Dirac versus Reduced Quantization of the Poincaré Symmetry in Scalar Electrodynamics

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Abstract: The generators of the Poincaré symmetry of scalar electrodynamics are quantized in the functional Schrödinger representation. We show that the factor ordering which corresponds to (minimal) Dirac quantization preserves the Poincaré algebra, but (minimal) reduced quantization does not. In the latter, there is a van Hove anomaly in the boost-boost commutator, which we evaluate explicitly to lowest order in a heat kernel expansion using zeta function regularization. We illuminate the crucial role played by the gauge orbit volume element in the analysis. Our results demonstrate that preservation of extra symmetries at the quantum level is sometimes a useful criterion to select between inequivalent, but nevertheless self-consistent, quantization schemes.

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1 Introduction

There has been a long standing debate in the literature concerning the Dirac versus reduced quantization of gauge theories. In Dirac quantization, one constructs quantum operators on the full space of fields prior to reducing to the physical degrees of freedom. The gauge constraints are then realized as operator constraints on physical states. Reduced quantization, on the other hand, as the name suggests, constructs quantum operators for physical observables only. Dirac quantization is simpler in the sense that the full space of fields is usually endowed with a flat configuration space metric. It has the disadvantage, however, of including supposedly unphysical information into the quantization scheme. It is well known that these two approaches to quantization generally lead to distinct quantum systems [1, 2, 3, 4, 5, 6], and that the difference can be understood as a factor ordering ambiguity involving the volume element on the gauge orbits [6]. Now it can happen that both approaches are self consistent, and so even though the respective Hamiltonians may yield different spectra, there is no internal criterion with which to select the correct factor ordering. This has been illustrated by Kuchař [3] using a finite dimensional model, which we will refer to as the helix model.

The purpose of the present paper is to examine in detail the quantization of a field theoretic version of Kuchař’s helix model, namely scalar electrodynamics in flat spacetime. An important distinction between the two models in the present context is that scalar electrodynamics contains a symmetry not present in the helix model: Poincaré invariance. Our principle contribution is to show that Poincaré invariance at the quantum level is sensitive to this factor ordering ambiguity, and provides a suitable internal criterion: minimal\(^1\) Dirac quantization.

\(^1\)This term is defined later.
passes, whereas minimal reduced quantization appears to fail.

The paper is organized as follows. In section 2 we review both the Lagrangian and Hamiltonian analyses of scalar electrodynamics, chiefly to introduce notation. This is followed in the next section by a discussion of the classical Poincaré symmetry: we write down the classical Poincaré charges on the full phase space, and verify that these generate the Poincaré algebra, up to ‘off-shell’ pieces which vanish on the constraint surface. Then in section 4 we quantize the Poincaré generators in the functional Schrödinger representation and show that minimal Dirac quantization preserves the Poincaré symmetry when acting on the physical state space. This involves showing that all potential van Hove anomalies [7, 8, 9] vanish, and that the off-shell pieces mentioned above annihilate physical states when quantized. In section 5 we turn to minimal reduced quantization, and demonstrate that it does not preserve the Poincaré symmetry: there exists a van Hove anomaly in, for example, the boost-boost commutator. This calculation involves zeta function regularization (via heat kernel techniques) [10, 11] of (the log of) the gauge orbit volume element. Finally, in section 6 we compare both Dirac and reduced quantizations (acting on the same physical state space) to clarify why minimal Dirac quantization succeeds, while minimal reduced does not. It is clear that the volume element on the gauge orbits plays a pivotal role.

2 Scalar Electrodynamics

The Lagrangian density for scalar electrodynamics in globally Lorentzian spacetime is

$$\mathcal{L} = \frac{1}{2} (D_{\mu} \phi)(D^{\mu} \phi) - U - \frac{1}{4} F_{\mu \nu} F^{\mu \nu},$$

(1)
where we use spacetime signature \((+---)\), and the indices \(\mu, \nu\) run from 0 to 3. \(\varphi := \xi + i\eta\) is a complex scalar field and \(U\) is a potential, for example a mass or self interaction term, which depends only on \(|\varphi|\). The covariant derivative is \(D_\mu := \partial_\mu + ieA_\mu\), with corresponding electromagnetic field strength \(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\).

Fixing an inertial frame, and using DeWitt’s condensed notation, the Lagrangian \(L(t) := \int d^3x L(t, \mathbf{x})\) can be cast into the form:

\[
L(\lambda, Q, \dot{Q}) = \frac{1}{2} G_{AB}(Q) \left( \dot{Q}^A - \lambda^a \phi^A_a(Q) \right) \left( \dot{Q}^B - \lambda^3 \phi^B_3(Q) \right) - V(Q), \quad (2)
\]

In the above, the configuration of the system at time instant \(t\) is represented by a point in the configuration space, \(M\), with coordinates

\[
Q^A := (A_i(\mathbf{x}), \xi(\mathbf{x}), \eta(\mathbf{x})), \quad (3)
\]

where the index \(A\) runs over discrete values (including the spatial index \(i = 1, 2, 3\)), as well as the continuum \(\alpha := x \in \mathbb{R}^3\). Repeated indices imply summation and/or integration, as appropriate. The overdot on the velocities \(\dot{Q}^A\) indicates time derivative, but the time argument has been suppressed. The time-like part of the vector potential plays the role of a Lagrange multiplier: \(\lambda^\alpha := -\epsilon A_0(\mathbf{x})\).

