Chiral and Color-superconducting Phase Transitions with Vector Interaction in a Simple Model

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We investigate effects of the vector interaction on chiral and color superconducting (CSC) phase transitions at finite density and temperature in a simple Nambu-Jona-Lasinio model. It is shown that the repulsive density-density interaction coming from the vector term, which is present in the effective chiral models but has been omitted, enhances the competition between the chiral symmetry breaking (\(\chi_{SB}\)) and CSC phase transition, thereby making the thermodynamic potential have a shallow minimum in a wide range of the correlated chiral and CSC order parameters. We find that when the vector coupling is increased, the first order transition between the \(\chi_{SB}\) and CSC phases becomes weaker and the coexisting phase in which both the chiral and color-gauge symmetry are dynamically broken gets to exist in a wider range of the density and temperature. We shall also show that there can exist two end points, which are tricritical points in the chiral limit, in the critical line of the first order transition in some range of the vector coupling. Although our analysis is based on a simple model, the nontrivial interplay between the \(\chi_{SB}\) and CSC phases induced by the vector interaction is expected to be a universal phenomenon and might give a clue to understand results obtained by the two-color QCD on the lattice.

\S1. Introduction

It is one of the central issues in hadron physics to determine the phase diagram of strongly interacting matter in the temperature\textit{(T)}-chemical potential\textit{(}\(\mu\)\textit{)} plane or \textit{T}-\(\rho_B\) plane with \(\rho_B\) being the baryonic density. In extremely hot and dense matter, the non-Abelian nature of QCD ensures that the colored quarks and gluons are not confined and the chiral symmetry is restored. The lattice simulations of QCD\textsuperscript{1,2)} show that the QCD vacuum undergoes the chiral and deconfinement transition at a temperature \(T_c\) around 150 \(\sim\) 175 MeV at vanishing chemical potential, with the order and the critical temperature being dependent on the number of the active flavors. Although there are several promising attempts\textsuperscript{3–9)} to make simulations of the lattice QCD with finite \(\mu\) possible, they are still not matured enough to predict a definite thing about the phase transition at finite \(\rho_B\) or \(\mu\). It is widely believed on the basis of effective theories\textsuperscript{10–12)} and chiral random matrix theory\textsuperscript{13)} that the chiral phase transition from the chiral-symmetry broken to restored phase is a first order

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at vanishing temperature. Furthermore, according to the recent belief, the critical
line of the first order chiral transition continues for smaller chemical potentials in
the $T$-$\mu$ plane and ends at some point with $T = T_e$ and $\mu = \mu_e$, which is called the
end point. We notice that the first order chiral transition is accompanied with a
jump in the baryon density.

The recent renewed interest in the color superconductivity (CS)\textsuperscript{14)--22) stimulated
intensive studies of the QCD phase structure at finite density in the low tempera-
ture region, which in turn are revealing a rich phase structure of the high density
hadron/quark matter with CS.\textsuperscript{23)--29) Possible relevance of CS on characteristic phe-
nomena observed for neutron stars are being lively discussed.\textsuperscript{30), 31) Some recent
studies have also suggested that experiments on the Earth using heavy-ion collisions
with a large baryon stopping can tell something about CS in the dense matter.\textsuperscript{32), 33)}

The purpose of the present paper is to reveal a new characteristics of the chiral
to color superconducting (CSC) transition based on a simple effective model incor-
porating the vector interaction by focusing on the implication of the density jump
accompanied with the chiral transition.

Low-energy effective models\textsuperscript{10)--12), 23), 24) are useful to study not only the chiral
transition but also CS in dense hadronic matter. For example, chiral models of
Nambu-Jona-Lasinio type\textsuperscript{34) which may be supposed to be a simplified version of
the one with the instanton-induced interaction well describes the gross features of
the $T$ dependence of the chiral quark condensates of the lightest three quarks as
given by the lattice QCD, and predict that the chiral transition at $\mu \neq 0$ is of rather
strong first order at low temperatures when the vector interaction is small.\textsuperscript{12), 35), 36) Chiral
effective theories show that the gap $\Delta$ of CS may become as large as 100 MeV
in relatively low densities where the phase change from chiral symmetry breaking
($\chi$SB) phase to CSC phase may also occur.\textsuperscript{23), 24)}

However, although many works on CS have been made with the use of effective
models, all of them have not taken into account the vector interaction\textsuperscript{36)--48) \footnote{The sign of the vector coupling is the same as that in Refs. 36)--46), but opposite to that in
Refs. 47), 48).}

\begin{equation}
\mathcal{L}_V = -G_V (\bar{\psi} \gamma^\mu \psi)^2.
\end{equation}

Our point is that such a vector interaction is chiral invariant and naturally appears in
the effective models derived from microscopic theories and, as we shall show, indeed
can give great effects on the chiral to CSC transition and the properties of the CSC
phase.

Although it may not be a common knowledge in the physics community, the
importance of the vector coupling for the chiral transition is known; i.e., the vector
coupling weakens the phase transition and postpones the chiral restoration.\textsuperscript{36), 42), 46) This is intuitively understood as follows.\textsuperscript{49) According to thermodynamics, when
two phases I and II are in an equilibrium state, their temperatures $T_{I, II}$, pressures
$P_{I, II}$ and the chemical potentials $\mu_{I, II}$ are the same;

\begin{equation}
T_I = T_{II}, \quad P_I = P_{II}, \quad \mu_I = \mu_{II}.
\end{equation}
If I and II are the chirally broken and restored phase with quark masses \( M_I > M_{II} \), the last equality further tells us that the chirally restored phase is in higher density than the broken phase, because \( \mu_{I,II} \) at vanishing temperature are given by \( \mu_i = \sqrt{M_i^2 + p_{F_i}^2} \), where \( p_{F_i} \) is the Fermi momentum of the \( i \)-th phase. Thus one sees that the chiral restoration at finite density is necessarily accompanied with a density jump to higher density state with a large Fermi surface, which in turn favors the formation of Cooper instability leading to CS.

