Tensor Integrals for Two Loop Standard Model Calculations

Dirk Kreimer
Dept. of Physics
Univ. of Tasmania
G.P.O.Box 252C
Hobart, 7001
Australia

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Abstract

We give a new method for the reduction of tensor integrals to finite integral representations and UV divergent analytic expressions. This includes a new method for the handling of the $\gamma$-algebra.

In calculating two loop corrections to the Standard Model one is confronted with two main problems. One is the analytical difficulty of integrals involving different masses. Often one is restricted to approximations as for example zero mass assumptions for light particles [1]. A further problem stems from the difficulty in the algebraic sector, where the increasing number of terms for the tensor structure has to be handled. For the case of two-point functions an elegant method is available to reduce all tensor integrals to a basic set of scalar integrals [2]. Nevertheless it is clear that these methods will run into difficulties when applied to three- or four-point functions. Also, for arbitrary mass cases, one is restricted to either numerical evaluations or asymptotic expansions [3]. This is due to the fact that for arbitrary mass cases it is not possible to express the generic scalar integrals through known special functions [4].

In this paper we will present a general method to express two-loop integrals in terms of finite integrals suitable to numerical evaluations plus a set of products of one-loop integrals containing the UV singular part. The method presented here is as well applicable to arbitrary tensor integrals but will prove especially powerful when applied to two-loop graphs directly, as it will bypass the whole tensor structure problem by constructing what can be called the characteristic polynomial of the graph.

The final result is appropriate for the use of integral representation generated from the ones in [5] but the method can also be applied with other choices for the generic integrals, as the way how we handle the tensor structure is independent from the analytical approach one uses.

The methods presented in the following apply to two-loop $n$-point Green's function for arbitrary $n$, but in examples we will restrict ourself to two-loop two- and three-point problems. We will demonstrate the use of our method on the topology of the following figure.

*email: kreimer@physwax.phys.utas.edu.au
A general two-loop integral has the following representation
\[ I = \int d^D k \frac{\delta^{(n_l)}(p_1)}{N_l M N_k}, \] (1)
where \((n_l, n_k)\) denotes the rank of its tensor structure in \(l, k\) loop momenta. Here \(N_l\) is the part of the denominator containing only propagators depending on \(l\), \(N_k\) the same for loop momentum \(k\) and \(M\) contains the propagators involving both loop momenta.

We omit in the following the cases where terms in the numerator cancel the \(M\) part of the denominator completely out as these terms give rise to products of one-loop integrals only.

Let us assume \(I\) has an overall degree of divergence \(\omega\) and degrees of divergence \(\omega_l, \omega_k\) for the subdivergences. To make this integral convergent we have to subtract its \(l\) resp. \(k\) sub-divergent behaviour as well as its overall divergence.

To this end let us define the following
\[
\hat{N}_l = N_l \Big|_{m_l=0,q_l=0}, \\
\hat{N}_k = N_k \Big|_{m_k=0,q_k=0}, \\
\hat{M} = M \Big|_{m_l=0,q_l=0}, \\
L_{j^i} = \frac{(\hat{N}_l \hat{M} - N_l M)^j}{(\hat{N}_l)^{j}}, \\
K_{j^s} = \frac{(\hat{N}_k \hat{M} - N_k M)^s}{(\hat{N}_k)^{s}},
\] (2)
where \([m_l=0,q_l=0]\) means to evaluate the corresponding expression with all masses and external momenta set to zero.

Now consider the replacement
\[
\frac{1}{N_l} \rightarrow \frac{1}{N_l} L_{j^i},
\]
which improves \(\omega\) and \(\omega_l\) at least by \(j^i\) and leaves \(\omega_k\) unchanged, and a similar replacement for \(k\). Consequently
\[
\frac{1}{N_l M N_k} \rightarrow \frac{1}{N_l M N_k} K_{j^s} L_{j^i},
\] (3)
results (at least) in an improvement
\[
\omega = \omega + j^i + j^k =: \omega', \\
\omega_l = \omega_l + j^i =: \omega'_l, \\
\omega_k = \omega_k + j^k =: \omega'_k.
\]