The kinetic energy term in the Lagrangian induces a flat metric on \(M\), with components

\[
G_{AB}(Q) := \begin{pmatrix}
\delta^i_j & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \delta(\mathbf{x} - \mathbf{y}) \quad (4)
\]

in the Cartesian coordinates \(Q^A\). The potential term in \(L\) is

\[
V(Q) := \int d^3x \left\{ \frac{1}{4} (F_{ij})^2 + \frac{1}{2} (\partial_i \xi - e A_i \eta)^2 + \frac{1}{2} (\partial_i \eta + e A_i \xi)^2 + U \right\}. \quad (5)
\]

Gauge transformations on \(M\) are generated by the ‘gauge vector fields’ \(\phi^A_\alpha = \phi^A_\alpha \partial / \partial Q^A\), whose components in the Cartesian coordinates are

\[
\phi^A_\beta(Q) = \left( -\frac{1}{e} \partial_\beta \delta(\mathbf{x} - \mathbf{y}), -\eta(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \xi(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) \right). \quad (6)
\]
The gauge vector fields are linearly independent (except on the $\xi = \eta = 0$ ‘axis’), and their Lie bracket algebra:

$$[\phi_\alpha, \phi_\beta] = f^\gamma_{\alpha\beta} \phi_\gamma = 0$$

(7)

is, of course, abelian.

The phase space $\Gamma = T^* M$. A straightforward Hamiltonian analysis (see, e.g. [13]) yields the canonical Hamiltonian

$$H(\lambda, Q, P) = \frac{1}{2} G^{AB}(Q) P_A P_B + V(Q) + \lambda^a C_a(Q, P),$$

(8)

in which $G^{AB}$ denotes the matrix inverse of $G_{AB}$. The momenta $P_A$, conjugate to $Q^A$, are (in uncondensed notation)

$$\Pi_{A_i(\mathbf{x})} = \frac{\delta L}{\delta A_i(\mathbf{x})} = \dot{A}_i(\mathbf{x}) - \partial_{x^i} A_0(\mathbf{x}) = F_{0i}(\mathbf{x}),$$

(9)

$$\Pi_{\xi}(\mathbf{x}) = \frac{\delta L}{\delta \xi(\mathbf{x})} = \dot{\xi}(\mathbf{x}) - \epsilon A_0(\mathbf{x}) \eta(\mathbf{x}),$$

(10)

$$\Pi_{\eta}(\mathbf{x}) = \frac{\delta L}{\delta \eta(\mathbf{x})} = \dot{\eta}(\mathbf{x}) + \epsilon A_0(\mathbf{x}) \xi(\mathbf{x}),$$

(11)

where $\delta/\delta \dot{A}_i(\mathbf{x})$, etc., denotes functional derivative.

The Lagrange multipliers $\lambda^a$ enforce the Gauss law constraints

$$C_a(Q, P) := \phi^B_a(Q) P_B = \frac{1}{\epsilon} \partial_{x^i} \Pi_{A_i(\mathbf{x})} - \eta(\mathbf{x}) \Pi_{\xi(\mathbf{x})} + \xi(\mathbf{x}) \Pi_{\eta(\mathbf{x})} \approx 0,$$

(12)

which defines a constraint surface $\Gamma_C \subset \Gamma$. It turns out that the $\phi_a$ are Killing vectors and that the potential $V$ is constant along the gauge orbits in $M$, conditions which, together with (7), are sufficient to guarantee that $\Gamma_C$ is invariant under time evolution.
3 Classical Poincaré Symmetry

Integrating the Lagrangian density $\mathcal{L}$ in (1) over spacetime gives the action, whose functional derivative with respect to the spacetime metric (evaluated at the globally Lorentzian metric, $\eta_{\mu\nu}$) yields the symmetric and conserved energy-momentum tensor

$$T^{\mu\nu} = (D^\mu\varphi)(D^\nu\varphi) - F^\rho_\mu F^\nu_\rho - \eta^{\mu\nu} L,$$  \hspace{1cm} (13)

where parentheses on the indices denotes symmetrization. Together with the Poincaré group of isometries, this then leads to the conserved Poincaré charges

$$\mathcal{P}^\mu := \int d^3 x T^{0\mu},$$  \hspace{1cm} (14)

$$\mathcal{J}^{\mu\nu} := \int d^3 x \left( x^{\mu} T^{0\nu} - x^{\nu} T^{0\mu} \right).$$  \hspace{1cm} (15)

As we shall see shortly, as dynamical variables on $\Gamma$ these constants of the motion canonically generate transformations on the classical states which realize the Poincaré algebra, at least when acting on the constraint surface $\Gamma_c$.

We find, in terms of the phase space variables,

$$T^{00} = \frac{1}{2} \left( \Pi^2_{A_i} + \Pi^2_\xi + \Pi^2_\eta \right) + \mathcal{V}(A_i; \xi, \eta),$$  \hspace{1cm} (16)

$$T^{0i} = -\Pi_{A_i} F_{ij} - \Pi_\xi (\partial_\eta \xi - \epsilon A_i \eta) - \Pi_\eta (\partial_\xi \eta + \epsilon A_i \xi),$$  \hspace{1cm} (17)

where $\mathcal{V}(A_i; \xi, \eta)$ is the integrand in (5). Thus the generator of time translation

$$\mathcal{P}^0 = \mathcal{C}_2 \left( \frac{1}{2} G^{-1} \right) + \mathcal{C}_0 (V)$$  \hspace{1cm} (18)

in condensed notation, where $\mathcal{C}_2 (S) := S^{A_1 \cdots A_s} (Q) P_{A_1} \cdots P_{A_s}$ denotes the homogeneous classical dynamical variable on $\Gamma$ associated with a symmetric contravariant valence $s$ tensor field $S$ on $M$ (cf (8)).
After an integration by parts the spatial translation generators turn out to be

\[
\mathcal{P}^k = \int d^3 x \left\{ -\Pi_A \partial_k A_i - \Pi_\xi \partial_k \xi - \Pi_\eta \partial_k \eta - e A_k C_\alpha \right\}
=: C_1^{(k)}X), \tag{19}
\]

where we read off the vector field components

\[
kX^A(Q) = (-\partial_{x^k} A_i(x), -\partial_{x^k} \xi(x), -\partial_{x^k} \eta(x)) - e \int d^3 z A_k(z) \phi_k^a(Q). \tag{20}
\]

