Cosmological Histories for the New Variables

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February 7, 1994

Abstract
Histories and measures for quantum cosmology are investigated through a quantization of the Bianchi IX cosmology using path integral techniques. The result, derived in the context of Ashtekar variables, is compared with earlier work. A non-trivial correction to the measure is found, which may dominate the classical potential for universes on the Planck scale.
1 Introduction

Attempts to apply quantum mechanics to the universe have for some time divided into two main schools focused on either the canonical approach or path integral methods. While these two approaches are driven by different conceptions of how a quantum theory of the universe is to be constructed and interpreted, progress in these directions has been circumscribed by the particular strengths and weaknesses of the two formalisms. These strengths and weaknesses allow each formalism to address, in rather different ways, key issues in quantum cosmology associated with the time reparametrization invariance. This reflects a lack of lack of clocks and observers “outside” the universe — the essential problem of quantum cosmology.

In the canonical formalism, the Dirac procedure gives us a prescription to construct physical states, define physical observables and propose, through the imposition of the reality conditions of the theory, a physical inner product. The strength of the canonical approach arises in the way that time reparametrization invariance, as well as the other gauge and diffeomorphism invariances of the theory can be treated directly, yielding a gauge invariant quantization. The existence, for the full theory of quantum gravity, as well as for models, such as 2 + 1 gravity, one and two killing field models, of exact results concerning physical states and diffeomorphism invariant states and operators speak of this strength.

On the other hand, the great weakness of the canonical approach is that physical observables are very difficult to construct explicitly because, both classically and quantum mechanically, observables must commute with the Hamiltonian constraint and necessarily freeze as constants of motion. This difficulty is real; it reflects the fact that physical operators which describe time evolution ought to be constructed as correlations between the degrees of freedom (one of which one would like to take as a clock)[15]. For example, suppose that we pick a condition that picks out a slicing of spacetime in to spacelike slices according to some degrees of freedom of the theory. Then we may define some observables that measure spatially diffeomorphism invariant information on these slices. For example, let $A(q, p)$ be a spatially diffeomorphism invariant quantity which measures an aspect of the geometry of spacetime (where $q$ and $p$ are coordinates on the phase space) and let $T(q, p)$ be another diffeomorphism invariant quantity which we will take as measuring time. Then for every possible value $\tau$ of this time observable there is a physical observable that measures what the value of $A(q, p)$ is on the spacial slice on which $T(q, p) = \tau$. We may note that as the condition that picks out the slices is expressed in terms of the degrees of freedom this procedure is completely gauge invariant because within any gauge one can specify these slices and evaluate the variables $A(q, p)$ and $T(q, p)$.

While simple to specify, to express such correlations explicitly in terms of functions on the phase space or operators on the physical states one must solve the dynamics of the theory. Thus, the construction of time reparametrization invariant observables in the canonical theory is a dynamical problem, which one cannot expect to solve without approximation procedures for theories outside of integrable and solvable systems.

Alternately, in this example using the path integral formalism we find that the problem of taking the expectation values of physical observables can be easily realized as soon as one has a measure and a set of histories that represent physically meaningful, gauge invariant amplitudes. For instance, the expectation value of the observable $A(T = \tau)$ can be simply given by summing (with the appropriate measure) paths weighted by the classical action and the value of the classical observable $A(q, p)$ on the slice when $T(q, p) = \tau$,

$$< \psi | \hat{A}(T = \tau) | \psi > = \frac{\int [d\mu(q, p)] A(T = \tau) e^{iS}}{\int [d\mu(q, p)] e^{iS}}. \quad (1)$$

By varying $\tau$ we can describe the evolution of the system in terms of time reparameterization invariant quantities. Though the path integral formalism steps by the difficulties of the canonical
approach, the path integral has complimentary difficulties. Setting aside interpretational issues, we lack a prescription which allows us to unambiguously find the set of histories, appropriate contours, and a measure $\mu(q, p)$ which implement gauge invariances and reality conditions.

We would like to suggest that the situation points to a mixed approach in which the physical, diffeomorphism invariant quantum states of the canonical theory are used as the starting point to define a path integral and measure, after which the dynamics of physical observables are computed with path integral techniques. This program, if it can be concretely realized, offers a possibility of an unambiguous formalism for quantum cosmology making use of the strong points of both the canonical and path integral approach.

To investigate this possibility, it would be very useful to have a working model of a quantum cosmology which has dynamics complicated enough that problems of constructing physical observables, inner product and path integral measure are non-trivial. However, the results ought to be simple enough that the path integrals for physically meaningful quantities could be computed by relatively simple numerical or approximation techniques. This is the first of two papers which aim to lay the groundwork to construct such a model of quantum cosmology based on the Bianchi IX spatially homogeneous spacetimes. In this paper we show that a gauge invariant measure can be constructed in this model, following the Faddeev-Popov procedure and using the new variables. In a companion paper we consider a physically meaningful canonical quantization procedure for Bianchi IX and show how, under what conditions, quantities defined through this canonical formalism can be expressed in terms of path integral expressions of the kind that are derived here.

