THE QUANTUM SYMMETRY
OF RATIONAL FIELD THEORIES

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Abstract. The quantum symmetry of a rational quantum field theory is a finite-dimensional multi-matrix algebra. Its representation category, which determines the fusion rules and braid group representations of superselection sectors, is a braided monoidal $C^*$-category. Various properties of such algebraic structures are described, and some ideas concerning the classification programme are outlined.


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1 Symmetries and observables

The notion of symmetry is one of the most fundamental concepts in science. An *internal*

symmetry leaves observable quantities invariant. It only changes the particular manner in which

a physical system is described and correspondingly represents a certain amount of redundancy.

Therefore it is often a good idea to eliminate internal symmetries so as to arrive at a description

in terms of observables only. But there are also various situations where, on the contrary, the

redundancy is highly welcome. For instance, often the use of redundant variables leads to

simpler coordinates and thereby facilitates a perturbative treatment. Another example arises

in the context of what I will call *quantum symmetry*. To give a first impression of what is

meant by this term, let me specialize to the specific case of conformal field theory: By a quantum

symmetry of a two-dimensional conformal field theory I mean an internal symmetry

whose representation theory reproduces the basis independent contents of the operator product

algebra, i.e. the fusion rules, and which is compatible with the duality properties of chiral blocks.

A model independent characterization of the notion of quantum symmetry will be given at the

beginning of section 3.

While conformal invariance is operational in the latter characterization, it is far from being

an essential ingredient. Rather, the type of quantum symmetries encountered in conformal field

theory turns out to be relevant to relativistic quantum field theory in general. To make this

remark more concrete, let me consider a specific axiomatic formulation of quantum field theory, namely the algebraic theory of superselection sectors [1–4], to which for short I will refer in the

sequel by the term *algebraic field theory*. In the algebraic field theory framework a theory is

described in terms of nets of von Neumann algebras $\mathcal{A}(\mathcal{O})$ which are indexed by specific open

subsets $\mathcal{O}$ of Minkowski space-time. Consequently, the proof of various results from algebraic

field theory on which I will rely below requires the mathematics of von Neumann algebras and

subfactors [5–8]; however, the essence of these results is in all relevant cases rather plausible

already from a more heuristic point of view. My perspective will therefore be to take these

results for granted and investigate their implications for quantum symmetry.

Classical physics can be described via a configuration space $X$ endowed with a measure $\mu$;

the dynamics is then described in terms of the elements of the algebra $\mathcal{L}^{\infty}(X, \mu)$ of bounded

measurable functions on $X$. In quantum physics, one trades the configuration space, respecti-

tively the commutative algebras of functions on it, for non-commutative $\ast$-algebras of operators.

The analogue of the measures $\mu$ are states (normalized positive linear forms) over the algebra.

A state induces a scalar product on the algebra, and the completion of the algebra with respect

to the associated norm is a separable Hilbert space $\mathcal{H}$ on which the algebra acts by multipli-
cation. Thus physical states correspond to the vectors $\psi$ of a Hilbert space $\mathcal{H}$; the analogue

of $\mathcal{L}^{\infty}(X, \mu)$ is then the field algebra $\mathcal{F}$ which is a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded

operators on $\mathcal{H}$. The vectors of $\mathcal{H}$ can be thought of as being created by acting with field

operators $f \in \mathcal{F}$ on a vacuum vector $\Omega$. In this context, the observables are the (self-adjoint)
elements of a $\ast$-subalgebra $\mathcal{A}$ of $\mathcal{F}$. Any measuring apparatus is contained in a bounded region

$\mathcal{O}$ of space-time, and hence there must exist local observable algebras $\mathcal{A}(\mathcal{O})$ of observables

measurable in $\mathcal{O}$. The local observable algebras associated to causally disconnected regions

commute among one another (Einstein causality). The total observable algebra $\mathcal{A} \subseteq \mathcal{F}$ is the

quasilocal $\ast$-algebra $\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$.