Here \( \gamma := z \) is included in the integration. Similarly, the spatial rotation generators

\[
\mathcal{J}^k := \frac{1}{2} [k^m n^l] \mathcal{J}^{mn}
= \int d^3 x [k^m n^l] \left\{ x^m [-\Pi_A \partial_n A_i - \Pi_\xi \partial_n \xi - \Pi_\eta \partial_n \eta - e A_n C_\alpha] - \Pi_A A_n \right\}
=: C_1^{(k)Y}, \tag{21}
\]

where \([k^m n^l]\) is the completely antisymmetric symbol in three dimensions, with \([123]=1\), and

\[
kY^A(Q) = [k^m n^l] (-x^m \partial_{x^l} A_i(x) - \delta_i^m A_n(x), -x^m \partial_{x^l} \xi(x), -x^m \partial_{x^l} \eta(x))
- e \int d^3 z [k^m n^l] z^m A_n(z) \phi_k^a(Q). \tag{22}
\]

Finally, the boost generators

\[
\mathcal{K}^k := \mathcal{J}^{0k} = -C_2 \left( \frac{1}{2} \mathcal{K} \right) - C_0 (kV) + t\mathcal{P}^k, \tag{23}
\]

where the boost tensors

\[
kK^{AB}(Q) = \begin{pmatrix}
\delta_{ij} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \frac{1}{2} (x^k + y^k) \delta(x - y), \tag{24}
\]
and the boost potentials

\[ kV(Q) = \int d^3x e^k \mathcal{V}(A_i; \xi, \eta), \tag{25} \]

which are analogous to the potential \( V(Q) \) in the Hamiltonian.

For future reference we record here some properties of the various tensors associated with the Poincaré charges. First, the Lie derivative with respect to \( \phi_a \) of every valence zero, one, and two tensor that occurs in \( \mathcal{P}^0, \mathcal{P}^k, \mathcal{J}^k, \mathcal{K}^k \) vanishes.\(^2\) This is sufficient (but not necessary) to guarantee that the Poincaré charges are classical observables, that is, gauge invariant on the constraint surface \( \Gamma_C \). Next, the boost tensors \( kK \) have field independent components in the Cartesian coordinates (see (24)), and so are, in fact, covariantly constant (cf (4)).\(^3\) Finally, the spatial translation and rotation vectors \( kX \) and \( kY \), while not Killing, are nevertheless divergence-free:

\[ \nabla \cdot kX = -\frac{1}{2} G_{AB} (\mathcal{L}_k G)^{AB} = \int d^3x d^3y \delta(x - y) \partial_x \delta(x - y) = 0 \tag{26} \]

since \( \partial_x \delta(x - y) \) is antisymmetric; similarly for \( kY \). As we shall see later, these results considerably simplify the Dirac quantization of the Poincaré algebra.

But first we must work out the algebra at the classical level. In terms of our previous notation the Poisson bracket can be expressed as

\[ \{C_a(S), C_t(T)\} = C_{a+t-1}([-S, T]), \tag{27} \]

where \([S, T]\) is the Schouten concomitant \([16]\) of \( S \) and \( T \). A straightforward, but lengthy calculation,\(^4\) paying careful attention to integrations by parts, yields

\[ \{\mathcal{P}^{\mu}, \mathcal{P}^{\nu}\} = 0 - e \int d^3z F^{\mu\nu} C_7, \tag{28} \]

\(^2\)For the explicit calculation refer to [14].

\(^3\)This implies they are Killing, and in involution, which, modulo terms that vanish on \( \Gamma_C \), is necessary for the Poincaré algebra to close.

\(^4\)Again, refer to [14].
\[
\begin{align*}
\{ J^\mu, P^\rho \} &= \eta^\rho_r \mathcal{P}^\mu - \eta^\mu_r \mathcal{P}^\rho - e \int d^3 z \ (z^\mu F^\rho - z^\rho F^\mu) C_\gamma, \\
\{ J^\mu, J^\rho \} &= \eta^\rho_r J^\mu - \eta^\mu_r J^\rho + \eta^\mu_r J^\mu - \eta^\rho_r J^\rho + e \int d^3 z \ (z^\mu z^\rho F^\sigma - z^\rho z^\mu F^\sigma + z^\mu z^\rho F^\mu - z^\rho z^\mu F^\rho) C_\gamma.
\end{align*}
\]

Since \( C_\gamma \approx 0 \) defines the constraint surface, we see that we have explicitly verified the classical Poincaré algebra for scalar electrodynamics; we are not aware of any similar calculation in the literature. Notice that since \( F^{0j}(\mathbf{z}) = -\Pi_{A_j}(\mathbf{z}) \), the ‘off-shell pieces’, which are linear combinations of the constraints, are linear and/or quadratic in the momenta, and these must be dealt with accordingly when we do Dirac quantization.

