{\phi'} - \frac{3}{4} \left( \frac{\phi''}{\phi'} \right)^2 \right] + \frac{1}{24} (\phi'^2(x) - 1). \tag{5}
\]
Thus there should be a generalization of Riemannian geometry where the metric is not a tensor, but rather has this inhomogeneous transformation law. In addition, the quantum generalization of the distance will no longer be a scalar function. Taking the distance to be the phase, \( W(x) \), of a solution to (2) leads to the transformation law:
\[
W \mapsto \phi \circ W, \quad \phi \circ W(x) = W(\phi(x)) - \frac{\hbar}{2i} \ln \phi'(x). \tag{6}
\]
As expected, in the limit \( \hbar \to 0 \ (b \to 0) \), \( q \) obtains the tensor transformation property, and \( W \) becomes a scalar.

The transformation law of \( q \) can also be understood as the co–adjoint action of the Virasoro algebra [9], just as the transformation law of the connection in Yang–Mills theory can be understood as the co–adjoint action of the affine Kac–Moody algebra.

The configuration space of our theory of gravity is \( \mathcal{Q} = \{ q : S^1 \to \mathbb{R} \} \), the space of periodic potentials or Hill’s operators. We then define a Riemannian metric on \( \mathcal{Q} \),
\[
||\delta q||^2 = \int \delta q^2(x) u_2^6(x) dx. \tag{7}
\]
Here \( u_2 \) is the solution to Hill’s equation with boundary conditions,
\[
u_2(0) = 1, \quad u_2'(0) = 0. \tag{8}
\]
This metric on \( \mathcal{Q} \) is invariant under the subgroup \( \text{Diff}_0(S^1) \) of diffeomorphisms that agree with the identity at \( x = 0 \) up to third order. (This restriction arises for reasons which will be explained later). This metric induces a Riemannian metric \( g \) on the three dimensional manifold \( \mathcal{M} = \mathcal{Q}/\text{Diff}_0(S^1) \). The wavefunctions of our model for gravity on the circle are postulated to be functions on \( \mathcal{Q} \) invariant under \( \text{Diff}_0(S^1) \): i.e., functions on \( \mathcal{M} \). The
hamiltonian the theory is the Laplace operator on $\mathcal{M}$ with respect to the induced metric $g$. These choices are justified by analogy with the canonical formalism of Yang–Mills theory on a circle [4].

We will compute the metric tensor on $\mathcal{Q}$ and its Laplace operator explicitly in an appropriate co-ordinate system. Also, we will show that the metric $g$ has two Killing vectors and one conformal Killing vector with constant scale factor. This implies [10] that $\mathcal{M}$ has a foliation as a one-parameter family of two dimensional manifolds of constant negative curvature. We will find a co-ordinate sytem that exploits this symmetry. Also we will show that the metric $g$ is in fact singular. The behaviour of the eigenfunctions of the Laplace operator near the singularities will be determined.

2. Analogy with Yang–Mills Theory on a Circle

In this section we follow the ideas of Segal [9] to construct the gravitational analogue of Yang–Mills theory on a circle. In the canonical formalism of Yang–Mills theory on a circle, [4] the dynamical variable is a Lie–Algebra valued 1-form (connection or gauge field) $A$. The invariance group (gauge group) $\mathcal{G}$ is the loop group, the set of functions from the circle to some compact Lie group $G$. Under the action of the gauge group, the gauge field transforms as follows:

$$A \rightarrow gAg^{-1} + \frac{1}{\epsilon}gdg^{-1}. \quad (9)$$

Here $\epsilon$ is the gauge coupling constant. (It is possible to set $\epsilon = 1$ by rescaling $A$, but we will find it convenient not to do this.) Infinitesimally,

$$\delta \lambda A = \frac{1}{\epsilon}d\lambda + [A, \lambda] \quad (10)$$

where $\lambda : S^1 \rightarrow \hat{\mathcal{G}}$ is an element of the loop algebra $\hat{\mathcal{G}}$. It is possible to understand this as the co-adjoint action of the central extension $\hat{\mathcal{G}}$ of the loop algebra. [9].
The central extension, $\hat{G}$, of the loop algebra consists of ordered pairs, $(\lambda, a)$, $\lambda \in \hat{G}$ and $a \in R$. The Lie bracket of $\hat{G}$ is
\[
[(\lambda, a), (\tilde{\lambda}, \tilde{a})] = ([\lambda, \tilde{\lambda}], \frac{1}{2\pi} \int \text{tr} d\lambda \tilde{\lambda}).
\] (11)
Here tr denotes an invariant inner product in $G$. Note that elements of the form $(0, a)$ are central; i.e., commute with every element. If we introduce a co–ordinate $0 \leq x \leq 2\pi$ on $S^1$ and a basis $J_{ma}(x) = (e^{imx}e_a, 0), k = (0, 1)$, (where $e_a$ is a basis in the finite dimensional algebra $G$) we can rewrite these commutation relations as:
\[
[J_{ma}, J_{nb}] = f_{ab}^c J_{m+n} c + ig_{ab}m\delta(m + n), \quad [J_{ma}, k] = 0.
\] (12)
Here $g_{ab} = \text{tr} e_a e_b$ is the inner product in the basis $e_a$. Thus $\hat{G}$ is an affine Kac–Moody algebra [11]. We will find it more convenient to think of it in terms of ordered pairs as above. The next step is to construct the representation of $\hat{G}$ on its dual space.

But first, we digress for a technical comment on the definition of the dual vector space of $\hat{G}$. In finite dimensions, the dual of a vector space isomorphic to itself, although there is no natural choice of such an identification. Each such choice is equivalent to the choice of an inner product. In infinite dimensions, (as for $\hat{G}$) in general, the dual of a vector space is not isomorphic to itself. If for example we assign to $G$ the topology of the space of smooth functions, its dual would be the space of distributions. This is too large for our purposes. It turns out that the best choice [9] is to work with the ‘smooth dual’, which consists of ordered pairs $(A, b)$, $A$ being a Lie–algebra valued 1–form on the circle and $b \in R$. Each $(A, b)$ can be thought of as defining the linear function
\[
< (A, b), (\tilde{\lambda}, \tilde{a}) > = \int A(x)\tilde{\lambda}(x) dx + \tilde{a} b
\] (13)
on $\hat{G}$. The co–adjoint action of $(\lambda, a)$ on $(A, b)$ is defined by the requirement that this linear function be unchanged when we change also $(\tilde{\lambda}, \tilde{a})$ by the adjoint action of $(\lambda, a)$.
\[
< \delta_{\lambda, a}(A, b), (\tilde{\lambda}, \tilde{a}) > + < (A, b), ([\lambda, a], (\tilde{\lambda}, \tilde{a})] >= 0.
\] (14)
That is,
\[
\langle \delta \lambda, a(A, b), (\tilde{\lambda}, \tilde{a}) \rangle = - \int \text{tr} A(x) [\lambda(x), \tilde{\lambda}(x)] dx - \frac{b}{2\pi} \int \text{tr} d\lambda \tilde{\lambda}
\]
(15)
which leads to
\[
\delta \lambda, a(A, b) = (- \frac{b}{2\pi} d\lambda - [A, \lambda], 0).
\]
(16)
Thus we find that \( b \) is unchanged under the action of the algebra; it can be treated as a constant parameter. If we identify \( b = \frac{2\pi}{e} \) we get the familiar transformation property of a connection under a gauge transformation.