Let us introduce the notation \(\omega' > 0 \iff \{\omega' > 0, \omega'_l > 0, \omega'_k > 0\}\). Let us further define
\[
L_{j^i} := \frac{1}{(N_l M)^{j^i}}, \\
K_{j^s} := \frac{1}{(N_k M)^{j^s}}
\] (4)
\[
\sum_{\ell=1}^{j^i} \binom{l}{j^i} (N_l M)^{j^i} (\hat{N}_l \hat{M})^{j^i-\ell},
\]
\[
\sum_{\ell=1}^{j^s} \binom{k}{j^s} (N_k M)^{j^s} (\hat{N}_l \hat{M})^{j^s-\ell},
\]
\[
\sum_{\ell=1}^{j^s} \binom{k}{j^s} (N_k M)^{j^s} (\hat{N}_k \hat{M})^{j^s-\ell}.
\]
where we emphasize that the sums start with \( l = 1 \) resp. \( k = 1 \).

We then have the algebraic identity

\[
\frac{T^{(m_1, n_k)}}{N_1 M N_k} = \frac{T^{(m_0, n_k)}}{N_1 M N_k} \left[ L^j_1 K^j_k \right] - \frac{T^{(m_1, n_k)}}{N_1 M N_k} \left[ L^j_0 K^j_k \right] \quad \forall j_1, j_k. \tag{5}
\]

Now there clearly exists for a given \( T \) and given \( N_1, M, N_k \) powers \( j_1, j_k \) such that \( \omega_1, \omega_2, \omega_3 \) are all positive, \( \omega_4 > 0 \), so that the first expression on the right hand side of equation (5) is finite. So it can be calculated in \( D = 4 \) dimensions. But by inspecting the sums

\[
L^j_1 = \frac{1}{(N_1 M)^{j_1}} \left[ \sum_{\omega = 0}^{j_1} \binom{j_1}{j_1} (-N_1 M)^{j_1} (\bar{N}_1 \bar{M})^{j_1-1} \right],
\]

\[
K^j_k = \frac{1}{(N_k M)^{j_k}} \left[ \sum_{\omega = 0}^{j_k} \binom{j_k}{j_k} (-N_k M)^{j_k} (\bar{N}_k \bar{M})^{j_k-1} \right],
\]

we can identify all terms in the expansion which fulfill \( \omega > 0 \). As these terms can be calculated in four dimensions they cancel against the corresponding terms of the second expression in the right hand side of equation (5). Note that to identify the divergent part a simple counting of dimensions of masses and exterior momenta in the numerator of equation (5) is sufficient.

So we define the following expressions

\[
\tilde{L}^j_1 = \frac{1}{(N_1 M)^{j_1}} \left[ \sum_{\omega = 0}^{j_1} \binom{j_1}{j_1} (-N_1 M)^{j_1} (\bar{N}_1 \bar{M})^{j_1-1} \right],
\]

\[
\tilde{K}^j_k = \frac{1}{(N_k M)^{j_k}} \left[ \sum_{\omega = 0}^{j_k} \binom{j_k}{j_k} (-N_k M)^{j_k} (\bar{N}_k \bar{M})^{j_k-1} \right],
\]

\[
\tilde{L}^j_0 = \frac{1}{(N_1 M)^{j_1}} \left[ \sum_{\omega = 0}^{j_1} \binom{j_1}{j_1} (-N_1 M)^{j_1} (\bar{N}_1 \bar{M})^{j_1-1} \right],
\]

\[
\tilde{K}^j_k = \frac{1}{(N_k M)^{j_k}} \left[ \sum_{\omega = 0}^{j_k} \binom{j_k}{j_k} (-N_k M)^{j_k} (\bar{N}_k \bar{M})^{j_k-1} \right], \tag{6}
\]

where \( \sum_{\omega \neq 0} \) means summation only over divergent contributions. We obtain

\[
\frac{T^{(m_1, n_k)}}{N_1 M N_k} = \frac{T^{(m_1, n_k)}}{N_1 M N_k} \left[ \tilde{L}^j_1 K^j_k \right] - \frac{T^{(m_1, n_k)}}{N_1 M N_k} \left[ \tilde{L}^j_0 \tilde{K}^j_k \right]. \tag{7}
\]