However, since the vector coupling includes the term \((\bar{\psi} \gamma^0 \psi)^2\), it gives rise to a repulsive energy proportional to the density squared i.e., \( G_V \rho_B^2/2 \) which is bigger in the restored phase than in the broken phase. Thus the vector coupling weakens and postpone the phase transition of the chiral restoration at low temperatures. Thus one expects naturally that \( L_V \) postpones the chiral restoration and the formation of CS to higher chemical potentials, and may alter the nature of the transition from the \( \chi_{SB} \) phase to the CSC phase drastically.

Is it legitimate to include the vector term like (1.1) in an effective Lagrangian? First of all, one should notice that the instanton-anti-instanton molecule model as well as the renormalization-group equation shows that \( L_V \) appears as a part of the effective interactions together with the ones in the scalar channels which is responsible for the chiral symmetry breaking (\( \chi_{SB} \)): The instanton-anti-instanton molecule model gives for the effective interaction between quarks

\[
L_{molsym} = G_{mol} \left\{ \frac{2}{N_c^2} \left[ (\bar{\psi} \gamma^a \psi)^2 + (\bar{\psi} \gamma^a i \gamma_5 \psi)^2 \right] - \frac{1}{2N_c^2} \left[ (\bar{\psi} \gamma^a \gamma_\mu \psi)^2 - (\bar{\psi} \gamma^a \gamma_\mu \gamma_5 \psi)^2 \right] + \frac{2}{N_c^2} (\bar{\psi} \gamma_\mu \gamma_5 \psi)^2 \right\} + L_8, \tag{1.3}
\]

where \( \tau^a = (\vec{\tau}, 1) \) and \( L_8 \) denotes the color octet part of the interaction which we shall not write down. Near the phase transition, the instanton molecules are polarized in the temporal direction, Lorenz invariance is broken and thus the vector interactions are modified as \((\bar{\psi} \gamma_\mu \Gamma \psi)^2 \rightarrow (\bar{\psi} \gamma_0 \Gamma \psi)^2\). Notice that the instanton-induced interaction breaks the \( U_A(1) \) symmetry. In reality, however, there should also exist \( U_A(1) \)-symmetric interactions such as the one-gluon exchange interaction or its low-energy remnant as

\[
L^0_{LL} = G^0_{ll} \left\{ (\bar{\psi}_L \gamma_0 \psi_L)^2 - (\bar{\psi}_L \gamma_i \psi_L)^2 \right\}, \tag{1.4}
\]

where \( \psi_L \) denote the left-handed quark field. It is shown that the strengths of the \( U_A(1) \)-symmetric and violating effective interactions are of the same order near the Fermi surface, using the renormalization group equation. Thus one sees that the vector interaction is there together with other chiral invariant terms which are usually used. Therefore, one may say that the previous works dealing with the \( \chi_{SB} \) to CSC phase transition are all incomplete because the vector interaction \( L_V \), which may alter the nature of the phase transition significantly, is not taken into account.

We shall show in this paper that the inclusion of the vector coupling induces a novel interplay between the \( \chi_{SB} \) and CS through the difference of the favoring baryon densities and changes both the nature of the phase transition and the phase
structure in the low temperature region drastically. The resultant phase diagram and the behavior of the chiral and diquark condensates as a function of \((T,\mu)\) will be found to have a good correspondence with those given in two-color QCD on the lattice, thereby our simple model gives a possible mechanism underlying the lattice results.

This paper is organized as follows: In the next section, the Lagrangian to be used is introduced. In \(\S\) 3, we shall give the thermodynamic potential and the self-consistency condition for the quark condensate and the pairing field. Numerical results are presented in \(\S\) 4. The final section is devoted to a summary and concluding remarks. The appendix summarises the effects of the vector interaction on the chiral transition when the CS is not incorporated.

\section{Model}

As a chiral effective model which embodies the vector interaction as well as the usual scalar terms driving \(\chi_{SB}\), we shall take a simple Nambu-Jona-Lasinio (NJL) model with two flavors \((N_f = 2)\) and three colors \((N_c = 3)\), following Ref.\(23)\). The NJL model may be thought as a simplified version of the one with the instanton-induced interactions and can be also derived by a Fierz transformation of the one-gluon exchange interaction with heavy-gluon approximation; see \(10), 11), 52)-54)\). This effective model has a merit to be used to investigate the chiral transition and CS simultaneously, hence their interplay. It was shown\(55)\) that the physical content given with the instanton model\(23)\) can be nicely reproduced by the simple NJL model with a simple three-momentum cutoff. This means that although there are several choices for high momentum cutoff which mimics the asymptotic freedom, the magnitude of the gap is largely determined by the strength of the interaction and insensitive to the form of the momentum cutoff.\(30)\) The Lagrangian density thus reads

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \tag{2.1}
\]

where

\[
\mathcal{L}_0 = \bar{\psi}(i\gamma \cdot \partial - m)\psi, \tag{2.2}
\]

with \(m\) being the current quark mass matrix \(m = \text{diag}(m_u, m_d)\), and

\[
\mathcal{L}_I = \mathcal{L}_S + \mathcal{L}_V + \mathcal{L}_C, \tag{2.3}
\]

with

\[
\mathcal{L}_S = G_S\{(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\tau\psi)^2\}, \tag{2.4}
\]

and

\[
\mathcal{L}_C = G_C\{(\bar{\psi}\gamma_5\tau_2\lambda_A\psi^C)(\bar{\psi}^Ci\gamma_5\tau_2\lambda_A\psi) + (\bar{\psi}\tau_2\lambda_A\psi^C)(\bar{\psi}^C\tau_2\lambda_A\psi)\}. \tag{2.5}
\]