The Bianchi IX model describes a family of cosmologies in which space is homogeneous, but the geometry has two dynamical degrees of freedom - measures of anisotropy. It has been studied extensively especially since the late 1960's when Misner found that, in a particular gauge, its dynamics can be expressed in terms of the motion of a particle in a time-dependent potential. Though it is a simple system with only two degrees of freedom, this model displays a surprisingly rich behavior even at the classical level. For instance, it has been shown that the Lyapunov exponent is greater than one for certain choices of time, meaning that the model is chaotic. (However, the Lyapunov exponent, a measure of the exponential separation of nearby trajectories in time, is not time reparameterization invariant). In the face of this it is unlikely that the theory can be exactly solved, making it an ideal candidate to test the program we have just discussed.

At the quantum level, although there does not exist, to our knowledge, a complete quantization of the Bianchi IX model in either a path integral or canonical formalism, a number of results have been found previously. An exact physical state has been found by Kodma using the new Hamiltonian variables of Ashtekar, which can even be transformed into the metric representation. Graham has constructed a supersymmetric solution to this Bianchi model. Numerical work, following the methods of Euclidean quantum cosmology shows qualitative agreement with the exact solutions - at early times the wavefunction is spread over the anisotropy space while at later times the wavefunction peaks at the isotropic model (closed FRW).

We find, perhaps not surprisingly, that the measure is non-trivial and dominates the weights of histories when the radius of the cosmology is comparable to or smaller than the Planck length. This indicates that quantum effects dominate the behavior of the cosmology near the classical singularity, as is generally expected. Using this result, we expect that it is now a straightforward numerical problem to compute the expectation value of physical observables in physical states.

We present the derivation in "geometrized units" in which $G = c = 1$. 

\[ G = c = 1. \]
2 Bianchi IX in the new variables

The new variables provide a complex chart on the phase space of general relativity with configuration variables, the connections $A^I_a$, and conjugate momenta, the densities $\tilde{E}^a_I$. Our notational convention denotes spatial indices as lower case latin letters, e.g. $a, b, c, ...$ and denotes internal indices as upper case latin letters, e.g. $I, J, ...$. Densities of weight one, such as the conjugate momenta, $\tilde{E}^a_I$, claim a tilde. The phase space is endowed with the structure given by

$$\{A^I_a(x), \tilde{E}^b_J(y)\} = i \delta^a_b \delta^I_J \delta^3(x, y).$$  \hspace{1cm} (2)

The more familiar metric is obtained from the frame fields, $E^a_I$, by defining triads on a three-manifold $\Sigma$, $E^a_I = \frac{1}{\sqrt{g}} \tilde{E}^a_I$, and by letting $h^{ij} = E^a_I E_{ij}$. As this chart is a complex one, to regain general relativity we must choose a section of the phase space in which reality conditions, such as

$$(h_{ij})^* = h_{ij} \hspace{1cm} (3)$$

and

$$(\dot{h}_{ij})^* = h_{ij} \hspace{1cm} (4)$$

hold.

In the 3+1 decomposition, with $\sigma$ chosen as the time parameter, the classical action is

$$I[A, E, N] = \int_{\Sigma} \int d^3 x \left(-i \tilde{E}^a_I A^I_a - N_a C^a\right).$$ \hspace{1cm} (5)

The asterisk, $^*$, is an index that runs from 0 to 6; it has one value for each constraint:

$$S := \varepsilon_{IJK} F^a_{b} \tilde{E}^a \tilde{E}^{bK} = 0$$ \hspace{1cm} (6)

$$G_I := D_a \tilde{E}^a_I = 0$$ \hspace{1cm} (7)

$$V_a := F^b_{ab} \tilde{E}^a_I = 0$$ \hspace{1cm} (8)

which are known as the scalar or hamiltonian, gauss, and vector or diffeomorphism constraints, respectively. The covariant derivative is associated with the connection $D_a f^I := \partial_a f^I + \varepsilon_{IJK} A_a J f^K$ and the curvature, $F^{ab} := \partial_a A^b + \varepsilon_{IJK} A_a J A^b K$. We investigate Class A Bianchi IX models. \hspace{1cm} (1)\footnote{The classification of Bianchi models involves splitting the structure constants of the Lie group of isometries into two irreducible pieces. Denoting these by $S^{LK}$ and $V_K$, the structure constants may be written as $C^I_{JK} = \varepsilon_{IJK} S^{LK} + \delta^I_J V_K$. Class A models are those for which $V_I = 0$.}

Homogeneity provides us with a foliation of spacetime into homogeneous space-like surfaces and gives each leaf a left invariant vector- one-form basis, $(v, \omega)$ in which to expand the new variables \cite{10}. On each leaf we can write

$$A^I_a = a^I_a(\sigma) \omega^a_S(x)$$ \hspace{1cm} (9)

and

$$E^a_I = e^a_I(\sigma) v^a_S(x).$$ \hspace{1cm} (10)