Still the first of the two terms on the right hand side of equation (7) can be calculated in \( D = 4 \) dimensions. We will now consider an example which will be useful to demonstrate the method. Consider Fig. (1) with propagators

\[
P_1 = l^2 - m_1^2, \\
P_2 = (l + q)^2 - m_2^2, \\
P_3 = (l + k)^2 - m_3^2, \\
P_4 = (k - q)^2 - m_4^2, \\
P_5 = k^2 - m_5^2,
\]
and choose

\[ T^{(2,1)} = l_y l_z k_x, \]

\[ N_1 = P_1 P_2, \]

\[ M = P_3, \]

\[ N_2 = P_4 P_5, \]

\[ \tilde{N}_1 = l^4, \]

\[ \tilde{M} = (l + k)^2, \]

\[ \tilde{N}_2 = k^4. \]

so that we have

\[
\frac{T^{(2,1)}}{N_1 M N_k} L^2 = \frac{T^{(2,1)}}{N_1 M N_k} \frac{2 T^{(2,1)}}{N_1 M N_k} + \frac{T^{(2,1)}}{N_1 M N_k} \frac{(N_1 M)^2 N_k}{(N_1 M)^2 N_k},
\]

\[
\frac{T^{(2,1)}}{N_1 M N_k} L^2 \quad - 2 \frac{T^{(2,1)}}{N_1 M N_k} + \frac{T^{(2,1)}}{N_1 M N_k} \frac{(N_1 M)^2 N_k}{(N_1 M)^2 N_k},
\]

\[
\frac{T^{(2,1)}}{N_1 M N_k} L^2 \quad = \frac{T^{(2,1)}}{N_1 M N_k} - \frac{T^{(2,1)}}{N_1 M N_k} \frac{2 (l \cdot q)}{N_1 M N_k},
\]

\[
\frac{T^{(2,1)}}{N_1 M N_k} L^2 \quad = - \frac{T^{(2,1)}}{N_1 M N_k} + \frac{T^{(2,1)}}{N_1 M N_k} \frac{(2 l \cdot q)}{N_1 M N_k}.
\]

where we used that \( P_1 = \tilde{P}_1 - m_1^2, \quad P_2 = \tilde{P}_2 + 2 l \cdot q + q^2 - m_2^2, \quad P_3 = \tilde{P}_3 - m_3^2. \)

\( \tilde{P}_1 \) means again the suppression of all masses and exterior momenta in the corresponding propagator.

We will continue later with this example and discuss now how to calculate the UV divergent part of equation (7), which is contained in the second term of the right hand side of this equation. These integrals are of the form

\[
\frac{T^{(m_i, n_k)}}{(l^2)^{(l + k)/2} N_k}, \quad \frac{T^{(m_i, n_k)}}{(k^2)^{(l + k)/2} N_k},
\]

or

\[
\frac{T^{(m_i, n_k)}}{(l^2)^{(l + k)/2} (k^2)^{(l + k)/2}} \equiv 0.
\]

The second equation, (11), is just a relabeling of (9).

The third equation, (11), vanishes identically in dimensional regularization. The integer powers \( i, j \) in the above equations are determined by \( j_1, j_2 \).

Let us comment on the vanishing of the integral in (11) here. It might happen that this cancellation is due to a mutual cancellation of an infrared and an ultraviolet divergence, as it appears in the integral \( \int d^4 l / l^4 \). Now the UV singular part of the integral in (11) will have its counterpart in the final integral representation in (7), where it is needed to cancel some other UV singular parts. So it seems that we have installed a spurious infrared singularity in our integral representation. This should then be regularized by some appropriate IR cutoff and this would install a corresponding scale in the integral in (11) which would make it non-vanishing. But these infrared singularities appear in the integral representation derived in [5] always as endpoint singularities. As this problem with spurious IR singularities can only appear when simultaneously \( n_1 > 0 \) and \( n_2 > 0 \) it will (in the Feynman gauge) be restricted to the two-loop boson self-energies. Nevertheless it is true in general that IR singularities appear as endpoint singularities in the method of [5] which avoids parametrizations. Let us study these endpoint singularities a little bit more. Investigating the case

\[ N_1 = P_1, \]
we find the following expression for the subtraction of UV divergences

\[
\frac{1}{N_1 M N_k} - \frac{1}{N_1 M N_k} - \frac{1}{N_1 M N_k} + \frac{M - m_3^2}{N_1 M M N_k} \tag{12}
\]

