Let us consider a continuous function $f(x)$ whose integral is known to be known. How can we find the Riemann integral of the function $f(x)$?

The Riemann integral of a function $f(x)$ over an interval $[a, b]$ is defined as:

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x_i$$

where $\Delta x_i = \frac{b-a}{n}$ and $x_i = a + i \Delta x$. The integral exists if the limit exists.

The Riemann integral is a fundamental concept in calculus and has many applications, such as in physics and engineering. It is also used in the formulation of the Fundamental Theorem of Calculus, which connects differentiation and integration.

Using a computer algebra system, we can evaluate the integral of a function numerically or symbolically. This is particularly useful for functions that are difficult to integrate analytically.

In conclusion, the Riemann integral is a powerful tool for finding the area under curves and solving many problems in mathematics and science.
transformation for qubits). We can define the Fourier transformation as an active operation on a quantum state \(|x\rangle\) as
\[
\mathcal{F}|x\rangle = \frac{1}{\sqrt{\pi}} \int dy \ e^{2\pi i xy} |y\rangle ,
\]
where both \(x\) and \(y\) are in the position basis. This has been used in developing error correction codes [16, 18] and Grover’s algorithm for continuous variables [20]. This Fourier transformation can be easily applied in physical situations. For example, when \(|x\rangle\) represents quadrature eigenstate of a mode of the electromagnetic field, \(\mathcal{F}|x\rangle\) is simply an eigenstate of the conjugate quadrature produced by a \(\pi/2\) phase delay.

Another useful gate on a continuous variable quantum computer is XOR gate (analogous to the controlled NOT gate for qubits but without the cyclic condition) defined as [18]
\[
|x\rangle|y\rangle \rightarrow |x\rangle|x + y\rangle .
\]

Further, we assume that given a classical circuit for computing \(f(x)\) there is a quantum circuit which can compute a unitary transformation \(U_f\) on a continuous variable quantum computer. If a quantum circuit exists that transforms
\[
|x\rangle|y\rangle \rightarrow |x\rangle|y + f(x)\rangle ,
\]
then by linearity it can also act on any superposition of quantum states. For example, if we evaluate the function on a state (1) along with another quantum state \(|z\rangle\), we have
\[
U_f(\mathcal{F}|x\rangle|z\rangle) = \frac{1}{\sqrt{\pi}} \int dy \ e^{2\pi i y} |y\rangle|z + f(y)\rangle .
\]

This shows that using quantum parallelism for idealized quantum computers one can evaluate all possible values of a function simultaneously with one application of \(U_f\).

Now, we present the Deutsch-Jozsa algorithm for a continuous variable quantum computer. The set of instructions for deciding the constant or balanced nature of function \(f(x)\) are given below

(i). Alice stores her query in a quantum register prepared in an ideal position eigenstate \(|x_0\rangle\) and attaches another qubit in a position eigenstate \(|\pi/2\rangle\). So the two qubits are in the state \(|x_0\rangle|\pi/2\rangle\)

(ii). She creates superpositions of quantum states by applying the Fourier transformation to the query qubit and the target qubit. The resulting state is given by
\[
\mathcal{F}|x_0\rangle|\pi/2\rangle = \frac{1}{\sqrt{\pi}} \int dx \ dy \ e^{2\pi i x y} |x\rangle|y\rangle .
\]

(iii). Bob evaluates the function using the unitary operator \(U_f\). The state transforms as
\[
\frac{1}{\sqrt{\pi}} \int dx \ e^{2\pi i x y} |x\rangle\mathcal{F}|\pi/2\rangle .
\]

Here, the key role is played by the ancilla quantum state \(|\pi/2\rangle\). To see how the function evaluation takes place consider the intermediate steps given by

\[
U_f(|x\rangle\mathcal{F}|\pi/2\rangle) = \frac{1}{\sqrt{\pi}} \int dy \ e^{2\pi i y} U_f(|x\rangle|y\rangle) = (-1)^{f(x)}|x\rangle\mathcal{F}|\pi/2\rangle .
\]

If the function \(f(x) = 0\) there is no sign change and if \(f(x) = 1\) there is a sign change. After the third step performed by Alice, she has a quantum state in which the result of Bob’s function evaluation is encoded in the amplitude of the quantum superposition state given in (6). To know the nature of the function she now performs an inverse Fourier transformation on her quantum state.

(iv). The qubit states after Fourier tranform is given by
\[
|q\rangle = \frac{1}{\sqrt{\pi}} \int dx \ dx' e^{2\pi i (x-x')/2} (-1)^{f(x')}|x\rangle\mathcal{F}|\pi/2\rangle .
\]

(v). Alice measures her qubit by projecting onto the original position eigenstate \(|x_q\rangle\). In an ideal continuous variable scheme the correct projection operator is defined as [23]
\[
P_{\Delta x_0} = \int_{x_0 - \Delta x_0/2}^{x_0 + \Delta x_0/2} dy \ |y\rangle\langle y| .
\]
As has been explained in [20, 23] if the observable has a continuous spectrum then the measurement cannot be performed precisely but must involve some spread $\Delta x_0$. Therefore, the action of projection onto the qutrit state after step (iv) is given by

$$P_{\Delta x_0} = \frac{1}{\pi} \int dx \int_{x_0 - \Delta x_0/2}^{x_0 + \Delta x_0/2} dy e^{2i(x-y)} f(x) f(y) \pi/2,$$

(10)

Now consider two possibilities. If the function is constant then the above equation reduces to $\pm |x_0\rangle \langle x_0|$. In simplifying we need to use the Dirac delta function $(1/\pi) \int dx e^{2i(x-y)} = \delta(x_0 - y)$. This means that if Alice measures $|x_0\rangle$ she is sure that $f(x)$ is definitely constant. In the other case, i.e., when the function is balanced she will not get the measurement outcome to be $|x_0\rangle$. In fact, in the balanced case the outcome is orthogonal to the constant case as the result gives zero. Therefore, a single function evaluation followed by a measurement onto $|x_0\rangle$ in a quantum computer can decide whether the promised function is constant or balanced. Unlike the qubit case, in the idealized continuous variable case the reduction in the number of query calls is from infinity to one.

In conclusion, we have generalised the primitive quantum algorithm (Deutsch-Jozsa algorithm) from the discrete case to the idealized continuous case. It may be worth mentioning that as in error correction codes for continuous variables [16], if one replaces the Hadamard transform and XOR gate by their continuous-variable analogs in original Deutsch-Jozsa algorithm for qubit case, then the idealized algorithm works perfectly. This theoretically demonstrates the power of quantum computers to exploit the superposition principle giving an infinite speed up compared to classical scenario. This idealized analysis has not considered the effects of finite precision in measurement or state construction and so whether it may be implemented experimentally remains an open question for further study. Part of the difficulty in extending this work in this direction is that defining an oracle for continuous variables appears to be a difficult task, one that we have carefully avoided here. An alternate way forward might be to consider some sort of “hybrid” approach involving both quants and qubits. This is precisely what Seth Lloyd considers in the following chapter.

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