These expansion coefficients may be viewed as $3 \times 3$ matrices. Homogeneity merrily reduces field theory to mechanics - from $9 \times 9$ degrees of freedom per spacetime event to $9 \times 9$ for each spatial section. The action simplifies to:

$$I[A, E, N] = \int_{\Sigma} \int d\sigma \left(-i \omega^a_S \dot{a}^I_a - N_a C^a\right).$$ \hspace{1cm} (11)
Here, $\Omega = \int_2 \omega \wedge \omega \wedge \omega = 16\pi^2$ is the volume element on $SU(2)$. (The Lagrange multipliers have been rescaled.) Henceforth, we will work in the unusual units $G = c = \Omega = 1$ meaning we measure fields in terms of this volume element and use a conversion factor of $\rho^2/G(\Omega)^{1/3}$ for energy. In terms of the expansion coefficients defined in Eqs. (9) and (10), the constraints, Eqs. (6), (7), and (8) become

$$S = \varepsilon^B_A \left(-\varepsilon^D_F a^A_D + \varepsilon^A_B a^D_F \right) \varepsilon^G_C$$

$$G_I = \varepsilon^{K}_{IJ} a^I_J$$

and

$$V_J = \varepsilon^{K}_{JK} a^K_J.$$  

We choose to fix the six gauge and diffeomorphism constraints by a "diagonal gauge" [10] to yield the extensively studied form of Misner in which the cosmology may be seen as a particle moving in $2 + 1$ dimensional spacetime with a time dependent potential. This choice parallels Misner’s $\beta_+\beta_-\beta_0$ diagonalization in the geometrodynamic framework (we will later translate our result into Misner’s notation for comparison).

We define

$$\epsilon_1 := \epsilon^1, \epsilon_2 := \epsilon^2, \epsilon_3 := \epsilon^3$$

and choose

$$\chi^I_J \equiv \epsilon^I_J = 0 \text{ for } I \neq J.$$  

Upon imposing these conditions the three Gauss constraints vanish, while the three remaining vector constraints require that the off-diagonal components of $a^I_J$ vanish as well. As above we define

$$a_1 := a^1, a_2 := a^2, a_3 := a^3.$$  

At the end of the kinematical gauge fixing, we are left with six canonical degrees of freedom per leaf.

At this kinematical level in which the Gauss and vector constraints have been solved, but the Hamiltonian constraint have not, the model is not difficult to quantize. The six canonical degrees of freedom can be taken to be diagonal components of the frame fields and (imaginary parts of) diagonal components of the connections. States, in the diagonal metric representation, may be expressed as functions of the $\epsilon$'s. The reality conditions, Eq. (3) and Eq. (4), are realized by the inner product,

$$\langle \psi(\epsilon) | \phi(\epsilon) \rangle = \int d^3 \epsilon \left( -F(\epsilon) \right) \psi(\epsilon) \phi(\epsilon).$$  

where

$$F(\epsilon) := \frac{\epsilon_1 \epsilon_2}{\epsilon_3} + \frac{\epsilon_2 \epsilon_3}{\epsilon_1} + \frac{\epsilon_3 \epsilon_1}{\epsilon_2}.$$  

Unfortunately this quantization cannot be used to compute physical quantities; reparametrization invariance remains. The the Hamiltonian constraint must be solved to reduce the physical phase space to four degrees of freedom.

To pull out the dynamics, we fix this reparametrization invariance by choosing the gauge condition:

$$\chi^0(\sigma) := \ln(\sqrt{\epsilon}) - \sigma = 0$$  

fixing $\sigma$ to be proportional to the volume of each spatial surface. This choice of parametrization is monotonic on half the history of any given classical Bianchi IX cosmology [5] (Bianchi IX expands from an initial singularity and then collapses in a final singularity [11]). An alternative choice exists which is monotonic on the whole of the evolution. This is to choose the time $\sigma$ to
be proportional to the momentum conjugate to the \(\ln(\sqrt{\hat{f}})\) — the trace of the extrinsic curvature of the slices of homogeneity. We have not studied the form of the path integral in this gauge in detail, but see no apparent reason why this could not be carried out.

To proceed to completely specify the canonical quantization in this gauge we should find a complete set of physical coordinates and momenta on the subspaces labeled by \(\sigma\). We do not know how to do this. Fortunately, the facility of the Faddeev- Popov ansatz allows us to compute the path integral.

## 3 Construction of the path integral

Ideally, we would provide a chart for the physical phase space, use this chart as the groundwork of a operator algebra, endow the space of states with an inner product, and produce dynamics through a Hamiltonian composed of these operators. We denote the canonical coordinates as \(\tilde{q}^i(\sigma)\) and \(\tilde{p}_i(\sigma)\) (where \(i = 1, 2\)) and denote the set of eigenstates by \(|\tilde{q}^i(\sigma)\rangle\) and \(|\tilde{p}_i(\sigma)\rangle\). As classically \(\tilde{q}^i(\sigma)\) and \(\tilde{p}_i(\sigma)\) are canonically conjugate, we would have \(|\tilde{q}^i(\sigma)\tilde{p}_j(\sigma)\rangle = \exp(i\tilde{q}^i\tilde{p}_j)(\sigma)\).

