INTRODUCTION

The collision of two black holes is expected to be an intense gravitational radiation source of gravitational wave energy. When coupled with a single-particle emission of the initial infall trajectory, we might infer that the gravitational wave is emitted by the black holes as they approach each other. This process is expected to be an important source of gravitational radiation, and understanding the details of the emission process is crucial for the design of gravitational wave detectors.

The quadrupole gravitational wave forms are also expected to be well modeled, with the quadrupole waveforms being particularly important for the detection of black hole coalescences. In this paper, we focus on the close limit of the black hole coalescence, where the black holes are close enough to interact with each other. We compare the numerical simulations of the black hole coalescence to the gravitational waveforms predicted by perturbation theory.

1. Theoretical Predictions

Theoretical predictions of the gravitational waveforms from black hole coalescences are based on the numerical simulations of the black hole dynamics. These simulations are based on the general relativity equations of motion, and they provide a detailed description of the black hole merger process. The gravitational waveforms predicted by these simulations are compared with the predictions of perturbation theory, and the agreement is typically excellent.

2. Experimental Measurements

The gravitational waveforms from black hole coalescences are detected by gravitational wave detectors, such as LIGO and Virgo. These detectors are designed to detect the minute distortions in space-time produced by the gravitational waves.

3. Conclusion

The study of black hole coalescences provides a unique opportunity to test general relativity and to improve our understanding of black hole dynamics. The comparison between theoretical predictions and experimental measurements is crucial for the validation of the theoretical models and for the design of new detectors.
the evolution results.

II. METHODOLOGY

In this paper, we will be concerned with two equal-mass nonrotating black holes, each with axisymmetric inward-pointing momentum \( P \) (the slice has zero net-momentum). Initial-data sets representing one or more black holes with individually specifiable linear and angular momenta are constructed using the conformal-imaging approach developed by York and coworkers [6]. The case of two black holes with axisymmetric momenta was implemented numerically by Cook [7]. The data sets used for this study were constructed using a code based on the Cadez-coordinate approach described in that work. The reader should refer there for details regarding the construction of the data sets and for further descriptions of the parameterization of the data sets described below. The separation of the holes is parameterized by \( \beta \), which is related to the bispHERical-coordinate separation parameter \( \mu_0 \) by the relation \( \mu_0 = \cosh^{-1}(\beta/2) \) in the case of equal-mass holes. The code is used to compute the inversion-symmetric (with minus isometry condition) extrinsic curvature \( K_{ij} \) and to solve the Hamiltonian constraint for the conformal factor \( \psi \). Once the full initial data is computed, several physical quantities characterizing the system are computed. Of interest in this paper are the ADM mass of the initial-data slice \( M \), the proper separation of the holes \( \ell \), and the masses of the individual holes \( m_1 = m_2 \) defined in terms of the area of the marginally outer-trapped surface associated with each hole. We also define the total or bare mass of the system by \( m = m_1 + m_2 \). Note that the difference between \( m \) and \( M \) is due to the binding energy of the system. Given an initial-data set, we use the boundary-value-problem method described in Paper I to locate all marginally outer-trapped surfaces surrounding the two holes (if they exist) and identify the apparent horizon(s). We should note that there is nothing unique about our choice of initial data for representing two colliding black holes. One could imagine, for example, considering data with Euclidean topology with the black holes represented by boosted matter collapsed inside its horizon. Our initial data was chosen for the convenience of its highly refined numerical treatment [7] and earlier physical exploration (Paper I).

Like Price and Pullin [4], for purposes of our analysis we treat the spacetime as a perturbation (but not necessarily a time-symmetric one) of Schwarzschild. First, we establish a Schwarzschild-like coordinate system around the two black holes in terms of the (background space) isotropic coordinates used in the numerical solution \( r = r_i(1 + M/2r_i^2) \). Note that the background space of the numerical solution can be directly parameterized by the isotropic radial coordinate \( r_i \), even though the numerical solution is found in Cadez coordinates. The total ADM mass of the slice \( M \) is used as the Schwarzschild background mass. Tortoise coordinates are also defined in the usual way: \( r_* = r + 2M \ln(r/2M - 1) \). Computation of wave perturbations involves the calculation of multipole amplitudes by surface integrals. These are performed over constant Schwarzschild radial-coordinate two-surfaces. The integrands involve the conformal factor \( \psi \), Schwarzschild-coordinate extrinsic-curvature components \( K_{ij} \), and their Schwarzschild-coordinate radial derivatives. Calculation of these quantities at their required locations is achieved with bi-cubic spline interpolations and a series of coordinate transformations.