## 4 Dirac Quantization of Poincaré Symmetry

We now proceed with Dirac quantization of the Poincaré charges, and subsequent verification of the Poincaré symmetry at the quantum level. The phase space \( \Gamma = T^* M \), and since \( M \) comes equipped with a (flat) metric, \( G \), it is natural to choose Schrödinger picture quantization with state space \( \mathcal{F} \) consisting of all smooth complex-valued functions on \( M \), and Hilbert space \( \mathcal{H}_{\text{dir}} := L^2(M, \mathcal{E}) \), the subset of those which are square integrable with respect to the volume form \( \mathcal{E} \) associated with \( G \).\(^5\) The quantization map is not unique; here we shall choose the simplest one:

\[
\begin{align*}
C_0(V) &= V \quad \mapsto \quad Q_0(V) = V, \\
C_1(X) &= X^A P_A \quad \mapsto \quad Q_1(X) = -i\hbar \left\{ X^A \nabla_A + \frac{1}{2}(\nabla_A X^A) \right\}, \\
C_2(K) &= K^{AB} P_A P_B \quad \mapsto \quad Q_2(K) &= (-i\hbar)^2 \left\{ K^{AB} \nabla_A \nabla_B + (\nabla_A K^{AB}) \nabla_B \right\} \\
&= (-i\hbar)^2 \nabla_A K^{AB} \nabla_B.
\end{align*}
\]

\(^5\)Of course the choice of volume form is arbitrary—and can be avoided altogether by using half-densities instead of wavefunctions [4]—but this choice is made for definiteness.
\[ C_\alpha(T) = T^{ABC} P_A P_B P_C \quad \longrightarrow \quad Q_\alpha(T) = (-i\hbar)^3 \{ T^{ABC} \nabla_A \nabla_B \nabla_C + \frac{3}{2} (\nabla_A T^{ABC}) \nabla_B \nabla_C - \frac{1}{4} (\nabla_A \nabla_B \nabla_C T^{ABC}) \}, \] (34)

where, as before, \( \nabla \) is the Levi-Civita connection on \( M \). This will be called ‘minimal quantization’ in that given the leading term (highest order in derivatives) the additional complimentary terms are the minimum ones necessary to make the operator self adjoint. (Notice that cubic operators occur in the commutator of two quadratic operators, and so are relevant for the Poincaré algebra).

In the spirit of Dirac [15] we now account for the constraints by quantizing them—on the same footing as any other observable linear in the momenta: \( \hat{C}_\alpha = Q_1(\phi_\alpha) \). The physical state space, \( \mathcal{F}_{\text{phys}} \subset \mathcal{F} \), is then defined as the collection of states \( \Psi_{\text{phys}} \) annihilated by the constraint operators:

\[ \hat{C}_\alpha \Psi_{\text{phys}} = 0 \, \forall \alpha \quad \Longleftrightarrow \quad \Psi_{\text{phys}} \in \mathcal{F}_{\text{phys}}. \] (35)

As emphasized by Kuchař [2], the choice of basis for the gauge vectors \( \phi_\alpha \) is arbitrary at the classical level, but that this breaks down at the quantum level, at least if one demands that the constraint operators be self adjoint. The trouble lies in the complimentary divergence term in (32), but can be eliminated by restricting to a preferred basis which is ‘compatible’ with the Hilbert space structure:

\[ \mathcal{L}_{\phi_\alpha} E = 0 \, \forall \alpha, \] (36)

i.e. in which the \( \phi_\alpha \) are divergence-free. This restriction is natural in the sense that (35) then implies \( \phi_\alpha \Psi_{\text{phys}} = 0 \, \forall \alpha \), i.e. \( \mathcal{F}_{\text{phys}} \) consists of gauge invariant complex-valued functions on \( M \).

Furthermore, given such a basis one is free to transform to any other basis whose elements are all divergence-free, i.e. taken from the set

\[ \mathcal{G} := \{ \mu = \mu^a \phi_\alpha \mid \nabla \cdot \mu = 0 \quad \Longleftrightarrow \quad \phi_\alpha \mu^a = 0 \}. \] (37)
In our case the Lagrangian provides a natural basis of \( \phi_a \) which are Killing, and so certainly satisfy (36). Furthermore, constraint operators constructed from elements of \( \mathcal{G} \) will be consistent (first class) iff \( \phi_a f^\gamma_{\alpha\beta} = 0 \), or, equivalently, \( f^\gamma_{\alpha\beta} = \phi_\beta f \) for some scalar \( f \). In our case the structure functions \( f^\gamma_{\alpha\beta} \) vanish (see (7)), and so this condition is trivially satisfied.

The Poincaré charges \( \mathcal{P}^0, \mathcal{P}^k, \mathcal{J}^k, \mathcal{K}^k \) contain pieces zero, first and second order in momenta, and are quantized accordingly using the minimal quantization scheme. In order for the resulting quantum Poincaré charges to be observables they must commute with the constraint operators \( \mathcal{Q}_1(\mu) \) (at least on \( \mathcal{F}_{\text{phys}} \)) for all \( \mu \in \mathcal{G} \).

Let the scalar \( U \) represent any of the Hamiltonian of boost potential \( V \) or \( kV \); we have

\[
\frac{1}{i\hbar} [\mathcal{Q}_0(U), \mathcal{Q}_1(\mu)] = \mathcal{Q}_0(-[U, \mu]).
\]

But \( -[U, \mu] = \mathcal{L}_\mu U \) vanishes since the potentials are constant along the gauge orbits. For a vector \( Z \), representing any of the spatial translation or rotation vectors \( kX \) or \( kY \), we find

\[
\frac{1}{i\hbar} [\mathcal{Q}_1(Z), \mathcal{Q}_1(\mu)] = \mathcal{Q}_1(-[Z, \mu]),
\]

where \( -[Z, \mu] = \mathcal{L}_\mu Z \). Using the fact that \( \mathcal{L}_{\phi\alpha} Z = 0 \) it is easy to show that \( \mathcal{L}_\mu Z \in \mathcal{G} \), so the right hand side of (39) annihilates \( \Psi_{\text{phys}} \).