This point of view on the gauge transformation can be generalized to another familiar algebra with a central extension, the Virasoro algebra. The gauge theory we construct this way can be thought of as a theory of gravity, since the gauge group is the group of diffeomorphisms. The analogue of the loop algebra \( \mathfrak{G} \) is the Lie algebra \( \text{Vect}(S^1) \) of vector fields on a circle,
\[
[u, \tilde{u}] = u \frac{d\tilde{u}}{dx} - \tilde{u} \frac{du}{dx}.
\]
(17)
Its central extension consists of ordered pairs \((u, b)\), with \( b \in R \). The commutation relations are
\[
[(u, b), (\tilde{u}, \tilde{b})] = ([u, \tilde{u}], \frac{1}{24\pi} \int \tilde{u} \frac{d^3 u}{dx^3} + \frac{du}{dx}] dx).
\]
(18)
The factor of \( \frac{1}{24\pi} \) is put in to agree with the conventional normalization of the central extension \[11\]. If we introduce the basis \( L_m = (-ie^{imx}, 0), c = (0, 1) \), we get the familiar commutation relations
\[
[L_m, L_n] = (n - m) L_{m+n} + \frac{i}{12} (m^3 - m) \delta(m + n), \quad [L_m, c] = 0
\]
(19)
of the Virasoro algebra.

From the co-adjoint action of the Virasoro algebra \[12\], \[13\], \[14\], we can find the analogue of the gauge transformation for our approach to gravity. Again, think of the
dual of the Virasoro algebra as consisting of ordered pairs \((q, b)\), where \(q\) is a real function on the circle and \(b \in R\). The pairing is

\[
< (q, b), (u, a) > = \int q(x)u(x)dx + ab. \tag{20}
\]

The co–adjoint action of \((u, a)\) on \((q, b)\) is defined by

\[
< \delta_{u,a}(q, b), (\tilde{u}, \tilde{a}) > + < (q, b), [(u, a), (\tilde{u}, \tilde{a})] >= 0. \tag{21}
\]

This means that

\[
< \delta_{u,a}(q, b), (\tilde{u}, \tilde{a}) > = -\int q(x)(u\tilde{u}' - \tilde{u}u')dx - \frac{b}{24\pi} \int \tilde{u}(u''' + u')dx \tag{22}
\]

which leads to

\[
\delta_{u,a}(q, b) = (-\frac{b}{24\pi}(u''' + u') + 2qu' + q'u, 0). \tag{23}
\]

Again, note that \(a\) has no effect on \(q\) and that \(b\) remains unchanged.

To see the geometrical meaning of this transformation, it is useful to consider the limit \(b = 0\). Then the inhomogeneous term disappears, and we find that \(q\) transforms as a tensor of order two. That is, \(q(x)dx^2\) is invariant under diffeomorphisms when \(b = 0\). Thus \(q\) can be thought of as a metric tensor on \(S^1\). In this case, all metrics are of course flat. The diffeomorphism invariant information contained in the metric is the length \(\int \sqrt{q(x)}dx\) of the circle.

We will be mostly interested in the case \(b \neq 0\). Thus our \(q\) has an inhomogenous transformation property,

\[
\delta_{(u,a)}q = -\frac{b}{24\pi}(u''' + u') + 2qu' + q'u. \tag{24}
\]

Our theory can therefore be thought of as a generalization of one–dimensional Riemannian geometry: the metric is not a tensor, but rather has an affine transformation law analogous to that of a connection. There should be analogues of this transformation in higher dimensions, but they are as yet too difficult to find.
Segal [9] has found the transformation law of \( q \) under finite diffeomorphisms as well. Remarkably, it involves the Schwarzian derivative which plays a central role in classical analysis. (See Ref. [15] for a geometric interpretation of the Schwarzian derivative)

If \( \phi : S^1 \to S^1 \) is a diffeomorphism, \( q \) transforms under it to \( \phi \circ q(x) \) where,

\[
\phi \circ q(x) = q(\phi(x)) \phi'^2(x) - \frac{b}{2\pi} \sigma_\phi(x). \tag{25}
\]

Here,

\[
\sigma_\phi(x) = \frac{1}{6} S_\phi(x) + \frac{1}{24} (\phi'^2(x) - 1)
\]

and \( S_\phi \) is the Schwarzian derivative of \( \phi \):

\[
S_\phi = \frac{1}{2} \frac{\phi'''}{\phi'} - \frac{3}{4} \left( \frac{\phi''}{\phi'} \right)^2. \tag{27}
\]

The quantity \( \sigma_\phi \) satisfies the identity

\[
\sigma_{\phi \circ \psi}(x) = \sigma_\phi(\psi(x)) \psi'^2(x) + \sigma_\psi(x) \tag{28}
\]

which is necessary in order that the action of the diffeomorphism on \( q \) satisfy the composition law.

The term \( \frac{1}{24}(\phi'^2 - 1) \) in the transformation law can be removed by adding a constant, \( -\frac{b}{48\pi} \) to \( q \). This amounts to a different choice of central term in the Virasoro algebra. However, our choice is more convenient, since then the isotropy group of \( q = 0 \) is the group of Mobius transformations. Under infinitesimal diffeomorphisms, the condition to leave \( q = 0 \) invariant is \( u''' + u' = 0 \), which has solutions \( u = (a - \bar{a}) L_0 + b L_1 - \bar{b} L_{-1} \), where \( L_m = -ie^{imx} \). This generates the subgroup of Mobius transformations on which the central term of the Virasoro algebra vanishes. The finite transformations which leave \( q = 0 \) invariant are of the form \( x \mapsto \phi(x) \),

\[
e^{i\phi(x)} = \frac{Ae^{ix} + B}{\bar{B}e^{ix} + \bar{A}}, \quad A\bar{A} - B\bar{B} = 1. \tag{29}
\]
This subgroup of diffeomorphisms is often also called $SU(1, 1)$ or $PSL_2(R)$. If we had used a different choice of the constant in $q$ (hence of $\sigma_\phi$), the point in $Q$ with $PSL_2(R)$ as the isotropy group would not have been the origin.

In the case of gauge fields the subgroup that preserves the trivial connection $A = 0$ consists of global or constant gauge transformations. In our case, from the above observation, the analogous subgroup is $PSL_2(R)$.

In the case of gauge theory, we are interested in the space of connections $\mathcal{A}$ modulo gauge transformations. However, only the subgroup $\mathcal{G}_0$ of gauge transformations that are equal to the identity at $x = 0$ acts without fixed points. The space $\mathcal{A}/\mathcal{G}_0$ is a manifold and the wavefunctions of pure gauge theory can be viewed as functions on this space. On a circle, this quotient space is finite dimensional: the only gauge invariant observable is the parallel transport operator (Wilson loop) around the circle. This quantity can be defined precisely using the differential equation (parallel transport equation)

$$\frac{d\psi}{dx} + eA(x)\psi(x) = 0. \quad (30)$$

This equation is invariant under the transformations $\psi(x) \mapsto \rho(g(x))\psi(x)$ (where $\rho$ is a representation of $G$), and $A \mapsto gAg^{-1} + 1/\epsilon gdgdg^{-1}$. It is enough to consider the fundamental representation, since solutions in the general case can be obtained by taking products of the fundamental solution. Although $A(x)$ is periodic, the solution is not, in general, periodic. The solution instead satisfies the condition

$$\psi(2\pi) = \rho(W(A))\psi(0) \quad (31)$$

where $W : \mathcal{A} \to G$ is the ‘parallel transport operator’ or Wilson loop. Given $W(A)$, $A$ is determined up to an action of the gauge group $\mathcal{G}_0$. In fact we can identify the quotient space $\mathcal{A}/\mathcal{G}_0$ with the group $G$. It is now possible to show that $\mathcal{A}$ is the total space of a principal fibre bundle

$$\mathcal{G}_0 \to \mathcal{A} \to G \quad (32)$$
with $G$ as the base space. Wavefunctions of Yang–Mills theory are functions on $A$ invariant under the action of $G_0$, hence are just functions on $G$.