The gauge-invariant function \( Q_{mn} \) is formed out of multipole projections of \( \psi \) and \( \psi_{,r} \) computed by numerical integrations over a coordinate two-sphere (cf. Refs. [8-10]). For this paper we compute only the case of \( Q \equiv Q_{20} \). We also require the Schwarzschild time-derivative of the gauge-invariant function \( \partial_t Q \). This time derivative is computed as

\[
\partial_t Q = a L_n Q,
\]

where \( L_n \) is the Lie derivative along the slice-normal congruence \( n \) and the factor \( a = \sqrt{1 - 2M/r} \) comes about from the transformation from the slice-normal time coordinate to the Schwarzschild time-coordinate. The Lie derivative of \( Q \) is calculated using the extrinsic curvature (and its radial derivative) via the definition

\[
L_n g_{ij} = -2K_{ij}.
\]

The gauge-invariant perturbation function and its time derivative, known as a function of radius surrounding the merged black hole, serves as initial data for integration of the Zerilli equation. The numerically generated initial perturbation is interpolated onto a fine grid (typically 8000-16000 zones) that is even in \( r_* \) and extends from \( r_* = -500M \) to \( r_* = 2000M \). The Zerilli equation (cf. Ref. [8,10,4]) is then integrated forward in time until the whole perturbation has been propagated to \( |r_*| = \infty \). Approximate asymptotic wave forms and energy fluxes are computed at large radii.

Our code for calculating the initial black-hole perturbation from numerically generated initial data was checked by comparing it against the time-symmetric results of Price and Pullin [4]. It should be noted that they analytically expanded the metric perturbation about Schwarzschild in powers of the parameter \( \epsilon = 1/|\ln \mu_1| \) and retain only the leading term in \( \epsilon \). In the limit of small separation, the initial perturbation we obtain numerically agrees closely with their analytic results (for \( \beta \lesssim 3.25 \) the agreement is better than 5%). Not surprisingly, for larger separations the differences become larger. For the horizon cutoff point of \( \beta = 4.17 \) or \( \mu_1 = 1.36 \) (the largest separation allowing an encompassing apparent horizon for time-symmetric initial data), the analytic prediction for the energy radiated is about 60% higher than the result from the full solution. It is interesting to note that neglecting the higher-order terms in \( \epsilon \) always seems to lead to a greater amount of radiated energy.
FIG. 1. The apparent horizon formation line. The inward linear momentum on each hole $P/m$ is plotted as a function of proper separation $\ell/m$.

III. WAVEmORs AND ENERGy FLUX

The perturbed black-hole approximation assumed in this paper is not valid if the two black holes have not merged (have no common event horizon). For separations small enough that an encompassing apparent horizon exists, we find that the addition of inward-pointing linear momenta makes the encompassing apparent horizon more spherical and the maximum radiation efficiency (defined as the ratio of $M$ minus the mass of the apparent horizon to $M$) decreases. Moreover, we find that the metric perturbation $Q$ always gets smaller. We, therefore, contend that our treatment of time-asymmetric initial-data sets with inward-pointing linear momenta is always at least as valid as the study of the time-symmetric solution.

In Paper I we located the horizon-formation line for boosted black-hole initial data. For a given separation parameter $\beta$, we searched for the smallest value of inward linear-momentum for which an encompassing horizon surrounded the holes. The momentum as a function of proper hole separation for this horizon line is displayed in Fig. 1. The horizon cutoff point mentioned previously, at $\beta = 4.17$ and $P = 0$, lies on this line. For larger values of $\beta$, it is necessary to give the holes inward momentum in order for an encompassing apparent horizon to exist. We note that this horizon line is only an estimate of where the actual encompassing event horizons will form. Along this horizon-formation line, we have computed the radiated energy for the initial-data sets using the gauge-invariant perturbation formalism and Zerilli-equation integration method described above. In Fig. 2, the radiated energy is plotted as a function of proper separation $\ell/m$. For the points shown, the inward momentum on each hole ranges from $P/m = 0.0355$ to $P/m = 1.738$. One interesting feature is that the radiation efficiency appears to saturate at about 2%, substantially below the maximum radiation efficiency based on area theorem arguments. This suggests that it may be impossible to obtain high radiation efficiency for black-hole collisions, even if they merge with very large momenta. As a gauge of what constitutes a large momentum, we estimate below the momenta of two holes at the moment of horizon formation assuming a parabolic infall from rest.