Finally, letting \( K \) stand for either the inverse metric, \( G^{-1} \), or any of the boost tensors, \( kK \), we have

\[
\frac{1}{i\hbar} [\mathcal{Q}_2(K), \mathcal{Q}_1(\mu)] = \mathcal{Q}_2(-[K, \mu]) + \hbar^2 \mathcal{Q}_0(W);
\]

\[
W = \frac{1}{2} \nabla_A (K^{AB} \nabla_B (\nabla \cdot \mu)).
\]
The $\hbar^2 Q_0(W)$ term is a van Hove anomaly (discussed more fully below), which in this case vanishes precisely because of the restriction (37). Furthermore,

$$ - \llbracket K, \mu \rrbracket = \mathcal{L}_\mu K = \mu^a \mathcal{L}_{\phi^a} K - 2\psi^a \otimes_S \phi^a, \quad (42) $$

where $\otimes_S$ denotes symmetrized tensor product. The term on the right hand side vanishes, and the Cartesian components of the vector fields $\psi^a$ are $\psi^a A = K^{AB} \nabla_B \mu^a$. Quantizing the remaining term yields an operator proportional to

$$ \nabla_A \psi^a B \phi^B \nabla_B + \nabla_A \phi^a B \psi^B \nabla_B. \quad (43) $$

The first term annihilates $\Psi_{\text{phys}}$, and the second is equivalent to

$$ (\nabla \cdot \phi_a) \psi^a + \phi_a \psi^a = [\phi_a, \psi^a] + \psi^a \phi_a. \quad (44) $$

Now the second term on the right hand side of this expression annihilates $\Psi_{\text{phys}}$, and, furthermore, the commutator vanishes:

$$ (\mathcal{L}_{\phi_a} \psi^a)^A = (\mathcal{L}_{\phi_a} K)^{AB} \nabla_B \mu^a + K^{AB} (\mathcal{L}_{\phi_a} \nabla \mu^a)_B = 0. \quad (45) $$

Thus, the quantum Poincaré charges are observables.

The next question to ask is whether or not they realize the Poincaré algebra when acting on physical states. There are two considerations: van Hove anomalies, and whether or not the minimal quantization of the off-shell pieces in (28 –30) produces operators which annihilate physical states. We discuss these in turn.

Since the work of Groenewold [8] and van Hove [7] it has been known that no map from classical to quantum observables exists which preserves the entire Poisson algebra.\footnote{See, e.g., [9] for a more precise statement.} For the minimal quantization map given in (31–34) van Hove
anomalies appear first in the quadratic-linear commutator: refer to (41), which applies for generic $K$, and $\mu$ replaced by a generic vector field, $Z$. But the only vector fields occurring in the Poincaré charges are the spatial translation and rotation vectors $^kX$ and $^kY$, which are both divergence-free (see 26), and so this particular van Hove anomaly is not present.

For the generic quadratic-quadratic commutator:

$$\frac{1}{\hbar} [Q_2(K), Q_2(L)] = Q_3([-K, L]) + \hbar^2 Q_1(Z),$$

(46)

$$Z^D := \frac{1}{2} \nabla_B \nabla_C [K, L]^{BCD} - \nabla_B A^{BD},$$

$$A^{BD} := K^{AB} L^{CD} \mathcal{R}_{AC} + (\nabla_C K^{AB})(\nabla_A L^{CD}),$$

$$-\frac{1}{3} \nabla_C (K^{AB} (\nabla_A L^{CD}) - K^{AD}(\nabla_A L^{CB})), -(K \leftrightarrow L).$$

(47)

For our example the van Hove term, $\hbar^2 Q_1(Z)$, vanishes because the Ricci tensor $\mathcal{R}_{AB}$ is zero ($M$ is flat) and the inverse metric and boost tensors are all covariantly constant. Thus the Poincaré algebra is free of van Hove anomalies under minimal Dirac quantization.

We now come to the quantization of the off-shell pieces in (28 30), of which there are essentially only two types. The first type has the form

$$-e \int d^3 z \ F^{ij} \gamma C_\gamma =: C_1(\mu^\gamma \phi_\gamma),$$

(48)

where (with $\gamma := z$) the scalars

$$\mu^\gamma := -e F^{ij}(z).$$

(49)

But since the electromagnetic field strength is gauge invariant, we certainly have $\phi_\gamma \mu^\gamma = 0$, so $\mu^\gamma \phi_\gamma \in \mathcal{G}$, in which case $Q_1(\mu^\gamma \phi_\gamma) \Psi_{phys} = 0$. 

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The second type has the form

$$- e \int d^8 z \ F^{0j} C_\gamma =: \mathcal{C}_2(\psi^\gamma \otimes \phi_\gamma),$$  

(50)

where the vector field components

$$\psi^{\gamma A} = (e \delta^j_0 \delta(z - \mathbf{x}), 0, 0).$$  

(51)

The quantization of this second type of term is exactly analogous to the discussion following (42), except the Lie derivative corresponding to (45) is

$$(\mathcal{L}_\phi \psi^\gamma)^A = - \int d^3 z \ d^3 y \left\{ e \delta(z - y) \frac{\delta}{\delta A_j(y)} \right\} \phi^A_\gamma(Q) = 0,$$  

(52)

since $\phi^A_\gamma(Q)$ has no dependence on the field $A_j(y)$ (and $\psi^\gamma A$ has no field dependence at all).

In conclusion, we see that the Poincaré algebra is, indeed, realized as quantum operators acting on $\mathcal{F}_{\text{phys}}$ using the minimal quantization scheme for Dirac quantization.

### 5 Reduced Quantization of Poincaré Symmetry

Classical reduction is readily achieved by choosing the complete set of gauge invariant functions

$$q^a = (B_i(\mathbf{x}), \rho(\mathbf{x})), $$  

(53)

where $B_i(\mathbf{x}) := A_i(\mathbf{x}) + \frac{1}{c} \partial_x \theta(\mathbf{x})$ and $\varphi(\mathbf{x}) =: \rho(\mathbf{x}) \exp i \theta(\mathbf{x})$, as coordinates on the reduced configuration space, $m$. Since the constraints are linear in the momenta, the reduced phase space $\gamma = T^* m$, with canonical coordinates $(q^a, p_a)$. An observable $\mathcal{C}_a(S)$ on $\Gamma$ maps to the corresponding physical variable $c_a(s) :=$
\[ s^{a_1 \cdots a_s}(p_{a_1} \cdots p_{a_s}) \] on \( \gamma \), where the tensor \( s \) on \( m \) is the (physical) projection of \( S \).