If the Hamiltonian which realizes evolution from fixed volume slice to fixed volume slice is \(h(\sigma)\), then we would have,

\[
< q_j^i(\sigma_f)|q'_i(\sigma_i) >_{phys} = K(q^i, \sigma_f; q'_i, \sigma_i) = \int [dq^i dp_i \exp i \int (\tilde{p} \tilde{q} - h(\tilde{p}, \tilde{q})) dt.]
\]

(21)

where the integral is over all possible physical trajectories which pass through the initial point \(q^i\), at volume \(\sigma_i\) and the final point \(q'_i\), at volume \(\sigma_f\). The brackets indicate that the measure is taken on each time slice of the skeletonization of the path integral. The brackets also include a factor of \(1/\sqrt{2\pi}\) for each differential. This notation will be used for the remainder of this paper.

This construction is only useful through its link to a path integral over the whole (unphysical) phase space. Denoting the coordinates and momenta of the whole phase space by \(q\) and \(p\) [1],

\[
K(q^i, \sigma_f; q'_i, \sigma_i) = \int \left[ \frac{dp^i dq_i}{2\pi} \frac{d\chi}{2\pi} \prod_i \delta(\chi^i) \langle C, \chi \rangle \right] \times \exp i \int (p \dot{q} - h(p, q) - N^\dagger \dot{C}, (p, q)) dt
\]

(22)

where the \(C\) are the constraints of the theory, the \(\chi^i(p, q)\) are the gauge choices. The key element of the link, the determinant, involves a Poisson bracket between constraints and gauge fixing conditions. The time coordinate \(\sigma\) is an arbitrary parametrization of the phase space trajectories and the initial and final conditions of the path integral are chosen to agree with those in Eq. (21). (The standard procedure for gauge theories described in [1] generalizes to the case in which time reparameterization invariance is one of the gauge invariances, so long as the choice is consistent[12].)

In our example of Binachi IX, the kinematical gauge symmetries are fixed by the choice of a diagonal gauge and the time reparameterization invariance is fixed by associating time with volume. Writing the physical coordinates (\(\tilde{q}\) and \(\tilde{p}\) above) as the anisotropies, \(\beta\), a transition element from \(\sigma_i\) to \(\sigma_f\) is generated by integrating the initial state with the kernel,

\[
|\beta|\psi(\sigma_f) = \int [d\beta] K(\beta, \sigma_f; \beta, \sigma_i) |\beta|\psi(\sigma_i).
\]

(23)

The kernel, the object we shall be concerned with from now on, may be written in the new variables as

\[
K(\epsilon_f, \sigma_f; \epsilon_i, \sigma_i) = \int [d\epsilon_i d\delta a [\{C^*, \chi^*\}] \delta(\chi^0(\epsilon, \sigma)) \delta(S)] \exp i \int_{\epsilon_i}^{\epsilon_f} d\sigma (-i\epsilon^{RT}a).
\]

(24)
Exponentiating the measure’s delta-function as
\[ \delta(S) = \int_{-\infty}^{\infty} \delta_\sigma dN e^{-i\frac{\sigma}{\epsilon} NS}, \]  
(25)
in which \( \delta_\sigma \) is the step size of the skeletonization of the path integral, we can write the kernel of Eq. (24) as
\[ K(\epsilon_f, \sigma_f; \epsilon_i, \sigma_i) = \int \left[ d^3 a \right] \exp i \int_{\sigma_i}^{\sigma_f} d\sigma \left( -i\epsilon^T \bar{a} - NS \right). \]  
(26)
with the action in the form of the (gauge-fixed) action of Eq. (11). In Eqs. (24) and (34) we performed the trivial integration over the off-diagonal pieces of \( a_2^f \) and \( \epsilon_f^I \), eliminating vector and diffeomorphism constraints and their associated gauge section \( \delta \)-functions. However, the measure contains contributions from both the kinematical gauge fixing and the time reparametrization gauge fixing (which is still explicitly indicated). The effects of our gauge choices can be computed explicitly,
\[ \{(C^*, \chi^*)\} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\epsilon_2 & 0 & \epsilon_3 \\ 0 & 0 & \epsilon_1 & 0 & 0 & -\epsilon_3 & 0 \\ 0 & -\epsilon_1 & 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_3 & 0 & -\epsilon_2 \\ 0 & 0 & -\epsilon_3 & 0 & 0 & \epsilon_1 & 0 \\ 0 & \epsilon_2 & 0 & -\epsilon_1 & 0 & 0 & 0 \\ F(\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]  
(27)
where \( F(\epsilon) \) is defined in Eq. (19). As the Hamiltonian constraint in Eq. (12) contains terms both linear and quadratic in the \( a_1^I \)'s and as the gauge fixing condition is a function of the \( \epsilon_f^I \)'s, it is possible that terms linear in the \( a_1^I \)'s appear in \( \{(C^*, \chi^*)\} \). However as a result of a serendipitous simplification, such terms cancel. The remaining constraint — the Hamiltonian constraint — is written
\[ S = (-a_1 + a_2 a_3) \epsilon_2 \epsilon_3 + (-a_2 + a_3 a_1) \epsilon_2 \epsilon_1 + (-a_3 + a_1 a_2) \epsilon_1 \epsilon_2. \]  
(28)
The imposition of this constraint is enforced by the integral over \( N \), which has a range from \(-\infty\) to \(+\infty\). The parameterization, Eq. (20), restricts the range of the \( \epsilon \)'s integration to the positive real axis \((0, +\infty)\). Meanwhile, restricting the \( \epsilon \) integral to the real axis satisfies the reality conditions of Eq. (3) that require that the three metric be real.