The last term \( \sim m_3^2 \) is the problematic one. The integral representation for it can be written as

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{|x|} \frac{1}{|x| + |y| + |x-y|}.
\tag{13}
\]

where we used \( \sqrt{x^2 + i\eta} = |x| \) in the limit \( \eta \to 0 \). Equation (13) has an apparent endpoint singularity at \( x = 0 \). But these endpoint singularities are a well defined distribution in the \( \eta \)-limit when interpreted as Hadamard’s Finite Part \( HFP \) [6]. Its relation to the propagator is known for long [7]. In fact, the distribution \( \int_{-\infty}^{\infty} dx / \sqrt{x^2 + i\eta} \) equals \( HFP[1/|x|] \). Using this, we handle these endpoint singularities with the help of \( HFP \) and find the corresponding integral representation

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{|x|} \frac{1}{|x| + |y| + |x-y| + x \leftrightarrow -x} + \ldots.
\]

Here the dots correspond to finite terms for which the endpoint singularity in \( x \) has canceled out after the \( y \) integration and so \( HFP \) reduces to an ordinary Riemann integral. As \( HFP \) is a local operation which modifies test functions only at the location of the singularity our UV behaviour remains unchanged and the sum in equation (12) is still UV finite. This can also be explicitly checked by working out all the different cases for the modulus function in equation (13) in detail. So the integral representation for equation (12) is free of IR and UV singularities. One can also use \( HFP \) to separate nonspurious IR singularities when they appear. As for the integral (11) to appear we need UV singular behaviour in both loop variables, it is clear that the above subtlety can only arise when there is a logarithmic UV divergence in the second loop integration, which is exactly the case for the above example.

In this sense the above example is generic and serves as a proof in general.

We can still maintain the vanishing of equation (11) by using \( HFP \) in our integral representations. This can also be interpreted as a consistency between \( HFP \) and DR.

Let us continue now our general discussion. We can do one integration in equation (9) leading to remaining integrals of the form

\[
C \int d^D k \frac{\tilde{T}}{(k^2)^p N_k}.
\tag{14}
\]

where \( \alpha \) is an integer only when \( D = 4 \) and \( \tilde{T} \) is some tensor in \( k \). The coefficient \( C \) will contain a pole in \( (D-4) \) in general. This and possible UV divergences of the \( k \) integration itself forbid to calculate these integrals in four dimensions.

Calculating the above integrals (14) for arbitrary \( D \) and \( \alpha \) is possible but a difficult analytical task. But there is an algebraic approach to these integrals. We will first transform the UV divergent integrals in equation (7) via a partial integration in such a way that the first \( i \) integration is UV finite and its UV divergences are shifted to the \( k \) integration. Then we use a method similar to the method above to split the final \( k \) integration once more into a finite integration, analytically computable in an easy manner and UV divergent integrals of massive tadpole structure. As a consequence everything is expressible by algebraic methods in one-loop functions.
Defining
\[ I^{(i,n,m)} := \int d^D l \frac{l_{\mu_1} \ldots l_{\mu_i}}{((l^2)^p((l + q)^2)^m)}, \]  
we have
\[ 0 = \int d^D l \frac{\partial}{\partial l_{\nu}} \frac{l_{\mu_1} \ldots l_{\mu_i}}{((l^2)^p((l + q)^2)^m)} = (D + i - 2n - m) I^{(i,n,m)} - m I^{(i-1,n,m+1)} + mk^2 I^{(i,n+1,m)}, \]
\[ j^{(i,0,m)} = j^{(i,m,0)} = 0 \quad \forall i, m. \]

As \( D + i \geq 2(n + m) \) for UV divergent integrals, we have \((D + i - 2n - m) \neq 0\). Therefore we can always solve equation (16) for \( I^{(i,n,m)} \) in the following.

We can repeat this procedure until we have expressed \( I^{(i,n,m)} \) through integrals of the form \( I^{(i,n',m')} \) with powers \( n', m' \) such that they are all finite in the \( l \) integration.

We will end up with integrals of the form
\[ \int d^D k \frac{k_{\mu_1} \ldots k_{\mu_i} (k^2)^\beta}{N_k}, \]
where again \( \beta \) is an integer only if \( D = 4 \) but all coefficients of these integrals are now finite in four dimensions.