An analogue of this construction can be found following Segal’s ideas. The analogue of $A$ is $Q = \{q : S^1 \to R\}$. The gauge group $G$ is replaced by the group $Diff(S^1)$ of diffeomorphisms of the circle. It acts on $Q$ by the above transformation. The analogue $Diff_0(S^1)$ of $G_0$ is given by diffeomorphisms of $S^1$ satisfying

$$\phi(0) = 0, \quad \phi'(0) = 1, \quad \phi''(0) = 0.$$  (33)

That is, they agree with the identity up to three derivatives. At least for infinitesimal transformations in $Diff_0(S^1)$, it is easy to see that there are no fixed points in $Q$. If the point $q \in Q$ is invariant under the action of a vector field $u$,

$$q'u + 2qu' - \frac{b}{24\pi}(u'' + u') = 0.$$  (34)

This third order equation has to be solved with the boundary conditions $u(0) = u'(0) = u''(0) = 0$. Clearly $u(x) = 0$ is the only solution. The proof in the case of finite transformations looks more complicated; we are not able to provide one. We expect the quotient space $Q/\text{Diff}_0(S^1)$ to be well–defined.

Now we look for the analogue of the parallel transport operator $W(A)$. Another observation of Segal is that the equation

$$D_q^{(2)} \psi = \psi'' + \hat{q} \psi = \psi'' + \left(-\frac{12\pi}{b}q + \frac{1}{4}\right)\psi = 0$$  (35)

is invariant if $q$ transforms as above and $\psi$ transforms as a density of weight $-\frac{1}{2}$. $\psi$ being a half density means that $\psi(x)(dx)^{-\frac{1}{2}}$ is invariant. Equivalently, $\psi \to \phi \circ \psi$, where

$$\phi \circ \psi(x) = \psi(\phi(x)][\phi'(x)]^{-\frac{1}{2}}.$$  (36)

More generally, there is a differential operator, $D_q^{(2s+1)}$, of order $2s+1$ which maps densities of weight $-s$ to those of weight $s+1$, for $s = 0, 1/2, 1, 3/2, ...$  [16]. These operators are
analogous to the covariant derivatives of gauge theory. We note in particular that the change of \( q \) under an infinitesimal diffeomorphism, \( \phi(x) = x + u(x) \), is proportional to:

\[
D_q^{(3)} u = u''' + u' - \frac{24\pi}{b} (2qu' + q'u).
\] (37)

where \( D_q^{(3)} \) is the differential operator which maps vector fields (weight = -1) to quadratic forms (weight = 2).

The solutions of the homogeneous equations \( D_q^{(2s+1)} u = 0 \) are just the products of the \( 2s \) solutions of \( D_q^{(2s)} u = 0 \). Thus, the densities of weight \(-\frac{1}{2}\) (which can be thought of as spinors) form the analogues of the fundamental representation of Diff\(_0(\mathbb{S}^1)\).

The solutions of the equation (known as Hill’s equation)

\[
\psi'' + \hat{q}\psi = 0
\] (38)

( with \( \hat{q} = -\frac{12\pi^2}{b} q + \frac{1}{4} \)) are not in general periodic, even though \( q \) itself is periodic. In the special case \( q = 0 \), there are anti-periodic solutions to Hill’s equation. In general, a basis \( u_1, u_2 \) of solutions, will change by a linear transformation \( M_q \) as we go from \( x = 0 \) to \( x = 2\pi \):

\[
\begin{pmatrix}
  u_1(x + 2\pi) \\
  u_2(x + 2\pi)
\end{pmatrix} = M_q \begin{pmatrix}
  u_1(x) \\
  u_2(x)
\end{pmatrix}.
\] (39)

A standard basis is defined by the boundary conditions,

\[
u_1(0) = 0, u_1'(0) = 1; \quad u_2(0) = 1, u_2'(0) = 0.
\] (40)

In this basis, matrix

\[
M_q = \begin{pmatrix}
  u_1'(2\pi) & u_1(2\pi) \\
  u_2'(2\pi) & u_2(2\pi)
\end{pmatrix}
\] (41)

is invariant under Diff\(_0(\mathbb{S}^1)\), since this subgroup preserves the boundary conditions. \( M_q \) is of determinant one, since the Wronskian of the two solutions is one. This is the analogue of the parallel transport operator \( W(A) \). (As Segal points out, we can get slightly more
information from the differential equation: an element of the universal covering group \( \hat{SL}_2(R) \), rather than one of \( SL_2(R) \). Our considerations will not distinguish between these.

The conjugacy class of \( M_q \) in \( SL_2(R) \) is invariant under the full diffeomorphism group, not just the subgroup \( \text{Diff}_0(S^1) \). This conjugacy class is labelled by the trace of \( M_q \). If this trace is greater than two, \( M_q \) is said to be hyperbolic. Then, \( M_q \) has real eigenvalues and the solutions to Hill’s equation, when extended to the real line, will blow up at either \( x = \infty \) or \( x = -\infty \). If the trace is equal to two, \( M_q \) cannot be diagonalized unless \( M_q = 1 \). There will be in this case an eigenvalue equal to one, and there will be a periodic solution to Hill’s equation. If \( M_q = 1 \), both solutions are periodic. If \( \text{tr}M_q < 2 \), \( M_q \) is said to be elliptic and the eigenvalues are complex numbers of modulus one. Then the solutions to Hill’s equation will remain finite when extended over the real line: they are similar to the Bloch waves in a solid.

As discussed above, we can regard Hill’s equation as the Schrödinger equation of a particle in a periodic potential, in which case we identify \( \frac{b}{12\pi} \hat{q}(x) = E - V(x) \), \( E \) being the total energy and \( V(x) \) is the potential energy. Then the parameter \( \frac{b}{12\pi} \) is analogous to \( \hbar^2 \). In the limit \( b \to 0 \), we can use the WKB (eikonal) approximation to find \( \text{tr}M_q = 2 \cos(\int_0^{2\pi} \sqrt{\hat{q}(x)}dx) \). Thus the invariant \( \text{tr}M_q \) is, in this limit, determined by the length of the circle with respect to the Riemannian metric \( \hat{q}(x)dx^2 \). We have therefore a direct correspondence between the diffeomorphism invariant quantities of the classical and quantum theories of the geometry of a circle.

Since the wavefunction \( \psi \) transforms as a density, its change in phase over the length of the circle is diffeomorphism invariant, and is therefore a function of \( \text{tr}M_q \). We can in fact make a choice of boundary conditions such that the total phase change over the length of the circle is \( \arccos[\frac{1}{2}\text{tr}M_q] \). Then the quantum length of the circle reduces to the classical
length in the limit $\hbar \to 0$. This choice of boundary conditions resolves the ambiguity in $\psi$ discussed earlier. For example, the quantum distance of the point $x$ from the origin is the phase of $\psi(x) = \psi(0, x)$, where $\psi$ satisfies Hill’s equation with the boundary conditions $\psi(0) = 1$ and $\psi(2\pi) = e^{\frac{i}{\hbar} \arccos(\frac{1}{4} \pi \Gamma M_e)}$.

The phase of the wavefunction $\psi$ will be real in the classically allowed regions ($E - V(x) > 0$), so it is in these regions that the quantum measure of distance will be real. In the classically forbidden regions ($E - V(x) < 0$) the quantum distance will take on imaginary values. Finally in those regions where $E - V(x) = 0$, the metric $\hat{q}(x)$ disappears, and all distances are null.

We find then that in the quantum theory the potential (or more precisely $\hat{q}(x) = \frac{1}{\hbar^2}(E - V(x))$), no longer transforms as a tensor, but still appears to have a geometrical meaning. It would be interesting to find higher dimensional analogues of this quantum generalization of the metric tensor.