In particular, the projected inverse metric

\[
g^{ab}(q) = \begin{pmatrix} \delta_{ij} - \partial_{x^i} \frac{1}{\sqrt{\rho(q)}} \partial_{x^j} & 0 \\ 0 & 1 \end{pmatrix} \delta(x - y). \tag{54}
\]

The other tensors involved in the Poincaré charges can similarly be projected onto \( m \), and the resulting reduced Poincaré charges on \( \gamma \) will obviously realize the Poincaré algebra (cf \((28 - 30)\) with \( C_\gamma = 0 \)), a fact which can be verified by direct calculation. For the purpose of our discussion it is sufficient to know only the projected boost tensors:

\[
m^a_k a^b(q) = \begin{pmatrix} \delta_{ij} x^m - \partial_{x^i} \frac{x^m}{\sqrt{\rho(q)}} \partial_{x^j} & 0 \\ 0 & x^m \end{pmatrix} \delta(x - y). \tag{55}
\]

We remark that these are Killing tensors, and are in involution with each other: \( \{ \alpha^a_k, \kappa^b_k \} = 0 \). Also it can be shown that the projected spatial rotation and boost vectors are Killing with respect to the metric \( g \) on \( m \), and so necessarily are (Levi-Civita) divergence-free.

We now quantize the reduced Poincaré charges, and attempt to verify the Poincaré symmetry at the quantum level. In analogy with the Dirac quantization considered earlier, we choose Schrödinger picture quantization with Hilbert space \( \mathcal{H}_{\text{red}} := L^2(m, e) \), where \( e \) is the volume form associated with the metric \( g \) on \( m \). As was the case with Dirac quantization, the choice of quantization map is not unique—especially now that the configuration space is not flat (see [12], and references therein).

But in order to compare Dirac and reduced quantization on an ‘equal footing’, we again choose minimal quantization (cf \((31 - 34)\)), which is also in keeping with tradition in the Dirac versus reduced quantization debate in the literature [1, 2,
In particular, for a physical variable quadratic in the momenta:

\[ c_2(k) = k^{ab} p_a p_b \mapsto \mathcal{Q}_2(k) := (-i\hbar)^2 \tilde{\nabla}_a k^{ab} \tilde{\nabla}_b = (-i\hbar)^2 \{ \partial_a k^{ab} \partial_b + k^{ab} (\partial_a \ln \omega) \partial_b \}, \]

where \( \tilde{\nabla} \) is the Levi-Civita connection on \( m \), \( \partial_a \) is the (functional) derivative with respect to \( q^a \), and \( \omega = \sqrt{\det g_{ab}} \) is the measure on \( m \) in the coordinates \( q^a \).

Now, as noted above, the classical Poincaré charges realize the Poincaré algebra, and this will automatically extend to the quantum level provided there are no van Hove anomalies. The first place such an anomaly might arise—with quadratic-linear commutators (cf (41) with \( \nabla \mapsto \tilde{\nabla} \), etc.)—it does not, since the projected translation and rotation vectors are divergence-free. This leaves a potential anomaly only with quadratic-quadratic commutators (cf (47)). However, instead of dealing with covariant derivatives\(^7\) and the Ricci tensor on \( m \) we calculate, in terms of \( \omega \):

\[
\frac{1}{(-i\hbar)^4} \{ \mathcal{Q}_2(k), \mathcal{Q}_2(l) \} = \llbracket [k, l] \rrbracket^{hcd} \partial_b \partial_c \partial_d + \frac{3}{2} \left\{ (\partial_b \llbracket k, l \rrbracket)^{hcd} + (\partial_b \ln \omega \llbracket k, l \rrbracket)^{hcd} \right\} \partial_c \partial_d \\
+ \left\{ k^{ab} \partial_d (\partial_b (v^d)) + u^b \partial_b (v^d) - (k \leftrightarrow l) \right\} \partial_d,
\]

(57)

for generic \( k \) and \( l \), where the ‘vectors’

\[
u^b := \partial_b k^{ab} + (\partial_a \ln \omega) k^{ab},\]

(58)

\[ v^d := \partial_d l^{cd} + (\partial_c \ln \omega) l^{cd}.\]

(59)

If \( k \) and \( l \) represent any of either the inverse metric or boost tensors then \( \llbracket [k, l] \rrbracket^{hcd} = 0 \), as noted earlier—a fact which must be true for the Poincaré algebra to close. Thus, for the quantum commutator to vanish, as it should, we require all components of the terms in braces in the last line of (57), which we will denote as \( \zeta^d \), to be zero.

\(^7\) Note that even if \( K \) is covariantly constant on \( M \), its physical projection \( k \) on \( m \) need not be.
For instance, the \((\text{quadratic-quadratic part of})\) the boost-boost commutator corresponds to taking \(k = \frac{m}{n}k, l = \frac{n}{n}k\), and we find, using (55), that only the \(d = \rho(w)\) component of \(\zeta\) is potentially nonvanishing:

\[
\zeta^w(w) = \int d^3 \mathbf{x} \mathbf{d}^3 \mathbf{y} y^m w^n \delta(\mathbf{x} - \mathbf{y}) \frac{\delta \ln \omega}{\delta \rho(\mathbf{x}) \delta \rho(\mathbf{y}) \delta \rho(w)} \\
+ \int d^3 \mathbf{y} y^m w^n \frac{\delta \ln \omega}{\delta \rho(\mathbf{y}) \delta \rho(\mathbf{y}) \delta \rho(w)} = (m \leftrightarrow n) \tag{60}
\]

(in uncondensed notation).