A distinctive feature of quantum symmetries in low-dimensional field theory is that they

correspond to *finite-dimensional* algebras. Recall that in classical physics the (finite) symmetry

transformations correspond to the elements of a group $G$, or rather to the associated group

algebra $\mathbb{C}[G]$. If $G$ is infinite, say a Lie group, then $\mathbb{C}[G]$ is infinite-dimensional. As it turns

out, this situation prevails in quantum theory as long as the dimensionality of space-time is

large enough. In contrast, in low-dimensional quantum field theory there are special systems,
the so-called rational field theories, for which the quantum symmetry is a finite-dimensional algebra which is generically not a group algebra. Heuristically, the possibility of having more general structures than group algebras can be understood by investigating the question of what happens to the group of symmetries in the course of quantization of a classical system. Namely, elements of the group \( G \) can be viewed as acting on the points of the configuration space. Since in quantum physics the configuration space is no longer present, the group elements are not needed any more either.

2 Superselection sectors

In algebraic field theory the observables are taken as the basic objects. Because of Einstein causality, observables must commute at space-like separations, or in other words, the statistics of observables is bosonic. One of the challenges of algebraic quantum field theory is the investigation of the possible statistics (i.e., the behavior with respect to permutations) of non-observable fields. The action of the field algebra \( \mathcal{F} \) on the vacuum vector \( \Omega \) is generically reducible; correspondingly, there are sub-Hilbert spaces of the Hilbert space \( \mathcal{H} \) that are orthogonal to each other. These subspaces are called superselection sectors; denoting them by \( \mathcal{H}_a \), one has the sector decomposition

\[
\mathcal{H} = \bigoplus_a \mathcal{H}_a .
\]

Observables act within the sectors. (This implies that the relative phases of vectors belonging to different subspaces \( \mathcal{H}_a \) cannot be observed. Hence, in contrast to the situation in quantum mechanics, the superposition principle is not valid universally, but holds only within sectors.) Thus each sector \( \mathcal{H}_a \) carries a representation \( \pi_a \) of \( \mathcal{A} \). The general representation theory of \( \mathcal{A} \) is rather wild, but fortunately in quantum field theory only a restricted class of representations is relevant. The precise requirements that these physical representations must meet depend on the specific axiomatic framework; in the present context, the relevant notion [1-4] is the one of DHR (Doplicher-Haag-Roberts) representations. Recall that a vacuum representation \( \pi_0 \) of \( \mathcal{A} \) is a positive-energy representation for which the associated Hilbert space \( \mathcal{H}_0 \) contains a unique (up to normalization) vector \( \Omega \), called the vacuum vector, which is cyclic and separating for \( \mathcal{H} \) and is invariant under the relevant space-time symmetry transformations, in particular under translations. Then by definition a DHR representation is a representation which is isomorphic to \( \pi_0 \) outside some bounded region, or in other words, is a ‘local excitation of a vacuum representation’. It will also be assumed below that the index of inclusion of the algebra \( \pi_0(\mathcal{A}(\mathcal{O})) \) in the commutant of \( \pi_0(\mathcal{A}(\mathcal{O}')) \) (\( \mathcal{O}' \) denotes the space-like complement of \( \mathcal{O} \)) with respect to \( \mathcal{B}(\mathcal{H}) \) is finite. DHR representations with this property are said to have finite statistics.

In more technical terms, DHR representations \( \pi_a \) are characterized by being unitarily equivalent to \( \pi_0 \) in the sense that

\[
\pi_a \cong \pi_0 \circ \rho_a ,
\]

where \( \rho_a : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(\mathcal{O}) \) are endomorphisms of the local algebras \( \mathcal{A}(\mathcal{O}) \) which act as the identity map on the von Neumann algebra generated by the local observables in \( \mathcal{O}' \). (This implies that the algebras generated by the local observables in different sectors are isomorphic, and the sectors can only be distinguished by global quantities, which are referred to as ‘superselection charges’.)

Several properties of DHR representations with finite statistics are relevant to the investigation of quantum symmetry. They are given in the following list.

- The representations are irreducible, and they appear in the sector decomposition (1) with
finite multiplicity. Accordingly \((1)\) can be rewritten as
\[
\mathcal{H} = \bigoplus \rho \mathcal{H}_\rho \times (\mathbb{C}^n_{\rho}),
\]
where the representations \(\pi_\rho\) corresponding to the Hilbert spaces \(\mathcal{H}_\rho\) are pairwise inequivalent, and where \(n_\rho < \infty\) are non-negative integers. The vacuum sector is non-degenerate, \(n_0 = 1\).