At each point on the horizon formation line, we have a value for the separation $\ell/m$ and momenta $P/m$ of the holes. Treating the black holes as point particles and using Newtonian dynamics, we can estimate the separation $(\ell/m)_0$ at which the holes were at rest:

$$\left(\frac{\ell}{m}\right)_0 = \frac{\ell/m}{1 - 16(P/m)^2(\ell/m)}.$$

Clearly, if we assume infall from rest, the maximum momenta the two holes can obtain at the point of horizon formation is estimated by the point on the horizon-formation line where the denominator of Eq. (3) vanishes, i.e. $(\ell/m)_0 \rightarrow \infty$. We find this point to be $P/m = 0.178$ and $\ell/m = 1.97$. At this point, the implied radiation efficiency is less than 0.1%. In the inset of Fig. 2, we show
FIG. 3. Waveforms from time-symmetric and asymmetric two-black-hole initial data. The top curve shows the quadrupole waveform from analysis of a time-symmetric data set with $\beta = 4.275$. The lower curve shows the quadrupole waveform from analysis of a time-asymmetric initial data set with the same value of $\beta$ and $P/m = 0.1225$. Both waveforms are extracted at a radius $r = 200M$.

The radiated energy from points on the horizon-formation line plotted as a function of the Newtonian estimate for their proper separation when at rest. We find striking agreement between our calculation of total energy radiated and the simulations of Anninos et al. [3]. For cases where their initial data had an encompassing horizon, it is clearly correct to compare with the perturbation analysis of Misner data. For cases with greater separation, the time-asymmetric analysis gives excellent results. For example, for $\mu_0 = 2.2$ or $(\ell/m)_0 = 3.97$, the radiation efficiency from perturbation theory of the corresponding horizon-line initial-data set is $E/M = 3.9 \times 10^{-4}$, as compared with $E/M = 1.7 \times 10^{-3}$ from the time-symmetric analysis of $\mu_0 = 2.2$ Misner data and $E/M \simeq 5.5 \times 10^{-4}$ from the fully relativistic simulations. A post-Newtonian calculation of the infall trajectory might improve this comparison.

In Fig. 3 we show a typical waveform from a boosted head-on collision observed at a radius of $r = 200M$. The case shown is for a separation of $\beta = 4.275$, $P/m = 0.1225$, and $\ell/m = 1.948$. For comparison we show the waveform from time-symmetric data with the same $\beta$. Both waveforms are dominated by normal mode oscillations within about 10M after the black-hole surface is causally apparent at the extraction radius. The addition of ingoing momentum considerably increases the amplitude of the oscillation and reverses the sign of the waveform. The presence of momentum (extrinsic curvature) on the initial slice causes a significant transient feature in the waveform; this qualitative effect is not seen by the numerical relativity simulations of time-symmetric initial data with $\mu_0 = 2.2$ because it takes finite time for the momentum component of the perturbation to propagate out to the extraction surface. It is possible that this feature will be present in the evolution of time-symmetric initial data starting at larger separations.

IV. DISCUSSION

Anninos et al. [3] have shown that for collisions resulting from large initial separations there is excellent agreement between the emitted energy and the well-known results for a test-particle falling into Schwarzschild corrected for equal-mass objects, finite infall distance, and horizon heating. Price and Pullin [4] demonstrated that a perturbation analysis of time-symmetric initial data could reproduce the results of the fully relativistic simulation in the case that the black holes have small initial separation. Here we have shown that the perturbation analysis can be extended to larger separations, including the regime in which point particle analysis is valid, by considering appropriate time-asymmetric initial-data sets. Adopting this perspective, one can accurately predict the total emitted energies over the entire range of separation, from the close limit to parabolic trajectories starting at infinity.

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