To implement the rest of the reality conditions, Eq. (4), we may choose a contour for the \( a_I \) integral that reflects these conditions. Recall that the \( A_I^I \)'s are complex variables which depend on the original canonical variables of relativity via \( A_I^I = \Gamma_0^I(E) + i K^I_0(E, \eta) \) where \( \Gamma_0^I \) is the \( SU(2) \) connection, \( K^I_0 \) is the extrinsic curvature, and \( \eta \) is the canonically conjugate momentum to \( E \). We have a similar relation for the diagonalized expansion coefficients,
\[ a_I = \gamma_I(\epsilon) - i\kappa_I. \]  
(29)
The reality conditions suggest that the \( \kappa_I \)'s may be taken as the independent variables in the path integral. The contours in Eq. (26) may then be taken along the imaginary \( a \) axes or, equivalently, performed for real \( a \). At each \( \sigma \)
\[ \int d^3 a d^3 \kappa = \int d^3 a d^3 a. \]  
(30)
After integrating by parts and discarding a complex boundary term \((-\epsilon^T \kappa |_{\epsilon_f} + \epsilon^T \kappa |_{\epsilon_i})\) the propagator of Eq. (26) becomes

\[
K(\epsilon_f, \sigma_f; \epsilon_i, \sigma_i) = \int_0^\infty [d^3 \epsilon] \int_{-\infty}^\infty [d^3 \kappa] \int_{-\infty}^\infty [d\sigma] dN \left[ \{C^\sigma, \chi_0^\sigma\}| \delta[\chi_0^\sigma(\epsilon, \sigma)]\right] \\
\times \exp i \left( \int_{\sigma_i}^{\sigma_f} d\sigma \kappa^T Q \kappa + \overline{\kappa}^T \kappa \right)
\]

(31)

where the coefficient of the quadratic term is

\[
Q = \frac{N}{2} \begin{pmatrix}
0 & \epsilon_1 \epsilon_2 & \epsilon_1 \epsilon_3 \\
\epsilon_1 \epsilon_2 & 0 & \epsilon_2 \epsilon_3 \\
\epsilon_1 \epsilon_3 & \epsilon_2 \epsilon_3 & 0
\end{pmatrix}
\]

(32)

and the coefficient of the linear term is

\[
b = \left( \begin{array}{c}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3
\end{array} \right) - iN \left( \begin{array}{c}
\epsilon_2 \epsilon_3 \\
\epsilon_1 \epsilon_3 \\
\epsilon_1 \epsilon_2
\end{array} \right).
\]

(33)

The integration over \(\kappa\) may be done; it is gaussian. This integral exists when the matrix \(Q\) is a real, positive matrix which is ensured by the reality conditions (Eq. (3)) and the choice of parametrization, respectively. The cross terms generated by the integration form a total derivative of the measure factor in the unphysical inner product, Eq. (18). Therefore, the integration preserves the configuration space - to - configuration space propagator under this inner product.

Performing the integration over \(\kappa\) then gives,

\[
K(\epsilon_f, \sigma_f; \epsilon_i, \sigma_i) = \int \left[ \sqrt{\epsilon^T \kappa \kappa^T \epsilon} dN \mu(\epsilon) \delta[\chi_0^\sigma(\epsilon, \sigma)] \right] \exp \left( -i \int_{\sigma_i}^{\sigma_f} d\sigma N L(\epsilon, \sigma, \sigma_i) \right) \]

(34)

with the measure;

\[
\mu(\epsilon) = \left( \frac{\epsilon_2 \epsilon_3}{\epsilon_1} \right)^2 \left( \epsilon_3^2 - \epsilon_2^2 \right) + \left( \frac{\epsilon_1 \epsilon_3}{\epsilon_2} \right)^2 \left( \epsilon_1^2 - \epsilon_3^2 \right) + \left( \frac{\epsilon_1 \epsilon_2}{\epsilon_3} \right)^2 \left( \epsilon_2^2 - \epsilon_1^2 \right)
\]