\(\triangleright\) It is possible to define a tensor product of (unitary equivalence classes of) the representations \(\pi_\rho\) of \(\mathcal{A}\); namely,
\[
\pi_\rho \times \pi_\eta \cong \pi_0 \circ \rho_\rho \circ \rho_\eta.
\]

\(\triangleright\) The product of sectors is completely reducible. Thus
\[
\pi_\rho \times \pi_\eta \cong \bigoplus_r (\mathbb{C}^N_{\rho \eta} \times \pi_r)
\]
for some non-negative integers \(N_{\rho \eta}^r\). I will refer to these integers as fusion rule coefficients. One has \(\pi_0 \times \pi_\eta \cong \pi_\eta\), i.e. \(N_{\rho \eta}^0 = \delta_{\rho \eta}\).

\(\triangleright\) To any representation \(\pi_\rho\) there is associated a unique representation \(\pi_{\rho^+}\), called *conjugate* to \(\pi_\rho\), such that the product of \(\pi_\rho\) with \(\pi_{\rho^+}\) contains \(\pi_0\), and then \(\pi_0\) appears precisely once in this product. In terms of fusion rule coefficients, \(N_{\rho\rho^+} = \delta_{\rho \rho^+}\).

\(\triangleright\) The composition of the endomorphisms \(\rho_\rho\) of \(\mathcal{A}\) is associative. Further, while the composition of endomorphisms is in general not commutative, it is still commutative up to unitary equivalence. As a consequence, the tensor product of DHR representations is associative up to isomorphism and commutative up to isomorphism. For the fusion coefficients, this implies \(N_{\rho^+}^s N_{\rho^+}^r = N_{\rho^+}^r N_{\rho^+}^s\) and \(N_{\rho}^s N_{\rho}^r = N_{\rho}^r N_{\rho}^s\).

\(\triangleright\) The matrices \(N_{\rho}\) with entries \((N_{\rho})_{pq}^r = N_{\rho_q}^r N_{\rho_q}^s\) are simultaneously diagonalized by a symmetric matrix \(S\) \([9]\). In general, this matrix may be degenerate, but in that case it is possible to enlarge \(\mathcal{A}\) such that the diagonalization matrix arising for the enlarged observable algebra is non-degenerate, and hence can be chosen to be unitary.

Further, the statistical properties of superselection sectors may be summarized as follows. For any natural number \(m\), each sector \(\pi_\rho\) carries an irreducible matrix representation of the braid group \(B_m\) on \(m\) strands. These representations are to a large extent characterized by two numbers, namely a by a phase \(\eta_\rho\) and by a positive real number \(d_\rho\); in the limiting case where the representations are in fact representations of the symmetric group \(S_m \subset B_m\) (permutation group statistics), these numbers correspond to the sign \pm 1 distinguishing bosons from fermions and to the dimensionality of the \(S_m\)-representation, respectively. With respect to statistics, the distinctive feature of algebraic field theory is that statistics is an *intrinsic* property of superselection sectors, in the sense that one can directly describe the numbers \(d_\rho\) and \(\eta_\rho\) in terms of the sectors. Recall that in quantum theory the statistics of fields is encoded in their commutation relations for space-like separation. In the algebraic field theory framework, this information is described by the so-called *statistics parameters* of the sectors. The inverse modulus of the statistics parameter, called the *statistical dimension*, reproduces the number \(d_\rho\), and the phase of the statistics parameter, called the *statistics phase*, coincides with the phase \(\eta_\rho\). The physical interpretation of the statistical dimension is that it measures the deviation from Haag duality, i.e. from the property \(\pi_\rho(\mathcal{A}(O')^c) = \pi_\rho(\mathcal{A}(O))\), where the prime on the algebra denotes the comutant with respect to \(\mathcal{B}(\mathcal{H})\); more precisely,
\[
d_\rho^2 = \text{Ind}[\pi_\rho(\mathcal{A}(O')^c) : \pi_\rho(\mathcal{A}(O))],
\]
4
is the index of the inclusion of $\pi_p(\mathcal{A}(\mathcal{O}))$ in $\pi_p(\mathcal{A}(\mathcal{O}'))$. Without essential loss of generality, the vacuum sector can be assumed to be Haag dual, and hence $d_0 = 1$. I will do so, and correspondingly identify $\mathcal{A}$ with its vacuum representation $\pi_0(\mathcal{A})$.