\(\mathcal{L}_V\) is given in (1.1). Here, \(\psi^C \equiv C\bar{\psi}^T\), with \(C = i\gamma^2\gamma^0\) being the charge conjugation operator, and \(\gamma_2\) and \(\lambda_A\)'s are the second component of the Pauli matrix.
representing the flavor SU(2)$_f$, and antisymmetric Gell-Mann matrices representing the color SU(3)$_c$, respectively. The scalar coupling constant $G_S = 5.5$ GeV$^{-2}$ and a three momentum cutoff $A = 631$ MeV are chosen so as to reproduce the pion mass $m_\pi = 139$ MeV and the pion decay constant $f_\pi = 93$ MeV with the current quark mass $m_u = m_d = 5.5$ MeV; we have assumed the isospin symmetry. It should be noted that the existence of the diquark coupling $G_C$ and the vector coupling $G_V$ do not affect to determine the pion decay constant and the chiral condensate. Although, there are several sources to determine the diquark coupling such as the diquark-quark picture of baryons, the instanton-induced interaction, the renormalization group analysis and so on, we shall take $G_C/G_S = 0.6$ that well reproduces the phase diagram obtained with the instanton-induced interaction. As for the vector coupling, we vary it as a free parameter in the range of $G_V/G_S = 0 \sim 0.5$ to see the effect of the vector coupling for the phase diagram. We remark that the vector coupling $G_V$ is given by $0.25G_S$ in the instanton-anti-instanton molecule model and $0.5G_C$ in the renormalization-group analysis; the range we employ for $G_V$ thus encompass these physical values.

§3. Thermodynamic Potential and Gap Equations

In this section, we calculate the thermodynamic potential in the mean-field approximation and derive the coupled gap equations for the chiral and diquark condensates.

The thermodynamic potential $\Omega$ is defined by

$$\Omega = -T \ln \text{Tr} \ e^{-\beta \hat{K}},$$

(3.1)

where $\beta = 1/T$ is the inverse temperature and $\hat{K} = \hat{H} - \mu \hat{N}$ with $\hat{H}$ and $\hat{N}$ being the Hamiltonian and the quark number operator, respectively. The expectation value of the quark number is given by

$$N_q = \langle \hat{N} \rangle,$$

(3.2)

where

$$\langle \hat{O} \rangle = \text{Tr} \ e^{-\beta (\hat{K} - \Omega)} \hat{O},$$

(3.3)

denotes the statistical average of $\hat{O}$. The quark number density is given by

$$\rho_q = N_q/V = \langle \bar{\psi} \gamma^0 \psi \rangle$$

(3.4)

where $V$ denotes the volume of the system and it is assumed that the vacuum contribution to the quark number is subtracted

*) The rotational invariance of the system which we assume implies that the spatial component of an expectation value $\langle \bar{\psi} \gamma^i \psi \rangle$ vanishes.
dynamic potential density as
\[ \rho_q = -\frac{\partial (\Omega/V)}{\partial \mu}. \] (3.5)

Since quarks have baryon number $1/3$, the baryon number density and chemical potential are given by $\rho_B = 1/3 \cdot \rho_q$ and $\mu_B = 3\mu$, respectively, where the iso-spin symmetry is assumed. We shall use the quark number density $\rho_q$ and chemical potential $\mu$ for the formulation, but $\rho_B$ and $\mu_B$ will be used in the presentation of the numerical results in §4.

To apply the mean-field approximation (MFA), we first assume that the system has a quark-antiquark condensate $\langle \bar{\psi}\psi \rangle$ and a diquark condensate $\langle \bar{\psi}^C i\gamma_5\tau_2 \lambda_2 \psi \rangle$, where $\lambda_A$ is restricted to $\lambda_2$ owing to the color SU(3) symmetry. In MFA, $\hat{K}$ is replaced by
\[ \hat{K}_{\text{MFA}} = \int d^3x \left[ \bar{\psi} (-i\vec{\gamma} \cdot \vec{\nabla} + (m + M_D) - (\mu - 2G_V\rho_q)\gamma_0)\psi + \frac{1}{2}(\Delta^*\bar{\psi}^C i\gamma_5\tau_2\lambda_2 \psi + \text{h. c.}) \right. \]
\[ \left. + \frac{M_D^2}{4G_S} + \frac{|\Delta|^2}{4G_C} - G_V\rho_q^2 \right]. \] (3.6)

Here, $M_D$ and $\Delta$ give the dynamically generated quark mass and the gap due to the CS, respectively;
\[ M_D = -2G_S\langle \bar{\psi}\psi \rangle, \quad \Delta = -2G_C\langle \bar{\psi}^C i\gamma_5\tau_2 \lambda_2 \psi \rangle. \] (3.7)

We notice here that $\mu$ in $\hat{K}_{\text{MFA}}$ appears in the combination
\[ \mu - 2G_V\rho_q \equiv \tilde{\mu}. \] (3.8)

Thus, the thermodynamic potential $\Omega_{\text{MFA}}$ in MFA per unit volume is calculated to be
\[ \omega(M_D, \Delta; T, \mu) \equiv \Omega_{\text{MFA}}/V \]
\[ = \frac{M_D^2}{4G_S} + \frac{|\Delta|^2}{4G_C} - G_V\rho_q^2 \]
\[ -4 \int \frac{d^3p}{(2\pi)^3} \left\{ E_p + T \log \left( 1 + e^{-\beta\xi_+} \right) \left( 1 + e^{-\beta\xi_-} \right) \right. \]
\[ + \text{sgn}(\xi_-) \epsilon_- + \epsilon_+ + 2T \log \left( 1 + e^{-\text{sgn}(\xi_-)\beta\epsilon_-} \right) \left( 1 + e^{-\beta\epsilon_+} \right) \right\}, \] (3.9)

where $E_p = \sqrt{p^2 + M^2}$, $\xi_{\pm} = E_p \pm \tilde{\mu}$ and $\epsilon_{\pm} = \sqrt{\xi_{\pm}^2 + \Delta^2}$, with
\[ M = m + M_D, \] (3.10)

being the total (constituent) quark mass and $\text{sgn}(\xi_-)$ is the sign function. Our thermodynamic potential reduces to those given in Refs. 23), 55) when $G_V = 0$. The