(35)

and the lagrangian;

\[
L(\epsilon, \sigma, \sigma_i) = \frac{2}{N} \left[ \left( \frac{\epsilon_1}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_2} + \frac{\epsilon_3}{\epsilon_3} \right)^2 - 4 \left( \frac{\epsilon_1 \epsilon_2}{\epsilon_1 \epsilon_2} + \frac{\epsilon_2 \epsilon_3}{\epsilon_2 \epsilon_3} + \frac{\epsilon_3 \epsilon_1}{\epsilon_3 \epsilon_1} \right) \right]
\]

\[-2N \left[ F(\epsilon)^2 - 4 \left( \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 \right) \right]
\]

(36)

The integral (34) is the main result of our analysis. It is manifest that the dynamics unfolds with a time-dependent potential though, the form of the potential is unclear. A more physical picture may be found by re-expressing our result in terms of anisotropy. To accomplish this and to compare with previous studies of the Bianchi cosmology we translate Eq. (34) back into Misner’s chart on the phase space of Bianchi IX using

\[
h_{ij} = e^{\alpha \beta} \left( e^{\beta \alpha} \right)_{ij} = \epsilon_{ij} \delta_{\alpha \beta}
\]

(37)

where \(\beta\) is the traceless matrix: \(\text{diag}(\beta_{++} + \sqrt{3} \beta_{--}, \beta_{+-} - \sqrt{3} \beta_{--}, -2 \beta_{--}).\) The Jacobian of this transformation from the \(\epsilon\) chart into the \(\beta\) chart is

\[
\left| \frac{\partial \epsilon_{ij} \epsilon_{kl}}{\partial \beta_{\alpha \beta} \delta_{\alpha \alpha}} \right| = 6(3)^{3/4}.
\]

(38)
The integral of Eq. (34) may then be written

\[
K(\beta_f, \sigma_f; \beta_i, \sigma_i) = \int_{-\infty}^{\infty} \left[ d\beta_+ d\beta_- d\sigma d\tau \delta_0^{\infty}(\delta - 3\alpha)e^{7/2}\mu(\beta_0) \right] \times \exp\left(2I \int d\sigma N \frac{1}{N} \left(12\beta_+^2 + 12\beta_-^2 - 3\alpha^2\right) - Ne^{2\alpha}U(\beta_0) \right)
\]

where \( \mu(\epsilon) = e^{-4\alpha}\mu(\beta_0) \) and

\[
\mu(\beta_0) = \left[e^{8\beta_+ \sin(4\sqrt{3}\beta_-)} + e^{-10\beta_+ \sinh(2\sqrt{3}\beta_-)} - e^{2\beta_+ \sinh(6\sqrt{3}\beta_-)}\right].
\]

Letting the delta function consume the \( \alpha \) integration we find

\[
K(\beta_f, \sigma_f; \beta_i, \sigma_i) = \int \left[ d\beta_+ d\beta_- \frac{dN}{\sqrt{\alpha}} 6\sqrt{3}e^{7/2}\mu(\beta_0) \right] \times \exp\left(2I \int_{\sigma_f}^{\sigma_i} d\sigma N \frac{1}{N} \left(12\beta_+^2 + 12\beta_-^2 - 1\right) - Ne^{2\alpha}U(\beta_0) \right)
\]

The propagator's form reveals its character. The action is that of Misner's Bianchi IX formulation of a particle moving in a time dependent potential while \( N \) has two roles, evolving the theory and preserving reparameterization invariance as it does for the relativistic particle[13]. The action contains the classical potential, \( V(\beta_0) := U(\beta_0) + 1 \) which takes the usual form[3]²

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Figure 1: The classical potential for Bianchi IX with contours spaced by powers of e with the first line at \( U(\beta_0) = -0.9 \). Three “channels,” one along the positive \( \beta_- \) axis, and the other two sloping diagonally in the negative \( \beta_- \) axis indicate the minima of the potential. The contours form a rough triangle with steep walls.
\[
U(\beta) = \frac{1}{3}e^{-8\beta} + \frac{4}{5}e^{-2\beta + \cosh(2\sqrt{5}\beta_\perp)} \\
+ \frac{2}{3}e^{\beta + \left(\cosh(4\sqrt{5}\beta_\perp) - 1\right)}.
\]

(42)

A contour plot of this potential appears in Figure 1. The evolution of the cosmology is seen as dynamics of a free particle reflecting off roughly triangular, exponentially steep “walls” shown in Fig. 1.

The nontrivial measure shows the mechanics of the gauge fixing procedure and the traces of integration. A weighting factor, constant in \(\beta_k\) but proportional to \(1/N\) appears in the action. Through the process of skeletonization, this weight may be expressed in the measure (on each time slice) as \(\exp\left(\frac{2\beta_k}{2N}\right)\). The \(\beta\)-dependent form of \(\mu(\beta_k)\), depicted in Figure 2, has a six-fold symmetry. The points of the point are the minima and so lend little support for wavefunctions peaked in these regions. In particular, the measure tends to support histories which do not peak in the channels of the classical potential and do not have maxima near the center of the triangular walls. Therefore, it seems that the effect of the measure is to splinter a wavefunction near the walls.