But the \( k \) integration is still UV singular, and to find an expression for these integrals we do a last replacement
\[ \frac{1}{N_k} = \frac{1}{N_k} (K^i - \bar{K}^i) \quad \text{where} \]
\[ K^i = (\bar{N}_k - N_k)^i \quad \text{and} \]
\[ \bar{N}_k := N_k \big|_{q=0}, \]
where it is understood that when expanding the numerators the sums run only over the divergent parts, as indicated by the \( \bar{\cdot} \) on the various \( K \). Note that we have chosen propagators with vanishing exterior momenta but non-vanishing masses to profit from symmetric integration properties and we remind the reader that we cannot set the masses to zero because this would result in vanishing tadpole integrals. So we end up with finite integrals and separated UV divergent integrals of the form
\[ \frac{k_{\mu_1} \ldots k_{\mu_i} (k^2)^\beta}{N_k}, \]
which are, as they are massive tadpole integrals, easy to evaluate in \( D \) dimensions. The finite integrals in (17) are related to standard massive one-loop integrals with integer powers of propagators.

Let us continue our example now. By inspecting equation (8) we see that we have to consider \( I^{(2,2,1)} \) and \( I^{(3,3,1)} \). The last one is already finite in the \( l \) integration. For the first one we have
\[ I^{(2,2,1)} = \frac{-1}{(D - 3)} \left[ \frac{2}{(D - 2)} k^2 j^{(2,1,3)} - k^2 I^{(2,2,2)} \right]. \]

The remaining integrals are of the form
\[ \int d^D k \frac{k_{\mu_1} \ldots k_{\mu_i}}{P_k P_3 (k^2)^\nu \frac{1}{m^2}}. \]

By using \( P_k = P_3 = P_4 - 2k \cdot q + q^2 \) we separate this according to the above equations (17). Note that as a consequence of our subtraction operation only terms with an even power
of \( k \) in the numerator give contributions to UV divergent integrals. The finite integrals are of the form
\[
\int d^n k \frac{k_m \ldots k_n (-2 k \cdot q + q^2)^p}{P_m P_n (k^2)^2 + \frac{m_n}{r}}.
\]
where only terms of even power in \( k \) contribute in the numerator.

Via partial fraction decompositions in \( k^2, P_1, P_2 \) this can be reduced to standard one-loop integrals with integer powers of propagators which are algebraically related to one-loop integrals with unit powers of propagators via mass derivatives.

All these reductions and separations can be handled by symbolic calculation routines. So it is possible to reduce an arbitrary two-loop tensor integral into finite integral representations and a standard set of one-loop integrations. For the examples discussed in this paper this was always possible using REDUCE [8] on a 486 notebook within less than 5 minutes. Note that even for the two-loop four-point function \( N_k \) is not worse than a one-loop three-point function (apart from the trivial topology arising from self-energy insertions), so that the resulting one-loop integrals are relatively simple. The knowledge of arbitrary tensor integrals of one-loop two- and three-point functions with integer powers of propagators is indeed sufficient for the calculation of two-loop problems by this method.

Having demonstrated the handling of the UV divergent part let us now discuss the integral representations for the finite integrals. In [5] an integral representation is given which transforms the integral to a twofold integral over the parallel space variables. This can be generalized to arbitrary tensor integrals which appear as polynomials in parallel and orthogonal space variables in the numerator. So we have to integrate out the orthogonal space variables. This can be easily handled by use of partial integrations and applying the residue theorem afterwards. But a little modification arises for the terms which subtract the UV divergence. Here we cannot use a partial fraction decomposition in the propagator as these propagators are equal. Doing the \( z \) and \( k_L \)-integration as in [5] we are left with
\[
\int d l_{\perp} \frac{l_i}{(l_i^2 - l_\perp^2)^2} \log \text{terms}; \quad i = 2, 3.
\]
But we can integrate these terms most easily by rewriting
\[
\frac{1}{(l_i^2 - l_\perp^2)^2} = \frac{1}{2} \frac{\partial^i}{\partial \mu^i} \left( \frac{1}{l_i^2 - l_\perp^2 - \mu} \right) |_{\mu = 0}.
\]
(18)

A similar remark applies to the three-point two-loop functions. The case of four-point functions will be discussed elsewhere, but will give no modifications in general [9]. Nevertheless we include the principal reduction of two-loop box integrals in the discussion below.