*) This is a familiar procedure in the $\sigma$-$\omega$ model.58)
quark density $\rho_q$ appearing in (3.9) is expressed as a function of the condensates $(M_D, \Delta)$ through the thermodynamical relation (3.5) as

$$\rho_q = 4 \int \frac{d^3p}{(2\pi)^3} \left\{ n(\xi_-) - n(\xi_+) - \frac{\xi_-}{\epsilon_-} \tanh \frac{\beta \epsilon_-}{2} + \frac{\xi_+}{\epsilon_+} \tanh \frac{\beta \epsilon_+}{2} \right\},$$

(3.11)

where $n(\xi_\pm)$ is the Fermi distribution function: $n(\xi_\pm) = 1/\exp\{\beta \xi_\pm + 1\}$. Equation (3.9) together with Eq.(3.11) gives the thermodynamic potential $\omega$ with the condensates $(M_D, \Delta)$ being the variational parameters at given $(T, \mu)$; the optimum value of them gives the absolute minimum of $\omega$.

The chiral and diquark condensates in the equilibrium state at given $(T, \mu)$ should satisfy the stationary conditions for the thermodynamic potential:

$$\frac{\partial \omega}{\partial M_D} \bigg|_\Delta = 0, \quad \frac{\partial \omega}{\partial \Delta} \bigg|_{M_D} = 0,$$

(3.12)

which are reduced to the self-consistency conditions for the two condensates,

$$M_D = 8G_SM \int \frac{d^3p}{(2\pi)^3} \frac{1}{E_p} \left\{ 1 - n(\xi_-) - n(\xi_+) + \frac{\xi_-}{\epsilon_-} \tanh \frac{\beta \epsilon_-}{2} + \frac{\xi_+}{\epsilon_+} \tanh \frac{\beta \epsilon_+}{2} \right\},$$

(3.13)

$$\Delta = 8G_C \Delta \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{\epsilon_-} \tanh \frac{\beta \epsilon_-}{2} + \frac{1}{\epsilon_+} \tanh \frac{\beta \epsilon_+}{2} \right\}.$$

(3.14)

Here we have utilized a chain rule

$$\frac{\partial \omega}{\partial M_D} \bigg|_\Delta = \frac{\partial \omega}{\partial M_D} \bigg|_{\Delta, \rho_q} + \frac{\partial \rho_q}{\partial M_D} \bigg|_\Delta \frac{\partial \omega}{\partial \rho_q} \bigg|_{M_D, \Delta},$$

(3.15)

and that for the $\Delta$-derivative together with the fact that Eq.(3.11) ensures the relation

$$\frac{\partial \omega}{\partial \rho_q} \bigg|_{M_D, \Delta} = 0.$$  

(3.16)

In the analogy with the BCS theory of the superconductivity, we call Eq.’s (3.13) and (3.14) the gap equations. Notice, however, that a solution of the gap equations may only give a local minimum or even maximum of the thermodynamic potential, and it is only a candidate of the optimal value of the condensates; one must check whether it gives the absolute minimum of the thermodynamic potential.

From the structure of the coupled gap equations and the thermodynamic relation (3.11) for $\rho_q$, one can extract some interesting properties of the condensates $(M_D, \Delta)$ as a function of $(T, \mu)$ and also of $(T, \rho_q)$.

(1) Once the absolute minimum of the thermodynamic potential and accordingly the optimal value of $(M_D, \Delta)$ are found at given $(T, \mu)$, the quark density $\rho_q$ is given by Eq. (3.11). The coupled gap equations (3.13) and (3.14) show us that the optimal value of $(M_D, \Delta)$ is a function of $T$ and $\mu$, thereby possible $G_V$ dependence is absorbed into $\mu$. Furthermore, if $(M_D(T, \mu), \Delta(T, \mu))$ is a solution of the coupled gap equations with $G_V = 0$, then

$$(M_D(T, \mu), \Delta(T, \mu)) \equiv (M_D(T, \mu - 2G_V \rho_q), \Delta(T, \mu - 2G_V \rho_q)),$$

(3.17)
is a solution with $G_V \neq 0$. Thus, the whole solution as a function of $\mu$ is shifted toward larger $\mu$ with an amount of $2G_V\rho_q$.

(2) Next, we shall examine how the solutions of the coupled gap equations behave as a function of $(T, \rho_q)$ instead of $(T, \mu)$. Let the 0-th order approximation of the condensates be given, then Eq.(3.11) gives $\tilde{\mu}$ as a function of $(T, \rho_q)$, i.e., $\tilde{\mu} = \tilde{\mu}(T, \rho_q)$. Thus the first order approximation of the condensates $(M_D, \Delta)$ is given as the solution to the coupled gap equation (3.13) and (3.14) which are only dependent on $T$ and $\tilde{\mu}$ but not on $\mu$ and $\rho_q$, separately; possible $G_V$ dependence is absorbed into $\tilde{\mu}(T, \rho_q)$. Thus, repeating this procedure, one sees that $(M_D, \Delta)$ becomes only a function of $T$ and $\rho_q$ and independent of $G_V$ because $\tilde{\mu}$, through which $G_V$ can affect the formulas, actually only plays a role of a dummy variable. Thus we have proved that there is no effect of the vector interaction on the behavior of the solution to the coupled gap equation as functions of $(T, \rho_q)$.