The measure factor, \(\mu(\beta_k)\), may be expressed as as a “quantum correction” to the classical lagrangian by exponentiating it giving a “quantum lagrangian” (including \(\tilde{\pi}\)),

\[
L_\phi(\beta_\pm, \dot{\beta}_\pm, N) = \frac{12}{N} \left(\dot{\beta}_\pm^2 + \dot{\beta}_k^2\right) - N e^{\frac{2\beta_k}{2N}} U(\beta_k) - i\tilde{\pi} V_\phi(\beta_k)
\]

(43)

\[\text{Note that in the literature one often finds written } V(\beta_k) := U(\beta_k) + 1. \text{ This is convenient because } V(\beta_k) \text{ is bounded from below, however what is important to remember is that the actual potential } U(\beta_k) \text{ is bounded from below by } -1.\]
with the measure \( \mu(\beta_\pm) \) expressed in terms of a quantum correction \( V_q \) to the potential as

\[
V_q(\beta_\pm) = \ln[\mu(\beta_\pm)]
\]

The expression of Eq. (41) is as far as we believe can be gone in the evaluation of the path integral of the Bianchi IX cosmology, without turning to approximation or numerical techniques. We may note that to evaluate either of these propagators, Eq. (41) or Eq. (34), it will be necessary to perform a Wick rotation. Since the “lapse,” \( N \), is used to exponentiate a delta function, the range of integration runs from \(-\infty\) to \(+\infty\). However, the construction requires us to include only forward (in the time parameter \( \sigma \)) propagating histories. To secure convergence in the anisotropy propagator Eq. (41) and in the quadratic integration Eq. (31) we must rotate to \( \sigma \rightarrow \sigma_N \), where

\[
\sigma_N = \sigma \exp \left( i \frac{\pi}{2} \right) \text{ when } N > 0 \text{ and } \sigma \rightarrow \sigma_N = \sigma \exp \left( -i \frac{\pi}{2} \right) \text{ when } N < 0,
\]

effectively excluding “backwards evolving” histories. Alternately, it’s possible to restrict the \( \dot{N} \) integration at the onset by using a \( \theta \)-function in the exponentiation, Eq. (25), as may be done in the path integral for the relativistic particle [13]. It is reassuring that the \( \dot{N} \) dependent rotations necessary to make the \( \alpha \) integration convergent will serve again to make the \( \beta_\pm \) integrals convergent. In particular, if the time is rotated to purely imaginary values we see that the classical part of the Euclideanized action is indeed positive definite. Moreover, the measure factor is purely real and positive, so that its presence does not disrupt this issue. Thus, as expected from general arguments [14], when the path integral is defined through a correct gauge fixing procedure there is no problem with the “runaway conformal modes” of the Euclideanized action. The analytic continuation provides a context to compare the two potentials as shown in Figure 3. For a fixed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{The classical potential and “quantum potential” superimposed for direct comparison. The effect of the measure is to suppress amplitudes in the three channels and at the center of the three walls of the classical potential. The contours (of both potentials) are spaced by multiples of two beginning at 0.1 (e.g. the first contour is drawn when the potentials have value 0.1, the second contour at 0.2, the third at 0.4, etc.)}
\end{figure}

value of the classical potential, as the cosmology approaches the initial singularity \( (\sigma \rightarrow -\infty) \),
the effect of the classical potential vanishes while the “quantum potential” remains. In fact this suppression occurs for any interval during the first half of the history of the cosmology (for $\sigma < 0$).

Unfortunately, there are drawbacks to analytically continuing the parameter $\sigma$ as may be seen in quantum action:

$$I_q(\beta_k, N) = -\int d\sigma \frac{2}{N} \left( 12\beta_+^2 + 12\beta_-^2 + \frac{1}{3} \right) + 2 N e^{i\pi/2} U(\beta_k) - V_0(\beta_k)$$  \hspace{1cm} (45)

in which $\sigma = i\sigma$ and so all variables displayed are real. As the potential term $N e^{i\pi/2} U(\beta_k)$ contains a parameter-dependent factor, when we continue to imaginary time the potential acquires an unfortunate oscillation. However, this may yield the expected result for as we go towards the initial singularity ($\sigma = -\infty$), a small change of $\sigma$ near the initial singularity causes this factor to oscillate wildly, effectively averaging the potential to zero. It is unclear whether the dynamics is correctly modeled by this continuation of $\sigma$.

But, there is another choice of analytic continuation. The analytic continuation of $\sigma$ may be seen as continuing the determinant of the metric the $\epsilon$'s which breaks our reality conditions. To avoid the oscillating potential and this problem we could instead continue $N$. This produces a positive classical lagrangian given by

$$I_q(\beta_k, N) = -\int d\sigma \frac{2}{N} \left( 12\beta_+^2 + 12\beta_-^2 - \frac{1}{3} \right) + 2 N e^{i\pi/2} U(\beta_k) - V_0(\beta_k)$$  \hspace{1cm} (46)

where all the displayed quantities are real. The form of the action is identical except one sign change; the weight factor proportional to $1/N$ changes sign. Since $g_{00} = -N^2$ this continuation would give us a Euclidean metric. A possible interpretation of this weight is that it gives weight to free particle histories or “Kasner” epochs. To see this we should return to the definition of the path integral in terms of skeletonized paths. In the limit of small, $N \ll 1$, for any fixed value of $\beta_k$, the factor $e^{2\pi/3} \beta_k$ gives weight to the kinetic energy term while the potential is negligible; this factor tends to support free particle histories. With the opposite sign in the $\sigma$ analytic continuation, the free particle histories are suppressed. This effect is not large (for the path integral is defined in the limit $\sigma_N \to 0$) but nevertheless the action with $N \gg 1$ reduces to the free particle. Until physical quantities are computed it is unclear precisely how these continuations are related.