Up to now we have studied particular tensor integrals only. Let us now discuss a method how to calculate tensor integrals more directly. We again use the notation of [5] for the parallel and orthogonal space variables and will discuss as a specific example the following graph, which appears in the flavour changing fermion self-energy. The two-loop radiative corrections to order \( O(a_s) \times g^2 \) are currently being investigated for these self-energies [10].

![Fig. 2: Again the master topology; the labels specify propagators and particles.](image)

Let us decompose the Clifford algebra
\[
\{ \gamma_\parallel, \gamma_{\perp} \} = 0,
\]
\[
\{ \gamma_\parallel, \gamma_\parallel \} = -k_\parallel \gamma_{\perp}^2,
\]
\[
\{ \gamma_{\perp}, \gamma_{\perp} \} = k_\parallel \gamma_{\parallel}^2.
\]
where we simply use the fact that we can split a loop momentum in DR in its component in the direction of the exterior momentum $q = q_{\parallel}$ and its orthogonal complement. The same splitting can be done for higher dimensional parallel spaces (the linear span of the exterior momenta). This gives rise to a corresponding splitting in the Clifford algebra which can also be written down in a covariant manner

\[
\begin{align*}
\gamma_{\parallel\mu} &= \frac{1}{q^2} q_{\parallel\mu}, \\
\gamma_{\perp\mu} &= \gamma_{\parallel\mu} - \frac{1}{q^2} q_{\parallel\mu}, \\
\gamma_{\parallel\mu} &= \frac{g}{q^2} q_{\parallel\mu}, \\
\gamma_{\perp\mu} &= \gamma_{\parallel\mu} - \frac{g}{q^2} q_{\parallel\mu},
\end{align*}
\]

The generalization to higher dimensional parallel spaces is clear.

We can now use this decomposition for a reordering where we bring all parallel space $\gamma$ matrices and the anticommuting $\gamma_5$ to the left so that we have

\[
\gamma_5^{n_1,n_2} \gamma_{\parallel,1} \ldots \gamma_{\parallel,i_1} \gamma_{\perp,1} \ldots \gamma_{\perp,i_2},
\]

for an $i$ dimensional parallel space and a rank $k$ tensor in the orthogonal space. The $n_i$ are the number of appearances of $\gamma_5$ and parallel space $\gamma$-matrices. Using $\gamma_5^2 = 1, \gamma_{\parallel}^2 = 1$ we can simplify the above expression. For the case that we do not have free Lorentz indices in our Green function we can also simplify the orthogonal space part of this expression. In this case we are allowed to replace the orthogonal space $\gamma_{\perp}$ matrices identically by metrical tensors $g_{\perp}$, $(g_{\perp}^{\mu\nu} = (D - i))$, even when we do not apply a trace to the string. Note that this means that all the lengthy trace and $\gamma$ algebra calculations of perturbation theory will not appear for the case of no free Lorentz indices. It also means that we only have to consider contributions with an even number of $\gamma$-matrices in the orthogonal space. To prove this statement note that as long as we have no free indices (after contracting out all double indices) $\gamma_{\perp}$-matrices can only be generated by the loop momenta $\ell$. As the only covariant available for the orthogonal space integration is the total symmetric combination of metric tensors $g_{\perp}$ (the Levi Civita tensor does not appear in symmetric integration) we only need the symmetric part of the Clifford algebra in orthogonal space. This corresponds to a replacement of $\gamma_{\perp}$ matrices by $g_{\perp}$ tensors, as all indices have to be contracted by a total symmetric tensor.

This method can be generalized. In the case that we cannot avoid free indices we still can use the above argument for the $\gamma_{\perp}$ matrices generated by orthogonal space components of loop momenta. But note that in most cases it is possible to contract free indices with metrical tensors or exterior momenta without losing information. For example, to determine the form factors appearing in radiative corrections of the fermionic vertex, one can contract with exterior momenta $p, p'$ say. Then one can use all advantages of the case of no free indices.