(3) Does it mean that there is no trace of the presence of the vector interaction in the phase diagram in the $T-\rho$ plane? The answer is no. The effect of the vector interaction manifests itself in the critical point or line when the transition is first or-
der. In this case, there are several solutions to the coupled gap equations Eqs. (3.13) and (3.14), corresponding to the local minima, maxima and even saddle points of \( \omega \); notice that these solutions correspond to different baryon densities. Since the thermodynamic potential (3.9) is explicitly dependent on \( G_V \) in a combination with the quark density, the relative magnitudes of the local minima change and can be altered with the vector interaction: In Fig.1, the right (left) figure in the upper panel shows the contour map of the thermodynamic potential \( \omega(M_D, \Delta) \) with \( G_V/G_S = 0.2 \) \( (G_V/G_S = 0) \) at \( T = 0 \) and \( \mu_B = \mu_{B0} = 1035 \) MeV, which is actually found to be the critical point. The thermodynamic potential \( \omega(M_D, \Delta) \) as a function of \( M_D \) at given \( \Delta = 0, 25, 50 \) and 80 MeV, i.e., the cross sections along the lines shown in the upper panels, are given in the lower panels; the solid (dashed) lines denote \( \omega \) with \( G_V/G_S = 0.2 \) \( (G_V/G_S = 0) \). One clearly sees that the vector interaction increases the thermodynamic potential in the small \( M_D \) region for every \( \Delta \); notice that the system with smaller \( M_D \) is in a higher density, as discussed in §1. Thus the absolute minimum given with \( \Delta \sim 50 \) MeV at a small \( M_D \) when \( G_V = 0 \) ceases to be even a local minimum with finite \( G_V/G_S \), and the local minimum at \( M_D \sim 300 \) MeV with \( \Delta \sim 0 \) in turn becomes the single local hence absolute minimum. Thereby the double-minimum structure disappears and the first order transition is altered to a crossover. In short, the critical temperatures and densities at which the transition from one local minimum to the other occurs are largely affected by the vector interaction, and the critical line of the first order transition in the \( T-\rho_q \) plane is changed with the vector interaction.

In passing, we remark that the \( \rho_q \) cannot be interpreted as a variational parameter with which the thermodynamic potential is minimized: Since Eq. (3.11) is obtained by a stationary condition Eq.(3.16), one might have imagined that the equilibrium state could be determined by searching the minimum point of the thermodynamic potential with \( \rho_q \) being a variational parameter together with \( M_D \) and \( \Delta \).

\(^{47}\) However, Eq. (3.16) is found to give a local maximum of the thermodynamic potential. That is, the absolute minimum of the thermodynamic potential in the \( M-\Delta-\rho_q \) space, if exists, does not give the thermodynamical equilibrium state.

\section*{§4. Numerical results and discussions}

In this section, we shall show the numerical results and discuss the effects of the vector coupling on the phase diagram, the \( T-\mu_B \) and \( T-\rho_B \) dependence of the order parameters.

\subsection*{4.1. Phase Diagram with no vector interaction}

As a preliminary for the discussion on the effects of the vector interaction, we first present the phase structure without the vector interaction. We shall show that a coexisting phase appears where the quarks with dynamically generated mass are color-superconducting. This is a manifestation of competition between the \( \chi_{SB} \) and CSC phase transition.

In Fig. 2a, the phase diagram in the \( T-\mu \) plane is shown. One can see that there are four different phases, i.e., the \( \chi_{SB} \) phase, the normal quark one, which we call...
the Wigner phase, the CSC phase, and a “coexisting” phase of χSB and CS; as seen from the upper small panel which is an enlargement of the part around the solid line near zero temperature, the last phase in which quarks with dynamically generated mass is color-superconducting, occupies only a small region in the $T$-$\mu_B$ plane near zero temperature with $\mu$ slightly smaller than $\mu_B = 1035$ MeV.

The critical line of a first and second order transitions are represented by a solid and dashed line, respectively; notice that there exists a dashed line in the upper small panel. We remark that there are three kinds of first order transitions: the χSB-Wigner, χSB-CSC and coexisting-CSC transitions.

An artificial critical line of the crossover chiral transition is also shown by the dash-dotted line on which the dynamical quark mass takes the same value as that at the end point $M_D = 186$ MeV so that the crossover critical line is connected continuously with the critical line of the first order transition at the end point $^\ast$). With this definition of the critical line for the crossover chiral transition, the critical temperature at vanishing chemical potential ($\mu = 0$) is found to be $187$ MeV, which is slightly larger than the critical temperature obtained in the simulations on the lattice QCD with two flavors.\(^2\)

In accordance with the widely accepted view,\(^{30,31}\) one sees that the chiral transition is a first order at low temperatures: The critical line of the first order transition stemming up from a point in the zero-temperature line terminates at

$$(T_c, \mu_{Be}) = (47, 990) \text{ MeV}.$$  

The figure also shows that the phase transition from the CSC to Wigner phase is a second order when $T$ is raised in our model where the gluon fields are not explicitly included.\(^{15}\) We have found that the χSB phase is transformed to the coexisting phase at very low temperatures by a second order transition when $\mu$ is raised, as

\(^{*}\) We have followed the criterion used in Ref. 36).
Fig. 3. The order parameters $M_D$ and $\Delta$ at $T = 0$ as a function of $\mu_B$ with $G_V = 0$. There is discontinuities of the order parameters at $\mu_B = 1035$ MeV. An enlarged figure near the critical point is also shown. The gap $\Delta$ is finite even in the region $\mu_B < \mu_B^0$.

shown in the small panel.