In order to evaluate \(\delta \ln \omega\) we first observe that (see (54))

\[
\det g^{ab} = \text{Det} \left[ \delta_{ij} - \partial_i \frac{1}{\rho^2} \partial_j \right] = \text{Det} \left[ \frac{1}{\rho^2} \left( -\frac{1}{\epsilon^2} \partial^2 + \rho^2 \right) \right], \tag{61}
\]

where \(\text{Det}\) denotes functional determinant, and \(\partial^2 := \partial_i \partial_i\). The last equality follows by decomposing the eigenvectors of the operator \(\delta_{ij} - \ldots\) into its transverse and longitudinal parts, and examining the eigenvalues. Hence

\[
\delta \ln \omega = \frac{1}{2} \delta \ln \text{Det} \left[ \frac{1}{\rho^2} \left( -\frac{1}{\epsilon^2} \partial^2 + \rho^2 \right) \right] = \frac{1}{2} \delta \text{Tr} \ln \left[ \frac{1}{\rho^2} \left( -\frac{1}{\epsilon^2} \partial^2 + \rho^2 \right) \right] \\
= \frac{1}{2} \delta \text{Tr} \ln \frac{1}{\rho^2} - \frac{1}{2} \delta \text{Tr} \ln \left( -\frac{1}{\epsilon^2} \partial^2 + \rho^2 \right) =: \delta I + \delta II, \tag{62}
\]

where we have assumed that the functional trace, \(\text{Tr}\), satisfies the usual cyclicity property, and that \(\delta \text{Tr} \ln A = \text{Tr} A^{-1} \delta A\) for any operator \(A\).

The first term is straightforward to evaluate:\footnote{We are being somewhat cavalier about regularization—simply because it is difficult to do much better—but we believe that our final conclusions still carry sufficient weight to be of interest, as we shall argue.}

\[
\delta I = \int d^3 \mathbf{x} \delta \rho(\mathbf{x}) \left\{ \frac{\delta (\mathbf{o})}{\rho(\mathbf{x})} \right\}, \tag{63}
\]

but the second term is more difficult. Following Hawking’s discussion \cite{10} on zeta function regularization we write

\[
\delta II = \left. \frac{d}{ds} \left[ \frac{1}{\Gamma(s)} \int d^3 \mathbf{x} \int_0^\infty d\tau \tau^s \delta D K(\mathbf{x}, \mathbf{x}, \tau) \right] \right|_{s=0}. \tag{64}
\]
Here the positive definite operator

\[ D := -\frac{1}{\epsilon^2} \partial^2 + \rho^2 + \epsilon, \]  

where \( \epsilon > 0 \) is a regulating ‘mass’ parameter. Its associated heat kernel, \( K(x, y, \tau) \), satisfies

\[ \frac{\partial}{\partial \tau} K(x, y, \tau) + DK(x, y, \tau) = 0, \]  

with initial condition \( K(x, y, 0) = \delta(x - y) \). \( D \) (and \( \delta D \) in (64)) act on the first argument of \( K \). In our case \( \delta D \) is simply \( 2\rho \delta \rho \).

As in, e.g., [11] we factorize:

\[ K(x, y, \tau) =: K_0(x, y, \tau)\Lambda(x, y, \tau) \]  

into a singular piece

\[ K_0(x, y, \tau) = \left( \frac{\epsilon}{4\pi \tau} \right)^\frac{3}{2} \exp \left( -\frac{\epsilon^2|\mathbf{x} - \mathbf{y}|^2}{4\tau} - \epsilon \tau \right), \]  

which satisfies (66) with \( \rho^2 \equiv 0 \), and initial condition \( K_0(x, y, 0) = \delta(x - y) \), and a regular piece, \( \Lambda(x, y, \tau) \). The latter contains the \( \rho \) dependence of \( K \), and satisfies

\[ \frac{\partial}{\partial \tau} \Lambda(x, y, \tau) + \frac{1}{\tau}(x^i - y^i)\partial_x^i \Lambda(x, y, \tau) = -\left( -\frac{1}{\epsilon^2}\partial_x^2 + \rho^2(x) \right) \Lambda(x, y, \tau), \]  

with initial condition \( \Lambda(x, x, 0) = 1 \). Again as in [11], we now expand

\[ \Lambda(x, y, \tau) = \sum_{n=0}^{\infty} a_n(x, y)\tau^n, \]  

and find that the coefficients \( a_n \) satisfy the recursion relation

\[ na_n + (x^i - y^i)\partial_x^i a_n + \left( -\frac{1}{\epsilon^2}\partial_x^2 + \rho^2(x) \right) a_{n-1} = 0, \quad n = 1, 2, \ldots \]  

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with $a_0(x, y) = 1$. In the coincidence limit we obtain

$$
a_1(x, x) = -\rho^2(x),
$$
$$
a_2(x, x) = \frac{1}{2} \left( -\frac{1}{3\epsilon^2} \partial_x^2 + \rho^2(x) \right) \rho^2(x),
$$
in agreement with [17].

Substituting these results into (64) yields an expression valid for $\text{Re}(s) > \frac{1}{2}$, which when analytically continued to $s = 0$ gives

$$
\delta II = \int d^3x \delta \rho(x) \left\{ -\sum_{n=0}^{\infty} c_n \rho(x) a_n(x, x) \right\},
$$
where the coefficients

$$
c_n = \left( \frac{e}{4\pi} \right)^{\frac{3}{2}} \frac{\Gamma(n - \frac{1}{2})}{\epsilon^{n+\frac{1}{2}}}. \tag{75}
$$

Now although $a_n(x, x)$ effectively goes like $1/n!$ (see (71)), the series in (74) nevertheless diverges as $\epsilon \rightarrow 0$, and must therefore be treated as a formal expansion in powers of $\epsilon$. (And note also that powers of $\epsilon$ are associated with derivatives of $\rho$).