The quantity

$$ F_q(x) = \frac{u_1(x)}{u_2(x)} \tag{42} $$

is a local diffeomorphism of $S^1$ to $RP^1$. It is not in general a diffeomorphism, since $F_q(2\pi) \neq F_q(0)$. In fact, $F_q(2\pi) = \frac{M_2}{M_{22}}$ is an invariant and under $\text{Diff}_0(S^1)$. Also, $F_q'(x) = \frac{1}{u_2(x)}$ transforms under $\text{Diff}_0(S^1)$ as a density of weight one. These are useful quantities in what follows.

3. The Hamiltonian

So far we have described the kinematical aspects of the theory, obtained by analogy with Yang–Mills theory. Now we will go beyond this and look for a dynamical principle, again by analogy with Yang–Mills theory.
The space $\mathcal{A}$ has a metric
\[ ||\delta A||^2 = \int \text{tr} \delta A(x) \delta A(x) dx \] (43)
invariant under the action of the gauge group. In this case, this metric is flat. The true configuration space is the quotient $G = \mathcal{G}/\mathcal{G}_0$, to which the above metric can be projected, since it is gauge invariant. The metric so obtained on $G$ is the usual Cartan–Killing metric, corresponding to the invariant bilinear $\text{tr}$ on the Lie algebra $\mathcal{G}$.

This leads to a natural classical dynamics, where the classical trajectories are geodesics with respect to this metric. In the case of Yang–Mills theory on a cylinder, this is precisely the dynamics determined by the Yang–Mills equations of motion. Of course, Yang–Mills equations do not in general describe geodesics. In 1+1 dimensions, the potential energy term in the hamiltonian is absent, so that the classical trajectories are geodesics. This can be most easily seen by considering the Yang–Mills action in first order form
\[ S = \int \text{tr} E(x) \frac{\partial A(x)}{\partial t} dx dt + \int \text{tr} A_0 \{dE + [A, E]\} dx dt - \frac{1}{2} \int \text{tr} E(x)^2 dx dt. \] (44)
The first term identifies $E$ and $A$ as canonical conjugates of each other. The second term imposes the first class constraint
\[ dE + [A, E] = 0 \] (45)
which generates the gauge transformations. The last term is the hamiltonian.

In quantum Yang–Mills theory, the wavefunctions are functions on the true configuration space $G$. The hamiltonian is just the Laplace operator on $G$. The resulting quantum dynamics can be easily solved using group theory [4].

To find an analogous theory invariant under the diffeomorphism group, we must look for a metric on $Q$. In the limit $b = 0$, the simplest choice is [17]
\[ ||\delta q||^2_0 = \int \delta q^2(x) q^{-\frac{2}{b}} dx. \] (46)
This already is not a flat metric on $Q$. For general value of $b$, we look for a metric of the form
\[ ||\delta q||^2 = \int \delta q^2(x)G_q(x)dx \] (47)
where $G_q$ has to transform as a density of weight $-3$ as $q$ transforms as in (25). We can build such a quantity from a solution to Hill’s equation, which transforms as a density of weight $-\frac{1}{2}$. However, the solution is not unique; a choice of boundary condition is necessary. We are not able to find a choice that is invariant under the full diffeomorphism group. If we pick the solution $u_2$ with the boundary condition
\[ u_2(0) = 1, \quad u'_2(0) = 0 \] (48)
the resulting metric
\[ ||\delta q||^2 = \int \delta q^2(x)u_2^2(x)dx. \] (49)
will be invariant under $\text{Diff}_0(S^1)$. This is because the boundary conditions on $u_2$ remain invariant under $\text{Diff}_0(S^1)$. This means that the point $x = 0$ has a special significance in our theory. (This might be natural if we regard $S^1$ as obtained by compactifying the real line, the point $x = 0$ then corresponds to infinity on the real line.) There could also be more complicated metrics that are invariant under the full diffeomorphism group, but we believe a first investigation should focus on the simplest possibility. Even this choice is not a flat metric on $Q$. It reduces to $||\delta q||^2$ (up to a constant scaling factor) in the limit $b \to 0$, as can be seen using the WKB approximation. Since $F_q' = \frac{1}{u_2^2}$, we can also write the metric as
\[ ||\delta q||^2 = \int \delta q^2(x)F_q'^{-3}(x)dx \] (50)

Due to the invariance under $\text{Diff}_0(S^1)$, we have an induced metric on the quotient space $M = Q/\text{Diff}_0(S^1)$. By analogy to Yang–Mills theory, we postulate that the wavefunctions of our theory are functions on $Q$ invariant under $\text{Diff}_0(S^1)$; i.e., functions on the quotient space $M$. The hamiltonian operator is the Laplacian on $M$ with respect to
the induced Riemannian metric. In the classical limit, the trajectories will be the geodesics of this metric. We do not yet know whether this dynamics can be derived from an action principle analogous to the Yang–Mills action.

We will now study the metric on \( M \) and the corresponding dynamics more explicitly.

4. The Riemannian metric on \( M \)

It is useful to think of the quotient \( M \) as the base space of a Principal Fibre Bundle \( \text{Diff}_0(S^1) \to Q \to M \). Although it is difficult to give rigorous definitions and proofs, this geometric language is very suggestive as in the case of Yang–Mills theory. A vertical vector at \( q \in Q \) represents an infinitesimal gauge transformation

\[
\delta_u q(x) = -\frac{b}{24\pi} D_q^{(3)} u = -\frac{b}{24\pi}(u''' + u') + 2qu' + q'u. \tag{51}
\]

A 1-form, \( \xi_q \), at \( q \in Q \) may be said to be horizontal if it annihilates all the vertical vectors at \( q \). This leads to the differential equation,

\[
D_q^{(3)} \xi_q = 0. \tag{52}
\]

If \( q = 0 \), the solutions are straightforward:

\[
\xi_0(x) = a_1 \sin x + a_2 \cos x + a_3. \tag{53}
\]

More generally, we note that the solutions to the equation (52) are given by products of solutions to Hill’s equation. (This can be verified by straightforward calculation, but can also be understood in terms of the construction of the sequence of operators \( D_q^{(n)} \) in Ref. [16].) Hence

\[
\xi_q(x) = au_2^2(x) + bu_1(x)u_2(x) + cu_1^2(x) = [a + bF_q + cF_q^2]F_q^{l-1} \tag{54}
\]

\( a, b, c \) being constants.
We will now show that, a horizontal vector $q \in \mathcal{Q}$ can be defined to be one that is orthogonal to all the vertical vectors. This defines a connection in the principal bundle $\text{Diff}_0(S^1) \to \mathcal{Q} \to \mathcal{M}$. In fact a horizontal vector can be obtained by operating on a horizontal 1–form by the metric (49):

$$\delta_h q(x) = F_q^3 \xi_q = [a + b F^1_q(x) + c F^2_q(x)] F^2_q(x).$$  \hspace{1cm} (55)

The length of such a horizontal vector can now be computed to be

$$||\delta_h q||^2 = \int_0^{2\pi} \delta_h q^2(x) F^{q-3}_q(x) dx = \int_0^\gamma [a + bx + cx^2]^2 dx$$  \hspace{1cm} (56)

where

$$\gamma = F_q(2\pi) = \frac{u_1(2\pi)}{u_2(2\pi)} = \frac{M_{12}}{M_{22}}.$$  \hspace{1cm} (57)

A horizontal vector can be thought of as tangential to the base manifold $\mathcal{M}$ of our Principal bundle. A point in $\mathcal{M}$ is parametrized by a matrix $M_q \in SL_2(R)$. A vector in $\mathcal{M}$ is an infinitesimal variation

$$\delta M_q = \begin{pmatrix} \delta u'_1(2\pi) & \delta u_1(2\pi) \\ \delta u'_2(2\pi) & \delta u_2(2\pi) \end{pmatrix}.$$  \hspace{1cm} (58)