To see the more detail of the coexisting phase, we show $\mu$ dependence of $M_D$ and $\Delta$ at $T = 0$ in Fig. 3. One can see that $M_D$ ($\Delta$) shows a discontinuous decrease (increase) at $\mu_B = \mu_B^0$, which clearly indicates a first order chiral (CSC) transition at this point. A notable point is that $M_D$ has a finite value even in the CSC phase because of the finite current quark mass; notice that $M_D$ is proportional to the chiral condensate and not the total (constituent) quark mass $M = m + M_D$. Although we shall not show the result here, we have checked that $M_D$ vanishes in the CSC phase in the chiral limit; nevertheless see Fig.7 for $G_V/G_S = 0.2$. On the other hand, there is a region in which $\Delta$ becomes finite in the $\chi$SB phase, which remains the case in the chiral limit. We have called this phase the coexisting phase.

We notice that a similar coexisting phase with ours was obtained in the instanton-anti-instanton molecule model,\textsuperscript{18) where the phase structure at $T \neq 0$ was not examined and the robustness of the existence of the coexistence phase is questioned because other calculations using a similar NJL-type chiral model\textsuperscript{23) in which the effective scalar coupling constant $G_S$ in our notation is relatively large in comparison with the effective diquark interaction $G_C$, did not exhibit such a coexisting phase \textsuperscript{*}). We have checked that if slightly larger $G_S$ is used, the coexisting phase does not exist even in our case. We shall show, however, that the vector interaction induces a competition between $\chi$SB and CSC phase transition, thereby the existence of the coexistence phase always becomes possible with a vector coupling. It is worth mentioning in this respect that the coexisting phase is obtained in the 2-color QCD on the lattice in a robust way.\textsuperscript{8)}

The phase diagram in the $T$-$\rho_B$ plane is shown in Fig. 2b. This phase structure is schematically presented in Fig. 4. Corresponding to the three types of first order transition mentioned above, there exist three mixed phases, which we call I, II and III, respectively: The I is a mixed phase of the $\chi$SB and Wigner phases, while the II and III are mixed phases of the $\chi$SB and CSC phases, and the coexisting and CSC phases, respectively. We remark that more various mixed phases are possible with incorporating the CSC phase than without it.

\textsuperscript{*} In Ref. 55), the full coupled gap equations for $M_D$ and $\Delta$ were not solved, which is necessary to find the coexisting phase.
Fig. 4. A schematic figure accounting for Fig. 2b. There are three mixed phases in the $T$-$\rho_B$ plane: The I is a mixed phase of the $\chi_{SB}$ and Wigner phases, while the II and III are the mixed phases of the $\chi_{SB}$ and CSC phases, and the coexisting and CSC phases, respectively.

So much for the phase structure without the vector interaction. When the vector interaction is included, the phase structure may be changed significantly, which we shall show is the case in the next subsection.

4.2. Phase structure with vector interaction

In this subsection, we discuss effects of the vector interaction on the phase structure of hot and dense quark matter by varying the vector coupling $G_V$ by hand in the range of $G_V/G_S = 0 \sim 0.5$. One will see that the vector interaction causes a nontrivial interplay between the $\chi_{SB}$ and CS to make the optimal condensates largely fluctuated in a combined way.

Fig. 5. The phase diagrams with $G_V/G_S = 0.2$ in the $T$-$\mu_B$ plane (a) and $T$-$\rho_B$ plane (b). The solid line represents the critical line of a first order phase transition, the dashed line a second order and the dot-dashed line a crossover, respectively. The phase structure in the $T$-$\mu_B$ plane with $G_V/G_S = 0.2$ is shown in Fig. 5a. The phase diagram consists of the $\chi_{SB}$, Wigner, CSC and coexisting phases, as in 2a. The dash-dotted line is a contour line at $M_D = 198$ MeV and supposed to denote the critical line of the crossover transition; the solid and dashed lines represent the critical lines of the first order and second order transitions, respectively, as in Fig. 2a. The corresponding phase diagram in the $T$-$\rho_B$ plane has the three mixed phases, I, II and III which have been seen in Fig. 4, as well as the $\chi_{SB}$, Wigner and CSC phases.
From these figures, the following points are notable:

1. The end point of the first order transition is moved toward a lower temperature and higher chemical potential,

\[(T_e, \mu_{Be}) = (27, 1056)\text{MeV}.\]

2. The chiral restoration is postponed toward larger \(\mu\). This is because the gap equations (3.13) and (3.14) are a function of \(T\) and \(\tilde{\mu}\) and thus explicit \(G_V\) dependence is absorbed into \(\tilde{\mu}\), as shown in Eq.(3.17). This means that \(\mu\) giving a fixed \(M_D\) is shifted toward larger value as \(G_V\) is increased.

3. The region of the coexisting phase becomes broader toward both \(T\) and \(\mu\) directions in the \(T\)-\(\mu\) plane, and hence also \(T\)-\(\rho\) plane. This feature is determined dominantly by the behavior of \(M_D\) in \(\chi\text{SB}\) phase. As an example, \(M_D\) together with \(\Delta\) as a function of \(\mu_B\) at \(T = 0\) is shown in Fig. 6; the same quantities in the chiral limit are shown in Fig. 7. One sees that there appears a small region of \(\mu_B\) smaller than but near \(\mu_B0\) in which \(M_D (\Delta)\) shows a gradual decrease (increase); accordingly a finite \(M_D\) and \(\Delta\) coexist in this region. One should here notice that although the coexistence in this sense is realized even when \(\mu_B > \mu_B0\) as seen in Fig. 6, \(M_D\) in the chiral limit vanishes identically in this region while the coexistence of \(M_D\) and \(\Delta\) remains at \(\mu_B < \mu_B0\) as seen in Fig. 7. In fact, this is also the case when \(G_V = 0\), as was noticed in §4.1. Thus, calling the phase coexisting makes sense. Anyway, the gradual change of the order parameters means that the first order transition is weakened. The decrease of \(M_D\) also implies that of the total quark mass \(M\), leading to a growth of the Fermi surface for a given \(\mu_q\). The larger the Fermi surface, the larger the gap \(\Delta\) owing to the BCS mechanism. Thus the region of the coexisting phase in the \(T\)-\(\mu_B\) plane becomes broader. This feature can be applied to the case at \(T \neq 0\). In short, the vector interaction promotes the formation of the coexisting phase. As mentioned before, the coexisting phase is obtained in the 2-color QCD on the lattice in a robust way.\(^8\) It would be interesting to explore a possible correlation between the appearance of the coexistence phase and the strength of the effective vector coupling extracted, say, from the baryon-number susceptibility\(^{44,59,60}\) as was done in Ref. 61).