Finally, working out the higher order variations of $I$ and $II$, and using these results in (60), yields

$$
\zeta^p(w) = \frac{2c_2}{3\epsilon^2} w^n \left\{ \frac{\delta \ln \omega}{\delta \rho} \rho(\partial_m\rho) + \left( \partial_m \frac{\delta \ln \omega}{\delta \rho} \right) \rho^2 \right\} - (m \leftrightarrow n) \tag{76}
$$

(plus higher order terms). It is instructive to note that if $\delta \ln \omega/\delta \rho \propto \rho^k$, the term in braces vanishes iff $k = -1$, a situation which corresponds precisely to term $I$ of $\ln \omega$ (see (63)). Term $II$, on the other hand, contributes a polynomial with higher powers of $\rho$ (and also derivatives of $\rho$): the leading order (in both $\epsilon$ and $\epsilon$) contribution is

$$
\zeta^p(w) = \frac{1}{(12\pi)^2} \epsilon \left( w^n \partial_{w^m} - w^m \partial_{w^n} \right) \rho^3(w), \tag{77}
$$
which does not vanish for generic \( \rho \). Also note that higher order terms in the expansion (74) do not contain this combination of \( \epsilon \) and \( \epsilon \), so no cancelation of this piece is possible.

Thus, the boost-boost commutator (and hence the Poincaré algebra as a whole) fails to be realized at the quantum level using minimal reduced quantization.

6 Discussion

We have thus shown that minimal Dirac quantization preserves the Poincaré symmetry of scalar electrodynamics, but that minimal reduced apparently does not. To better understand how this comes about it is instructive to determine what the Dirac-quantized Poincaré charges look like acting on physical states, so they can be compared on the same footing with their reduced counterparts.

For instance, direct calculation using (4), (33), and (53) shows that the kinetic energy operator \( Q_2(\frac{1}{2}G^{-1}) \), acting on \( \Psi_{\text{phys}} \in \mathcal{F}_{\text{phys}} \), is equivalent to \( Q_2(\frac{1}{2}g^{-1}) \) (see (56)), except with \( \partial_a \ln \omega \) replaced by an object we call, similarly, \( \partial_a \ln \omega' \), whose only nonvanishing component is

\[
\frac{\delta \ln \omega'}{\delta \rho(\mathbf{x})} = \frac{\delta(o)}{\rho(\mathbf{x})}. \tag{78}
\]

In fact, the analogous statement applies for the entire set of Poincaré charges: minimal Dirac quantization (acting in \( \mathcal{F}_{\text{phys}} \)) is identical in form with minimal reduced quantization, except with \( \partial_a \ln \omega' \) in place of \( \partial_a \ln \omega \), a difference which corresponds to retaining only the first term, \( \delta I \), in (62). (Compare (78) with (63)).

This means, for instance, that the quadratic-quadratic commutator in minimal Dirac quantization has the same form as (57), but when applied to the
boost-boost commutator is easily seen to yield \( \zeta = 0 \), i.e. no anomaly, as expected from the results of section 4. We remark that, although the term \( \delta I \) in (63) contains \( \delta(\omega) \), and so is not regulated, it is common to both the Dirac and reduced approaches, and the (independent) results of section 4 support the proposition that this term does not cause a problem with the Poincaré algebra. Rather, it is the additional term, \( \delta II \), present in reduced quantization—in particular, those pieces involving derivatives of \( \rho \), which begin to appear with the \( n = 2 \) term in (74)—that causes a van Hove anomaly.

In fact, we observe that \( \exp(-II) \) is nothing but the volume element, \( \sqrt{\det \gamma_{\alpha\beta}} \), on the gauge orbits, where the metric

\[
\gamma_{\alpha\beta} := G_{AB} \phi_A^A \phi_B^B = \left( -\frac{1}{\epsilon^2} \partial_\alpha^2 + \rho^2(x) \right) \delta(x - y).
\]  

(79)

Now in must be emphasized that, in general, the minimal Dirac and minimal reduced quantization schemes are not equivalent\(^9\) [1, 2, 3, 4, 5, 6]: Having transformed to a common Hilbert space, the inequivalence manifests itself in the quadratic operators as a factor ordering ambiguity involving precisely the above volume element on the gauge orbits [6]. Furthermore, for a given model it can happen that both the Dirac and reduced factor orderings are self consistent—the relevant example here being Kuchar’s helix model [3] (which is a finite dimensional analogue of scalar electrodynamics). So even though the Hamiltonians might have different spectra, which could, in principle, be measured, there may be no internal physical criterion with which to select the correct factor ordering, as happens in the helix model [3].

The significant point here is that scalar electrodynamics has an additional symmetry—the Poincaré symmetry—and, at the quantum level, this symmetry is sensitive to this difference in factor ordering (or presence of \( \sqrt{\det \gamma_{\alpha\beta}} \)), suggesting,

\(^9\)For example, the respective Hamiltonians have different spectra, in general.
in fact, that minimal Dirac quantization in correct, and minimal reduced is not (at least in this case).

This result also supports previous work [18, 12] suggesting a preference for minimal Dirac over minimal reduced because of the natural similarity of the former with several curved-space quantization schemes proposed in the literature.

In general, then, demanding the preservation of a sufficiently nontrivial classical symmetry at the quantum level may serve as a useful internal physical criterion with which to select amongst inequivalent factor orderings, as we have demonstrated here. It might also be instructive to find a suitable finite-dimensional model with which to demonstrate this point, free of any regularization complications.

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