4 Conclusion

We used the Faddeev-Popov prescription to construct a path integral for the Bianchi IX quantum cosmology. Our strategy is to begin with a classical dynamical system on a 9 + 9 dimensional complex phase space defined by the constraints (Eqs (6), (8), and (7)) and define the measure of the path integral which follows from the Faddeev-Popov ansatz. We choose contours of integration which ensure that phase space histories satisfy the reality conditions, corresponding to the physical condition that the metric of spacetime is real and of Minkowskian signature.

We find, as a result, the configuration space – anisotropy space – path integral Eq. (41) for the Bianchi IX cosmology in the gauge in which time is parameterized by the volume of space. RQuantum corrections to the potential arise in performing gauge fixing with the Faddeev-Popov method yielding the non-trivial effects of the measure. We may make several comments about its form.

First of all, it is interesting to note that the effect of the measure is independent of time, and hence the volume of the universe. However, the relative importance of the classical and quantum parts of the potential are time dependent as the classical potential is multiplied by the
factor $e^{2\sigma/3}$. By graphing $\mu(\beta_\pm)$, as in Fig. 1, for a fixed $\sigma$ we may see where the effects of the measure are important.

Comparing the quantum and classical potentials (as we do in Fig. 3) the quantum effects are negligible for large anisotropies except for times much less than the Planck time. Furthermore, relative region of the $\beta_+$, $\beta_-$ plane where the quantum effects dominate grows smaller as time increase because of the factor of $e^\sigma$ in the classical potential. Of course, the fact that the quantum potential diverges for vanishing anisotropy merely reflects the fact that the measure vanishes, when the anisotropy $\beta_-$ vanishes. This means that the $\beta_-$ - isotropic evolutions have vanishing measure for all time, which implies that the quantum state must have a finite spread in anisotropy. We may note that this effect is stronger than a simple uncertainty principle effect in that it might be expected to keep the "wave function of the universe" spread in anisotropy. The measure actually vanishes when $\beta_-=0$, which is the configuration space of the Taub model. This suggests that reductions from one quantum cosmological model to another are not always appropriate.

To further understand the effects of the measure in the path integral, it is necessary to finish the evaluation of the path integral. As we have seen that there is an analytic continuation which makes the integral in Eq. (41) real and convergent, there should be no problem with defining the integral through standard Monte-Carlo techniques, or by semiclassical techniques.

In particular, given this kernel one can proceed directly to the evaluation of the expectation values of gauge invariant, and hence physically meaningful, quantities. For example, any quantity of the form $F(\beta_+(\sigma))$ measures correlations of the anisotropies, defined at slices with particular volumes, is gauge invariant and meaningful[15]. Quantities like this have been evaluated successfully in a variety of cosmological models including $2+1$ gravity[16], Gowdy models[17] and the Bianchi I model[18] and were found to give physically meaningful results. These models were all exactly solvable, so that expectation values of some physical quantities could be computed exactly. We believe that with path integrals of the form of Eq. (41), in which gauge invariance is guaranteed by the construction, it is possible to extend the calculations of physically meaningful quantities in quantum cosmology to cases such as the Bianchi IX cosmology.

While the Faddeev-Popov technique guarantees, if correctly carried out, that the resulting path integral is gauge invariant and so represents a physical amplitude, exactly which physical amplitude it corresponds to cannot be defined outside of the context of a consistent quantization that takes into account the special circumstance of quantum cosmology. These include the fact that the universes described by the Bianchi cosmological model each live for a finite time, and that the lifetime of each universe is a function of the initial conditions. Because the gauge invariance includes time reparametrization invariance one must be careful to be sure that one is neither under or overcounting gauge independent configurations in the path integral. Another way to say this is that any quantization of a cosmological models can only give sensible answers to physical questions about evolution in time if the answers are formulated in a manner that is both diffeomorphism invariant and takes into account the fact that any given classical or quantum universe may no longer exist after a certain span of time. We take up such issues in the companion paper.

ACKNOWLEDGEMENTS

We would like to thank Abhay Ashtekar, Don Marolf, Nenad Manojlovic and Jorge Pullin for conversations during the course of this work, and also Chris Isham, Karel Kuchar, Enzo Marinari and Carlo Rovelli for conversations about the possibility of defining quantum cosmological models through path integrals. This work has been supported by the National Science Foundation under grants PHY 9306246 to Syracuse and Penn State Universities.
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