![Fig. 6. The order parameters \(M_D\) and \(\Delta\) as a function of \(\mu\) at \(T = 0\) with \(G_V/G_S = 0.2\).](image)

The characteristics (1) and (2) of the effects of the vector interaction were known for the chiral transition without the CSC transition incorporated.\(^{36,42,46}\) An account of the phase structure without the CSC transition is presented in Appendix A as a reference.
However, when the interplay between the $\chi_{SB}$ and CS enhanced with the vector interaction is taken into account, the variation of the phase diagram becomes not so simple for larger $G_V$. In Fig. 8a, we show the phase diagram in the $T$-$\mu$ plane with $G_V/G_S = 0.35$. It is noteworthy that there appears two end points at both sides of the critical line of the first order transition. Accordingly, the coexisting-CSC transition at low temperatures becomes a crossover transition. We have checked that the crossover transition becomes a second order in the chiral limit, hence a tricritical point appears instead of the end point of the first order transition. As far as we know, this is the first time when it is shown that the critical line of the first order transition for the chiral restoration can have another end point in the low temperature side, implying that the transition from the $\chi_{SB}$ to CSC phase at low temperatures become a crossover (second order in the chiral limit). Nevertheless it is noteworthy that the two-color QCD on the lattice at nonzero temperature and chemical potential gives a similar phase diagram; see Fig. 1 of Ref. 8). Again, the lattice result might be interpreted in terms of the effective vector coupling, which deserves to explore to understand the underlying physics.

The corresponding phase diagram in the $T$-$\rho$ plane is shown in Fig. 8b. Its schematic phase structure is represented in Fig. 9. The II and III phases correspond to the mixed phases of the $\chi_{SB}$ and CSC phases, and the coexisting and CSC phases, respectively as in Fig. 4. Notably, the I phase does not exist anymore.

To examine the mechanism of the appearance of the two end points in detail, we show the thermodynamic potentials in the $M_D-\Delta$ plane for various $T$ and $\mu$ in Fig. 10. In the lowest panels, the thermodynamic potential at $T = 5$ MeV is shown: We see only one local hence the absolute minimum point which varies continuously as $\mu$ is increased. This implies that the phase transition is a crossover at $T = 5$ MeV. At higher temperatures, however, the thermodynamic potential gets to have two local minima near the critical point, as shown in the second and third row panels for $T = 12$ MeV and $T = 15$ MeV, respectively, and the phase transition becomes a first order. At further higher temperatures, the double-minimum structure cease to exist and the thermodynamic potential has only one local minimum again, as
shown in the uppermost panel for $T = 22$ MeV, and the phase transition return to a crossover.

In our model calculation, the two-end-point structure of the phase diagram appears for finite $G_V$ but in a narrow range of $G_V/G_S$, i.e., $0.33 \lesssim G_V/G_S \lesssim 0.38$. We should also notice that even when the phase transition is a first order, the height of the bump between the two local minima of the thermodynamic potential per particle is so small that it is found to be comparable with or smaller than the temperature. It means that thermal fluctuations which is ignored in the mean-field approximation taken in this work may easily destroy the two-end-point structure. What we have found is that the inclusion of the vector interaction makes the minimum of the thermodynamic potential shallow in the $M_D-\Delta$ plane, suggesting the significance of fluctuations of the chiral and diquark condensates in a combined manner. The incorporation of the thermal fluctuations is beyond the scope of this work.

When we take a larger $G_V$ than $0.38G_S$, the first order transition disappears and is changed to a crossover transition completely. Figure 11a shows the phase diagram in the $T-\mu$ plane with $G_V/G_S = 0.5$ as a typical example in this case. The dashed line denotes the second order transition. The dash-dotted line represents an artificial crossover line on which $M_D = 200$ MeV. The corresponding phase diagram in the $T-\rho$ plane is shown in Fig. 11b. One can see that there is no first order transition, hence no mixed phase. As is pointed out in §3, the vector interaction affects the phase
Fig. 10. The contour of the thermodynamic potential in the $M_D-\Delta$ plane for various $(T, \mu_B)$ around the critical point of the first order transition. The difference of the values of $\omega$ between the adjacent solid lines is $1.5 \times 10^9$ MeV$^4$. As shown in the second and third row panels, there appear two local minima at $T = 12$ MeV and $T = 15$ MeV with $\mu_B$ near the critical value, which indicates that the phase transition is first order at these temperatures. On the other hand, as shown in the bottom and top panels, there always exists only one local hence the absolute minimum at $T = 5$ and 22 MeV, which minimum moves continuously as $\mu_B$ is increased, implying that the phase transition is a crossover.
As a nice summary of the effects of the vector interaction on the phase structure of hot and/or dense quark matter, we show three-dimensional plots of the dynamical quark mass $M_D$ and the gap $\Delta$ in the $T-\mu$ and $T-\rho$ plane in Fig. 12. The thick line represents the critical line of the first order transition. The dotted points indicate the end points. One sees that $M_D$ decreases more smoothly for larger $G_V$ in the $T-\mu$ plane. It is clear that the $G_V$ dependence of $M_D$ and $\Delta$ in the $T-\rho$ plane appears only in the critical region of the first order transition.