The corresponding horizontal vector in $\mathcal{Q}$ (the ‘horizontal lift’) can be found using the first order perturbation of the Hill’s equation:

$$D_q^{(2)} \delta u_i = -\delta_h \hat{q} u_i \hspace{1cm} i = 1,2$$  \hspace{1cm} (59)

The boundary conditions are $\delta u_i(0) = \delta u'_i(0) = 0$. This equation can be solved using the Green’s function of Hill’s operator:

$$G_q(x, y) = \begin{cases} u_1(y) u_2(x), & x \leq y \\ u_1(x) u_2(y), & x \geq y \end{cases}$$  \hspace{1cm} (60)

so that

$$\delta u_i(x) = -\int_0^{2\pi} dy \ G_q(x, y) \delta_h q(y) u_i(y) + A_i u_1(x) + B_i u_2(x) \hspace{1cm} i = 1,2$$  \hspace{1cm} (61)
where $A_i$ and $B_i$ are determined by the boundary conditions on $\delta u_i$. We will get

$$A_i = 0 \quad B_i = \int_0^{2\pi} dx u_1(x) \delta_h q(x) u_i(x), \quad i = 1, 2.$$  \hspace{1cm} (62)

Thus $\delta_h q$ at $q \in \mathcal{Q}$ projects to the vector

$$\delta M_q = \Delta M_q$$  \hspace{1cm} (63)

where

$$\Delta = \begin{pmatrix} -\epsilon_{21} & \epsilon_{11} \\ -\epsilon_{22} & \epsilon_{12} \end{pmatrix}$$  \hspace{1cm} (64)

with $\epsilon_{ij} = \epsilon_{ji} = \int_0^{2\pi} dx \ u_i(x) \delta_h q(x) u_j(x) \quad i,j = 1, 2$.  \hspace{1cm} (65)

Using the earlier expressions for $\delta_h q$ and for $F_q$ in terms of $u_1$ and $u_2$, we get

$$\Delta = a \Gamma_1 + b \Gamma_2 + c \Gamma_3$$  \hspace{1cm} (66)

where the 2x2 matrices $\Gamma_i$ can be written in the following way:

$$
\begin{pmatrix}
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3
\end{pmatrix} =
\begin{pmatrix}
\gamma & \gamma^2/2 & \gamma^3/3 \\
\gamma^2/2 & \gamma^3/3 & \gamma^4/4 \\
\gamma^3/3 & \gamma^4/4 & \gamma^5/5
\end{pmatrix}
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{pmatrix}.
$$  \hspace{1cm} (67)

Also, $\{\tau_i\}$ gives a basis for $\overline{SL_2(R)}$:

$$\tau_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (68)

Now we can find explicitly the metric tensor on $\mathcal{M}$ obtained by projecting from $\mathcal{Q}$. It is useful to identify $\mathcal{M}$ with $\overline{SL_2(R)}$ locally, and introduce the usual Maurer–Cartan forms,

$$\delta M_q M_q^{-1} = \omega^1 \tau_1 + \omega^2 \tau_2 + \omega^3 \tau_3.$$  \hspace{1cm} (69)

We can now find the length of $\delta M$ by expressing it in terms of $\delta_h q$ and using (49):

$$||\delta M_q||^2 = G_{ij} \omega^i \omega^j.$$  \hspace{1cm} (70)
Here $G$ is the $3 \times 3$ matrix, whose inverse is given by

$$G^{-1} = \begin{pmatrix}
\gamma & \gamma^2/2 & \gamma^3/3 \\
\gamma^2/2 & \gamma^3/3 & \gamma^4/4 \\
\gamma^3/3 & \gamma^4/4 & \gamma^5/5
\end{pmatrix}$$

so that

$$G = 3 \begin{pmatrix}
3/\gamma & -12/\gamma^2 & 10/\gamma^3 \\
-12/\gamma^2 & 64/\gamma^3 & -60/\gamma^4 \\
10/\gamma^3 & -60/\gamma^4 & 60/\gamma^5
\end{pmatrix}.$$  

Thus the metric tensor on $\mathcal{M}$ induced by (49) is

$$g = G_{ij}\omega^i \otimes \omega^j.$$  

5. Symmetries of the Metric on $\mathcal{M}$

Our model quantum gravity on a circle has the Laplacian of the metric on $\mathcal{M}$ as the hamiltonian. Clearly, to understand this better we must exploit the symmetries of the metric. We will show next that there are two Killing vectors and one conformal Killing vector. Methods of classical differential geometry [10] can now be used to find a co–ordinate system adapted to the metric. Unfortunately we will not be able to separate the variables in the eigenvalue equation for the Laplacian; but we will achieve a considerable simplification. We will also discover that the metric on $\mathcal{M}$ has singularities. This is an indication that our quantum gravity theory has divergences, which may require a renormalization.

The Maurer–Cartan forms $\omega^i$ are invariant under the right multiplication by any element of $SL_2(R)$. If the coefficients $G_{ij}$ had been constants, any such right action would have been an isometry of our metric $g$. Although $G_{ij}$ are not constant in our case, they only depend on the variable $\gamma$. Hence a right translation that leaves $\gamma$ invariant, will be an isometry. Now,

$$\gamma = \frac{M_{12}}{M_{22}}.$$
is unchanged under right multiplication by elements of the form

\[
\begin{pmatrix}
\alpha & 0 \\
\beta & \alpha^{-1}
\end{pmatrix}.
\]

(75)

Thus there are two independent Killing vectors, corresponding to infinitesimally small transformations of the above type. Furthermore, note that the matrix elements of \( G_{ij} \) are monomials in \( \gamma \) of varying degrees. This leads to a homothety (conformal isometry with constant scale factor) of the metric (see below).

Let us define the vector fields

\[
W_i f(M_q) = \lim_{t \to 0} \frac{f(M_q[1 - t \tau_i]) - f(M_q)}{t}, \quad i = 1, 2, 3
\]

(76)

Then, \( W_1 \) and \( W_2 \) are Killing vectors. It will also be convenient to introduce the generators of the left action (right invariant vector fields)

\[
V_i f(M_q) = \lim_{t \to 0} \frac{f([1 + t \tau_i]M_q) - f(M_q)}{t}, \quad i = 1, 2, 3
\]

(77)

These satisfy

\[
[V_i, V_j] = f_{ij}^k V_k, \quad [V_i, W_j] = 0, \quad [W_i, W_j] = f_{ij}^k W_k
\]

(78)

where \( f_{ij}^k \) are the structure constants of \( SL_2(R) \), using the basis \( \{ \tau_i \} \). The vector fields \( V_i \) are dual to the Maurer–Cartan forms \( \omega^i \),

\[
i_{V_i} \omega^j = \delta^j_i.
\]

(79)

We can parametrize \( M_q \) by the co-ordinate system \((\alpha, \beta, \gamma)\), defined by

\[
M_q = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix}
\]

(80)

This is valid near the identity element of \( SL_2(R) \). Our manifold \( \mathcal{M} \) agrees with \( SL_2(R) \) only in a local neighborhood of the identity, so this is sufficient. We will now find a series
of co-ordinate transformations aimed at finding a co-ordinate system that exploits the symmetries of $g$ fully.