Our example was calculated using this method by installing the proposed ordering algorithm via simple `LET` rules in REDUCE. Comparing this with the same calculation using the high energy packet in REDUCE, it was found that the method we suggest here was orders of magnitude quicker. Even for the one-loop case we can report on a calculation of two- and three-point functions in the unitary gauge in the SM where significant advantages were obtained for the $\gamma$ algebra [12].
This method fails for the anomaly. The interesting covariant is \( \epsilon_{\mu_1 \ldots \mu_d} p_1^{\mu_1} p_2^{\mu_2} \) and there exists no tensor constructed from symmetric metrical tensors and exterior momenta \( p_1, p_2 \) to give a nonvanishing contraction with this covariant. One nevertheless can calculate the resulting trace by this method. But then one has to consider the case of \( \gamma_5 \) with six \( \gamma \) matrices, two in the linear span of \( p_1, p_2, \) two in the orthogonal complement but still in four dimensional Minkowski space and two with arbitrary indices, which gives rise to the non-cyclic trace of [11].

Continuing with our example we are left with the tensors \( 1, \gamma_5, \frac{d}{d}, \gamma_5 \partial \). Each of these tensors is multiplied by an integral representation for the finite part, which is of the form (using (7), (18) and the results in [5])

\[
\frac{1}{(P_1 - P_2)(P_3 - P_4)} \left[ c_{25} L_{325} - c_{15} L_{315} + c_{14} L_{314} - c_{24} L_{324} \right] \\
- \frac{\partial}{\partial \mu} [((\epsilon(3, 0) + \epsilon(2, 1) + \epsilon(2, 0) + \epsilon(1, 1)) L_{\mu=0} - l \cdot q \frac{\partial^2}{\partial \mu^2} [(\epsilon(3, 0) + \epsilon(2, 1)) L_{\mu=0} \\
- \frac{\partial}{\partial \mu} [(\epsilon(0, 3) + \epsilon(1, 2) + \epsilon(0, 2)) L_{\mu=0} + k \cdot q \frac{\partial^2}{\partial \mu^2} [(\epsilon(0, 3) + \epsilon(1, 2)) L_{\mu=0}].
\]

We list the explicit coefficients \( c_{25}, c_{15}, c_{24}, c_{14}, c(i, j) \) in an appendix. For the case of Fig. (2) it took only about 20 seconds in REDUCE to generate them for the complete graph.

Let us summarize the situation for the most interesting two-loop graphs. We assume that the corresponding calculations are done in a renormalizable gauge, though it is not necessary to restrict to this case.

Two-point functions:
We have two topologies. For the trivial self-energy insertions we can always arrange that power counting improves by \( 2 n_i \) (assuming \( i \) is the subloop momentum), as we can always arrange exterior momenta not to flow through the subloop. For this topology we then have \( n_k = 0 \) always, while \( n_i = 1 \) or \( n_i = 2 \) corresponding to \( \omega = 1 \) or \( \omega = 2 \), that is corresponding to boson or fermion self-energies.

For the other (master) topology of our example we have \((n_k = 0, n_i = 2)\) or vice versa if \(\omega = 1\) and \((n_i = 2, n_k = 1)\) or vice versa if \(\omega = 2\).

Three-point functions:
We have three topologies, the ladder topology, the crossed topology and the self-energy insertion topology. For the last one we always have \((n_i = 1, n_k = 0)\) while the other ones reduce to \((n_k = 0, n_i = 1)\) or \((n_k = 0, n_i = 2)\), depending what kind of vertex one considers (the fermionic one has only logarithmic overall divergence, while e.g. triple boson couplings are linear divergent). Note that for the crossed topology, as \( M \) contains two propagators, the UV divergent one-loop integrals reduce to two-point functions and so this more complicated topology is simpler in its divergent structure. This corresponds to the fact that it has no proper subd Haversion, while the ladder topology has a vertex type subd Haversion and thus involves three-point one-loop functions.

Four-point functions:
We need four exterior bosons to have a UV divergence of logarithmic degree. We then have \(\omega = 0\) and the case \((n_i = 1, n_k = 0)\) is always sufficient. Again the complicated topologies involve simpler one-loop integrals.