§5. Summary and concluding remarks

We have investigated effects of the vector coupling on the chiral and color superconducting phase transitions at finite density and temperature in a simple Nambu-Jona-Lasinio model by focusing on the implication of the density jump accompanied with the chiral transition. We have shown that the phase structure is largely affected by the vector interaction especially near the critical line between the chiral symmetry breaking ($\chi_{SB}$) and color superconducting (CSC) phase: The first order transition between the $\chi_{SB}$ and CSC phase becomes weaker as the vector coupling is increased, and there can exist two end points of the critical line of the first order restoration in some range of the parameters; the two end points become tricritical points in the chiral limit. Our calculation has shown that the repulsive vector interaction enhances the competition between the $\chi_{SB}$ and CS leading to a degeneracy in the thermodynamic potential in the $M_D-\Delta$ plane. This implies that there exists gigantic fluctuations of the order parameters in a correlated way near the critical region, and suggests a necessity of a theoretical treatment incorporating the fluctuations, which is, however, beyond the scope of this work. We have found that the coexisting phase, where the quarks with a dynamically generated mass are color-superconducting, appears in a wide range of $\mu_B$ and $T$. We have emphasized that the appearance of such a coexistence phase becomes robust hence universal by the inclusion of the vector interaction. We have also shown that the repulsive vector interaction postpones the
transition from the chirally broken to color superconducting phase toward larger $\mu_B$.

Although our analysis is based on a simple model, our finding that the vector interaction enhances the competition between the $\chi_{SB}$ and CSC phase transition is universal and should be confirmed and further studied with more realistic models and on the lattice QCD. In fact, similar phase structure with ours are obtained in the two-color QCD on the lattice,\(^7\),\(^8\) in which there appear two tricritical points related to the chiral and CSC transitions, and also the coexisting phase in a wide range of the temperature and chemical potential. These results might be intuitively understood in terms of the effective vector coupling which can be extracted by calculating the baryon-number susceptibility.

We have confined ourselves in the two-flavor case in this work. Needless to say, it is very interesting to examine the effects of the vector interaction in the three-flavor case, thereby on the color-flavor locked phase.\(^{25\text{--}28}\)

T.Kunihiro thanks David Blaschke for telling him some references related to this work. This work is partially supported by the Grants-in-Aid of the Japanese Ministry of Education, Science and Culture (No. 12640263 and 14540263).

Appendix A

--- Effects of vector interaction on chiral phase transition ---

In this appendix, we summarize how the chiral transition is affected by the vector interaction in the case where the CS is not incorporated. Although this problem was examined by some authors\(^{46\text{,}23}\) its coherent summary has not been given in the literature.

The phase diagram of the chiral transition in the $T$-$\mu$ plane is shown in Fig. 13: Here we have used the same Lagrangian (2.3) as the one used in the text but with $G_C$ switched off. One can see the following features from Fig. 13:

(i) The chiral restoration is delayed toward larger $\mu$ as $G_V$ is increased.
(ii) $G_V$ acts to move the end point toward lower $T$ and larger $\mu$.
(iii) The chiral restoration eventually turns to a crossover transition for large $G_V$.

The feature (i) can be understood as follows. The Fermi momentum $p_F = \sqrt{\mu^2 - M^2}$ becomes large (small) for small (large) $M$, where $M = m + M_D$ is a constituent (total) quark mass, and so does the density $\rho$ at the fixed $\mu$ (see Fig. 14). Since the vector interaction gives rise to a repulsive energy proportional to the density squared, $G_V \rho^2$, a system with a smaller density is favored when $G_V$ is present. Thus one can see when $G_V$ is finite, the larger $M$ is favored. We show the thermodynamic potential $\omega$ as a function of $M_D$ with various $G_V$ in Fig. 15. We see that the thermodynamic potential at small chiral condensate $M_D$ is raised as $G_V$ is increased owing to the repulsion of the vector interaction. Accordingly, the chiral restoration is postponed toward large $\mu$ as $G_V$ is increased.

Figure 15 also shows that the first order transition is weakened as $G_V$ is increased: One sees from the far left panel that the thermodynamic potential with $G_V = 0$ has two local minima and there exists a bump between these minima. This two-local minima structure becomes obscure and the local minima get to closer as
$G_V$ is increased; see the $G_V/G_S = 0.2$ case (short-dashed line) in the middle panel. Such two-local minima structure disappears with $G_V = 0.3$ for all $\mu_B$.

References

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Fig. 12. The left panels show three-dimensional plots of the order parameters $M_D$ (solid lines) and $\Delta$ (dashed lines) as a function of $(T, \mu_B)$ with various $G_V/G_S$, while the right panels of $(T, \rho_B)$. The thick line corresponds to the first order transition and the circles denote the end points of them. Notice that the behaviors of the order parameters as a function of $(T, \rho_B)$ do not depend on $G_V$ except for the region of the mixed phases in accordance with the discussion given in §3.
Fig. 13. The \(G_V\) dependence of the phase diagram for the chiral transition in the \(T-\mu\) plane. The solid line represents the critical line of the first order transition. The dash-dotted line denotes an artificial critical line of the crossover transition which is determined with the same condition as that in the text.

Fig. 14. A conceptual diagram accounting for the relation between the total quark mass \(M = m + M_D\) and the density \(\rho\) at given \(\mu_q\). The quark density in the equilibrium state becomes small for larger \(M\) with \(\mu_q\) fixed.

Fig. 15. The thermodynamic potential at \(T = 0\) as a function of \(M_D\) for \(G_V = 0, 0.1, 0.2, 0.3\). For smaller \(G_V\), \(\omega\) has two local minima reflecting the first order transition. As \(G_V\) is increased, \(\omega_D\) at small \(M_D\) becomes large and the local minimum at smaller \(M_D\) disappears, thereby the chiral transition turns to a crossover.