The Killing vectors $W_1, W_2$ satisfy

$$[W_1, W_2] = -2W_1. \tag{81}$$

There is then \cite{10} a pair of variables $(x^1, x^2)$ such that

$$W_1 = e^{2x^2} \frac{\partial}{\partial x^1}, \quad W_2 = \frac{\partial}{\partial x^2}. \tag{82}$$

One such co-ordinate system is given by

$$x^1 = -\beta/\alpha, \quad x^2 = -\ln \alpha, \quad x^3 = \gamma \tag{83}$$

for $\alpha > 0$. We then find the right-invariant vector fields:

$$V_1 = (1 - 2\gamma x^1) \frac{\partial}{\partial x^1} - \gamma \frac{\partial}{\partial x^2} + \gamma^2 \frac{\partial}{\partial \gamma}$$

$$V_2 = 2x^1 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - 2\gamma \frac{\partial}{\partial \gamma} \tag{84}$$

$$V_3 = \frac{\partial}{\partial \gamma}$$

and the dual 1-forms:

$$\omega^1 = dx^1 - 2x^1 dx^2$$

$$\omega^2 = \gamma dx^1 + (1 - 2x^1 \gamma) dx^2$$

or

$$\omega^3 = \gamma^2 dx^1 + 2\gamma(1 - \gamma x^1) dx^2 + d\gamma \tag{85}$$

$$\omega^2 = \gamma \omega^1 + dx^2$$

$$\omega^3 = d\gamma + 2\gamma \omega^2 - \gamma^2 \omega^1. \tag{86}$$

Consider the vector field

$$D = \gamma \frac{\partial}{\partial \gamma} - x^1 \frac{\partial}{\partial x^1}. \tag{87}$$

It generates a scale transformation in which $\gamma$ has degree one, $x^1$ has degree $-1$ and $x^2$ has degree zero. Then, $\omega^1$ has degree $-1$, $\omega^2$ has degree zero and $\omega^3$ has degree one. That

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is, $\omega^i$ has degree $i - 2$. Since $G_{ij}$ is a monomial in $\gamma$ of degree $-(i + j - 1)$, it follows that the metric tensor $g = G_{ij}\omega^i \otimes \omega^j$ is of degree $-3$. That is,

$$\mathcal{L}_D g = -3g. \quad (88)$$

Thus $D$ is a conformal Killing vector with constant scale factor (or infinitesimal homothety). Thus our theory admits a scale invariance. It would be interesting to see if this scale invariance is related to the divergences that arise in the quantum theory (see below). Also, there could be a connection to the renormalization theory of quantum mechanics [18].

The surfaces $\gamma=$constant are two dimensional sub–manifolds of constant curvature. This is because, $W_1$ and $W_2$ are tangential to these manifolds and form two independent Killing vectors satisfying (81) [10]. However, the vector $\frac{\partial}{\partial \gamma}$ is not orthogonal to this surface. If we find a vector $N$ that is orthogonal, we will be able to bring the metric into block–diagonal form. It is not difficult to find a vector that is normal to $W_1$ and $W_2$:

$$N = (1 - \frac{3x^1\gamma}{2}) \frac{\partial}{\partial x^1} - \frac{3\gamma}{4} \frac{\partial}{\partial x^2} + \frac{3\gamma^2}{5} \frac{\partial}{\partial \gamma}. \quad (89)$$

The integral curves of this vector field are,

$$x^1 = \frac{10}{9\gamma} + \frac{\xi^1}{\gamma^{3/2}}, \quad x^2 = -\frac{5}{4} \ln\gamma - \frac{1}{2}\xi^2. \quad (90)$$

Here, we regard $\gamma$ as the parameter that varies along the curve. The quantities $\xi^1, \xi^2$ are constants of integration, which distinguish between different integral curves. If we use $\gamma, \xi^1, \xi^2$ as the co–ordinates, the metric will be block–diagonal. Moreover, if we trade $\gamma$ for

$$\rho = -\frac{2\sqrt{5}}{3\gamma^{3/2}}, \quad (91)$$

the metric will take the form

$$g = d\rho^2 + A\rho^4 d\xi^1 d\xi^1 + 2\rho^3 h d\xi^1 d\xi^2 + \frac{\rho^2}{A}(h^2 + q) d\xi^2 d\xi^2. \quad (92)$$
Here,
\[ h = A\rho \xi^1 + B, \quad q = AC - B^2. \]  
(93)

And, the constants \( A, B, C \) are:
\[ A = -\frac{93}{200}, \quad B = \frac{27}{5\sqrt{5}}, \quad C = \frac{43}{5} \]  
(94)

We also note for future use the contravariant elements:
\[ g^{11} = \frac{1}{q^4}(h^2 + q) \]
\[ g^{12} = g^{21} = -\frac{h}{q^3} \]  
(95)
\[ g^{22} = \frac{A}{q^2} \]

and:
\[ \tilde{g} \equiv dctg_{ij} = q\rho^6 \]  
(96)

As mentioned previously the two-dimensional submanifolds \( \rho = \text{constant} \), which contain the two Killing vectors, are in fact surfaces of constant negative curvature. We find that the intrinsic curvature of these surfaces is \( R_{(2)} = -A/q\rho^2 \).

This is the simplest form to which we can bring the metric. The Killing vectors \( W_1, W_2 \) now take the form
\[ W_1 = e^{-2\xi^2} \frac{\partial}{\partial \xi^1}, \quad W_2 = -2 \frac{\partial}{\partial \xi^2}. \]  
(97)

Finally, (93)-(95) enable us to write the action of the Laplacian on \( \Psi \in C^\infty(M) \). It is clear that translations in \( \xi^2 \) are symmetries of the Laplacian, so that we can assume that the wavefunctions satisfy
\[ \partial \Psi / \partial \xi^2 = K_2 \Psi. \]  
(98)

and let \( \Psi(\xi^1, \xi^2, \rho) \mapsto e^{K_2 \xi^2} \Psi(\xi^1, \rho) \). Then, we can express the Laplacian as:
\[ \Delta \Psi = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left( \sqrt{\tilde{g}} g^{ij} \frac{\partial}{\partial x^j} \Psi \right) \]
\[ = \frac{1}{\rho^3} \rho^3 \frac{\partial}{\partial \rho} \left( \rho^3 \frac{\partial \Psi}{\partial \rho} \right) + \frac{h^2 + q}{Aq^{4} \xi^1} \frac{\partial^2 \Psi}{\partial \rho^2} - \frac{2h(K_2 - 1)}{q} \frac{\partial \Psi}{\partial \xi^1} + \frac{AK_2(K_2 - 1)}{q\rho^2} \Psi \]  
(99)
This is a partial differential operator in two variables, and we are not able to separate the variables any further. However, the form is simple enough that we can study some of its qualitative behaviour. We can also find some special solutions to the resulting Schrödinger equation.

6. Metric Singularities and Wavefunction Behavior in Their Vicinity

We have seen that the metric coefficients (either covariant or contravariant) have singular behavior for \( \rho \to 0 \) and for \( \rho \xi^1 \to \infty \). We have also calculated the intrinsic curvature of the \( \rho = \text{constant} \) surfaces and found that it blows up for \( \rho \to 0 \). These facts alone however do not imply the presence of true metric singularities. The singular behavior of the metric coefficients may just be due to the choice of coordinate system. The singular behavior of the intrinsic curvature of some submanifolds may be cancelled by a similarly singular behavior of the submanifolds’ extrinsic curvature. For this reason we have performed the somewhat lengthy task of calculating the Ricci scalar curvature of \( g \) explicitly, finding it to be:

\[
R = \frac{1}{\rho^2} \left( R_4 h^4 + R_3 h^3 + R_2 h^2 + R_1 h + R_0 \right)
\]

where \( R_i \) are constants.