We see that only for boson self-energies we have the case \((n_i \neq 0, n_k \neq 0)\) so that only for them do we have to use \( HF P \) explicitly. Note also that all \((n_i = 1, n_k = 0)\) cases reduce to a very simple subtraction which simply sets the exterior momenta and masses to zero in the subtracted term, and this is sufficient especially for the two-loop fermionic vertex
corrections.

We conclude that an easy separation of divergences for two-loop functions can be achieved. We saw that for the subtraction of the UV divergences of the real two-loop part very simple one-loop functions were sufficient. Especially only terms of order \(1/(D-4)\) were used. This was not unexpected as the order \(1/(D-4)^2\) poles of a two-loop graph can be obtained from products of one-loop functions. This is true in general for the leading divergence in a \(n\)-loop calculation. So we will find the leading divergences in the terms where the \(M\) part of the denominator is cancelled, giving rise to decoupled one-loop integrals. For our example in Fig.(2) we list the corresponding expressions in the appendix. As the individual terms of equation (7) correspond to terms subtracting \(l\) divergences, \(k\) divergences and overall divergences, one might wonder if it is possible to implement the whole renormalization program (that is the addition of one-loop counterterm graphs) at this level to obtain even simpler expressions. We intend to investigate this question in the future.

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Appendix

In this appendix we list the relevant polynomials in \(l_0, k_0\) for the final integral representation of the graph of Fig.(2) explicitly. Our notation follows [5]. We list these results to give an idea of the complexity resp. simplicity of the integral representation for a complete graph. We see that the integral representation, compared with the pure integral representation for the scalar master function, is similar in its qualitative and quantitative difficulty. Let us first list the propagators of Fig.(2):

\[ P_1 := l_0^2 - m_i^2, \quad P_2 := (l - q)^2, \]
\[ P_3 := (l + k)^2 - m_i^2, \quad P_4 := (k + q)^2 - m_i^2, \]
\[ P_5 := k_0^2 - m_i^2. \]

We define \(q := \sqrt{q^2}\) and \(g := q/l_0\).

\[ c_{15} := 8 \cdot \gamma_3 g \cdot (-2 \cdot q \cdot l_0 - k_0 + m_i^2 \cdot l_0 + m_i^2 \cdot k_0 - m_i^2 \cdot k_0 + m_i^2 \cdot l_0 + m_i^2 \cdot k_0)
- 4 \cdot \gamma_5 \cdot m_j \cdot (-2 \cdot q \cdot l_0 - 2 \cdot q \cdot k_0 - m_i^2 + 3 \cdot m_i^2 + m_j^2)
+ 8 \cdot g \cdot (-2 \cdot q \cdot l_0 - 2 \cdot q \cdot l_0 \cdot k_0 + m_i^2 \cdot l_0 + m_i^2 \cdot k_0 - m_i^2 \cdot k_0
+ m_i^2 \cdot l_0 + m_j^2 \cdot k_0) + 4 \cdot m_j \cdot (2 \cdot q \cdot l_0 + 2 \cdot q \cdot k_0 + m_i^2 - 3 \cdot m_i^2 - m_j^2) \]

\[ c_{14} := 8 \cdot \gamma_3 g \cdot (-2 \cdot q \cdot l_0)
- q^2 \cdot k_0 - 2 \cdot q \cdot l_0 - 4 \cdot q \cdot l_0 \cdot k_0 - 2 \cdot q \cdot k_0^2 + m_i^2 \cdot l_0 + m_i^2 \cdot l_0
+ m_i^2 \cdot k_0 + 4 \cdot \gamma_5 \cdot m_j \cdot (q^2 - 2 \cdot q \cdot l_0 + 2 \cdot m_i^2 + m_j^2)
+ 8 \cdot g \cdot (-2 \cdot q \cdot l_0 - q^2 \cdot k_0 - 2 \cdot q \cdot l_0 - 4 \cdot q \cdot l_0 \cdot k_0 - 2 \cdot q \cdot k_0^2
+ m_i^2 \cdot l_0 + m_j^2 \cdot l_0 + m_i^2 \cdot k_0) + 4 \cdot m_j \cdot (-q^2 + 2 \cdot q \cdot l_0 - 2 \cdot m_i^2 - m_j^2) \]
where we used the symbol \(OO\) to denote the term where the \(M\) part of the denominator is cancelled, so that it corresponds to a squared one-loop graph.
References


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