Thus the metric \( g \) does indeed have singularities at \( \rho = 0 \) and \( \rho \xi^1 = \infty \). We can investigate the effect on wavefunctions of the singularity at \( \rho = 0 \) by examining the Schrödinger equation, \( \Delta \Psi = E \Psi \), in the limit \( \rho \to 0 \). Using (99) we find that for \( \rho \xi^1 \to 0 \) the Schrödinger equation approaches:

\[
\frac{1}{\rho^3} \frac{\partial}{\partial \rho} \left( \rho^3 \frac{\partial \Psi_0}{\partial \rho} \right) + \frac{C}{q \rho^4 (\partial \xi^1)^2} - \frac{2B(K_2 - 1)}{q \rho^3} \frac{\partial^2 \Psi_0}{\partial \xi^1} + \frac{AK_2(K_2 - 1)}{q \rho^2} \Psi_0 = E \Psi_0
\]

i.e., \( \Psi(\rho, \xi^1) \sim \Psi_0(\rho, \xi^1), \quad \rho \xi^1 \to 0. \)
This equation for $\Psi_0(\rho, \xi^1)$ is separable. We may assume that $\partial \Psi_0/\partial \xi^1 = K_1 \Psi_0$ such that (101) becomes:

$$\Psi_0'' + \frac{3}{\rho} \Psi_0' + \left( \frac{C K_1^2}{\rho^4} - \frac{2 B K_1 (K_2 - 1)}{\rho^3} + \frac{A K_2 (K_2 - 1)}{\rho^2} \right) \Psi_0 = E \Psi_0$$  \hspace{1cm} (102)

upon letting $\Psi_0(\rho, \xi^1) \rightarrow e^{K_1 \xi^1} \Psi_0(\rho)$.

Consider first the case $K_1 = 0$, such that $\Psi_0(\rho, \xi^1)$ is independent of $\xi^1$. Then (102) is related to Bessel’s equation and has the two independent solutions:

$$\Psi_0(\rho) = \begin{cases} \frac{1}{\rho} J_{\nu}(i \sqrt{E} \rho), & \text{if } \nu \notin Z \\ \frac{1}{\rho} J_{\nu}(i \sqrt{E} \rho), \frac{1}{\rho} N_{\nu}(i \sqrt{E} \rho), & \text{if } \nu \in Z \end{cases}$$ \hspace{1cm} (103)

where $\nu = \sqrt{1 - AK_2 (K_2 - 1)}/q$.

In fact (see (99) if $\partial \Psi/\partial x^1 = 0$), the solutions in (103) are exact wavefunctions, satisfying $\triangle \Psi_0 = E \Psi_0$.

In the case where $\Psi(\rho, \xi^1)$ is not independent of $\xi^1$, $\Psi(\rho, \xi^1) \neq \Psi_0(\rho, \xi^1)$. We can, in this case, solve (102) asymptotically (i.e. for $\rho \rightarrow 0$) to find the leading behavior of wavefunctions near the $\rho = 0$ singularity. (102) gives:

$$\Psi_0'' + \frac{3}{\rho} \Psi_0' \sim -\frac{C K_1^2}{\rho^4} \Psi_0, \quad \rho \rightarrow 0$$ \hspace{1cm} (104)

Solving (104) asymptotically we find:

$$\Psi_0(\rho) \sim e^{\pm \sqrt{\frac{C}{-\nu} K_1}}, \quad \rho \rightarrow 0.$$  \hspace{1cm} (105)

So that, finally:

$$\Psi_0(\rho, \xi^1) \sim e^{K_1 \xi^1} e^{\pm \sqrt{\frac{C}{-\nu} K_1}}, \quad \rho \xi^1 \rightarrow 0$$  \hspace{1cm} (106)
which shows that wavefunctions having an $\xi^1$ dependence ($K_1 \neq 0$) generically have an essential singularity at $\rho = 0$.

7. Some Further Directions

We record here some ideas that should be interesting to pursue further.

Although Hill’s equation

$$\psi'' + \hat{q} \psi = 0 \quad (107)$$

is invariant under $\text{Diff}(S^1)$, the eigenvalue problem

$$\psi'' + \hat{q} \psi = \lambda \psi \quad (108)$$

is not invariant. This is because the LHS transforms as a density of weight $\frac{3}{2}$ while the RHS is of weight $-\frac{1}{2}$. The determinant of Hill’s operator then, need not be diffeomorphism invariant: it could have an anomaly. This anomaly can be calculated by the zeta–function method [19] and we find it to be zero. Thus the Hill determinant is a diffeomorphism invariant function of $q$ and can be written in terms of $\text{tr} M_q$. Assuming diffeomorphism invariance, we can show that,

$$\det [-D_q] = \text{tr} M_q - 2. \quad (109)$$

An argument for this would be note that if $D_q$ has no kernel, $q$ can be transformed to a constant $\hat{q}_0$ by a diffeomorphism (see below, [9], and [20]). The LHS can be evaluated to be

$$\prod_{n \in \mathbb{Z}} [n^2 - \hat{q}_0] = (-\hat{q}_0) \prod_{n=1}^{\infty} n^4 \prod_{n=1}^{\infty} [1 - \frac{\hat{q}_0}{n^2}]^2. \quad (110)$$

The first infinite product is divergent but can be given a meaning by zeta function regularization,

$$\prod_{n=1}^{\infty} n^4 = e^{-\zeta'(0)} = (2\pi)^2, \quad (111)$$
where
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \] (112)
is the Riemann zeta function.

Now recall the standard formula
\[ \frac{\sin z}{z} = \prod_{n=1}^{\infty} \left[ 1 - \frac{z^2}{n^2 \pi^2} \right]. \] (113)

With \( z = \pi \sqrt{\hat{q}_0} \) this can be used to evaluate the last factor, so that
\[ \det [-D_{\hat{q}_0}] = -4 \sin^2[\pi \sqrt{\hat{q}_0}] = \cos 2\pi \sqrt{\hat{q}_0} - 1. \] (114)

We can solve the differential equation easily to determine the monodromy operator \( M_{\hat{q}_0} \) and get
\[ \text{tr} M_{\hat{q}_0} = 2 \cos 2\pi \sqrt{\hat{q}_0}. \] (115)

Thus we get the above result for constant \( q \). Through diffeomorphism invariance, the identity follows for any \( q \) for which \( D_q \) is invertible, because such a \( q \) can transformed to a constant. If \( D_q \) is not invertible, both sides vanish and the identity is still true.

Another way to construct a diffeomorphism invariant from \( q \) is to consider the nonlinear eigenvalue problem
\[ \psi'' + \hat{q} \psi = -\mu \psi^{-3}, \quad \int \psi^{-2} dx = 1 \] (116)

This nonlinear equation is invariant under \( \text{Diff}(S^1) \) since \( \psi^{-3} \) is a density of weight \( \frac{3}{2} \). The normalization condition on \( \psi \) also is invariant. If such a \( \psi \) exists that is smooth, periodic and positive everywhere, there is a diffeomorphism \( \phi_q \in \text{Diff}(S^1) \) which will transform \( q \) to a constant \( \hat{q}_0 \). This can be verified by putting,
\[ \psi(x) = (2\pi)^{-\frac{1}{2}} \phi_q^{-\frac{1}{2}}(x), \quad \hat{q}(x) = \hat{q}_0 \phi_q^{2}(x) + S_{\phi_q}(x), \quad \mu = -2\pi \hat{q}_0. \] (117)
Conversely, whenever such a $\phi_q$ exists, we can solve the above nonlinear eigenvalue problem. In Ref. [9] a sufficient condition for the existence of $\phi_q$ is given: $M_q$ should lie on a one-parameter subgroup. This happens whenever $D_q$ is invertible, for then, $M_q$ is diagonalizable. Thus in these situations $\psi$ will exist. Moreover, if $q_0$ so determined is not equal to $\frac{m^2}{4}$ for $m \in \mathbb{Z}$, the solution $\psi$ is unique, and therefore $\phi_q$ is also determined uniquely modulo an additive constant corresponding to a rotation. If $q_0 = \frac{m^2}{4}$ there are an additional periodic solutions, which can be easily determined in the co-ordinate system where $q = q_0$:

$$
\psi^2(x) = C + \sqrt{C^2 - 4\pi^2 \cos m(x - x_0)}, \quad \mu = -\pi^2 m^2. \quad (118)
$$

Here $C$ and $x_0$ are constants of integration.

It would be interesting to give an independent proof of the existence of the solution to the above nonlinear equation. It can be viewed as the Euler–Lagrange equation of the variational problem

$$
\mu = \inf_{\psi} \left\{ \int [\psi'^2(x) - \dot{q}(x)\psi^2(x)]dx; \int |\psi|^{-2}(x)dx = 1 \right\}. \quad (119)
$$

This problem is many ways analogous to the Yamabe problem [21] of differential geometry. Recall that if $(M, g)$ is a compact $m$-dimensional Riemannian manifold, ($m \geq 3$ the Yamabe variational problem is

$$
\mu = \inf \left\{ \int [|d\psi|^2 + aR\psi^2]dV_g; \int |\psi|^r dV_g = 1 \right\}. \quad (120)
$$

Here

$$
a = \frac{m - 2}{4(m - 1)}, \quad r = \frac{2m}{m - 2} \quad (121)
$$

and $dV_g$ is the volume measure defined by $g$ and $R$ the scalar curvature. The variational equation is,

$$
\Delta \phi + aR\phi = -\mu \phi^{r-1}. \quad (122)
$$
Whenever such a \( \phi \) exists and is positive, there is a new metric \( \hat{g} = \phi^{r-2}g \) with constant scalar curvature.

The operator \( \Delta + aR \) is the conformal Laplacian which maps scalar densities of weight \( \frac{m}{2} - 1 \) to those of weight \( \frac{m}{2} + 1 \). The homogenous equation

\[
[\Delta + aR]|_\psi = 0
\]

(123)
is conformally invariant while the corresponding linear eigenvalue problem is not.

The Yamabe problem is of course trivial when \( m = 1 \): all Riemann metrics are then flat. (When \( m = 2 \) the analogous equation is the Liouville equation). Yet, our nonlinear eigenvalue problem (116) is in many ways a one–dimensional limiting case of the Yamabe problem. Hill’s operator \( \frac{d^2}{dx^2} + \hat{q} \) maps densities of weight \(-\frac{1}{2}\) to those of weight \( \frac{3}{2} \), which is thus analogous to the conformal Laplacian, if we put \( m = 1 \) in the above formulae. Also, the Yamabe eigenvalue problem becomes our problem (116) if we replace the conformal Laplacian by Hill’s operator and put \( m = 1 \). In this point of view, \( q \) is analogous to the Ricci scalar rather than metric and \( \psi \) determines the transformation that reduces it to a constant.

We have argued that for generic \( q \), there is a unique \( \psi \) that solves the eigenvalue problem. This allows us to define a new metric on the space \( Q \):

\[
||\delta q||^2 = (2\pi)^3 \int \delta q^2(x)\psi^8(x)dx = \int \delta q^2(x)\phi^{-3}(x)dx.
\]

(124)
This new metric is invariant under the full group Diff(\( S^1 \)). This metric is also worth a detailed study. In particular, we believe it will be of interest in understanding the geometry of the space of one–dimensional potentials in quantum mechanics as well as quantum gravity on a circle.

Although we have mainly discussed the generic case, the case where the isotropy group of \( q \) is smaller than \( S^1 \) (when it cannot be transformed to a constant) or larger
(when $\hat{q}_0 = \frac{m^2}{4}$ and the isotropy group is three dimensional, the $m$-fold cover $SL_2^m(R)$ of $PSL_2(R)$) are also of interest. The analysis of the latter case is quite straightforward; there is a two parameter family of solutions to the eigenvalue problem (116) as described above. If $C > 2\pi$ this solution is positive and will define a diffeomorphism. There is a three parameter family of diffeomorphisms that will reduce such a $q$ to a constant, $q_0 = \frac{m^2}{4}$. The corresponding co-adjoint orbit will be $\text{Diff}(S^1)/SL_2^m(R)$. The case where the isotropy group does not contain rotations, $S^1$, is more complicated; their study will involve a detailed understanding of the Mathieu equation.

The Hill’s determinant can be thought of in terms of a Gaussian path integral:

$$Z_0[q] = \int \mathcal{D}\psi e^{-S[\psi, q]} = \det -\frac{1}{\beta}[D_q]$$  \hspace{1cm} (125)

where

$$S_0[\psi, q] = \frac{1}{2} \int_0^{2\pi} [\psi'^2 - \hat{q}(x)\psi^2] dx.$$  \hspace{1cm} (126)

The path integral is over all configurations of period $2\pi$. This is a free one-dimensional quantum field theory. We can ask if there is a way to add interactions without destroying diffeomorphism invariance. The answer is,

$$Z_\mu[q] = \int \mathcal{D}\psi e^{-S_\mu[\psi, q]}$$  \hspace{1cm} (127)

where,

$$S_\mu[\psi, q] = \frac{1}{2} \int_0^{2\pi} [\psi'^2 - \hat{q}(x)\psi^2 - \mu\psi^{-2}] dx.$$  \hspace{1cm} (128)

If $\mu = 0$, this reduces to the expression of the Hill’s determinant. The classical equation of this field theory is the nonlinear eigenvalue problem we discussed earlier. It is more difficult to see if this generalization of the Hill’s determinant is diffeomorphism invariant directly, since there is no analogue for the zeta function argument in the full interacting theory. One approach would be a perturbation theory in $\mu$. But, if we assume that $Z_\mu(\hat{q})$ is invariant, it is possible to evaluate this path integral in the case of generic $q$. 

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Then we can transform to co-ordinates where $\hat{q} = \hat{q}_0$. Then this becomes the action of a point particle in a harmonic oscillator plus inverse square potential. This one dimensional quantum field theory has been studied by Parisi and Zirilli [22]. (See also Gupta and Rajeev [18].) In fact, by viewing this in the canonical language, where $\psi$ would be the position variable and $x$ would be time,

$$Z_\mu[\hat{q}_0] = \text{tr} e^{-2\pi H_\mu}$$

(129)

where the hamiltonian is

$$H_\mu = -\frac{1}{2} [\frac{d^2}{d\psi^2} + \hat{q}_0 \psi^2 + \mu \psi^{-2}].$$

(130)

The eigenvalues of this hamiltonian are (after correcting an algebraic error in Ref. [22])

$$E_n = \sqrt{(-\hat{q}_0)[n + 1 - \frac{1}{2} \sqrt{(1 - 4\mu)]}$$

(131)

when $q_0 < 0, \mu < \frac{1}{4}$. In this range we can get the partition function,

$$Z_\mu(q) = Z_\mu(q_0) = \frac{1}{2 \sinh[\pi \sqrt{(-q_0)]}} e^{\pi (\sqrt{1-4\mu} - 1) \sqrt{-q_0}}.$$  

(132)

This agrees with $\det \frac{1}{2} [-D_\mu]$ when $\mu = 0$. For values of $q_0$ and $\mu$ at which the above argument does not apply, we can define $Z_\mu(q_0)$ by analytic continuation. Note that there is a branch point at $\mu = \frac{1}{4}$, which is the critical value observed in Ref. [18]. When $\mu > \frac{1}{4}$, we must perform a renormalization in order to get a sensible ground state, as described in Ref. [18]. This would be necessary to get a meaningful value of $Z_\mu$ as well.

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