One can actually compute the statistics parameter of a sector $\pi_p$ rather directly from the associated endomorphism $\rho_p$. Namely, one has $\Phi_p(\varepsilon_{\rho_p}) = (-\ln \rho_p \partial_\rho) \mathbf{1}$. Here $\varepsilon_{\rho_p}$ is the ‘statistics operator’ which describes the noncommutativity of the composition of the endomorphisms $\rho_p$ and $\rho_q$, and $\Phi_p$ is a ‘left-inverse’ of $\rho_p$, i.e. a positive linear map with the property that $\Phi_p \circ \rho_p$ is the identity map on $\mathcal{O}$. Further, the index (6) of von Neumann algebras coincides with the index

$$\text{Ind}[\mathcal{A}(\mathcal{O}) : \rho_p(\mathcal{A}(\mathcal{O}))]$$

of C*-algebras.

In the special case that one is actually dealing with a (two-dimensional) conformal quantum field theory, it is straightforward to make contact to the framework of the bootstrap formulation of these theories. The chiral half of a (unitary) conformal field theory corresponds to a quantum field theory on a circle $S^1$ which is a compactified light ray of two-dimensional Minkowski space. At a heuristic level, one then has the following correspondences (compare [10]; for a more detailed analysis, see [11-15]):

- The elements of the local observable algebras $\mathcal{A}(\mathcal{O})$ correspond to bounded functions of operators of the form $\int_a^b dz \: f(z) W(z)$, where $f$ is a test function with support in $\mathcal{O}$ and $W$ an element of the enveloping algebra of the chiral symmetry algebra $\mathcal{W}$.

- The superselection sectors correspond to the physical representations of $\mathcal{W}$, or in other words, to the chiral families (headed by primary fields) of the theory.

- The vacuum sector corresponds to the algebra $\mathcal{W}$ itself, i.e. to the identity primary field.

- The statistical dimension of a sector coincides with the quantum dimension of the associated primary field, and the statistical phase with $\exp(2\pi i \Delta)$, with $\Delta$ the conformal dimension of the primary.

- The rationality property means that the number of primary fields is finite.

- The composition of sectors corresponds to the fusion rules of the conformal field theory which describe the basis independent contents of the operator product algebra (recall that the fusion rules must not be mixed up with the ordinary tensor products of representations of the chiral algebra $\mathcal{W}$).

- Choosing the observable algebra such that the matrix which diagonalizes the fusion rules is symmetric and unitary is equivalent to choosing the maximally extended chiral algebra, i.e. incorporating all purely holomorphic primary fields with integer conformal dimension into $\mathcal{W}$.

3 Quantum symmetry

The above considerations motivate the following terminology.

A quantum symmetry is a symmetry structure $\mathcal{H}$ which allows for an intrinsic determination of the fusion rules and of the statistics of superselection sectors.
In view of the results presented in the previous section, one of the characteristic properties of a quantum symmetry is thus that the representation theory of the observable algebra $A$ should coincide with the representation theory of the quantum symmetry $H$. In more mathematical terms, the physical representations of $A$ can be understood in terms of the category of representations of $H$. (This is sometimes rephrased by saying that the representation category of $H$ should be isomorphic to the category of physical representations of $A$. One should keep in mind, however, that according to standard terminology the objects of a representation category are \textit{finite-dimensional} matrix representations; thus in the context of $A$ which has no faithful finite-dimensional representations, this use of category theoretic terms is a bit non-standard.) As already mentioned, the qualification 'physical' for the representations of $A$ is essential. Its precise meaning depends on the concrete physical situation one is interested in; in the present context, a physical representation is a DHR representation with finite statistics.

By construction, the 'gauge charge' structure describing the representation labels with respect to the quantum symmetry $H$ is thus in one-to-one correspondence with the superselection structure of the theory, and it is natural to identify the two concepts of charge [16]. In terms of the Hilbert space $H$, this implies that the sector decomposition (1) gets generalized to (3), or more precisely, to

$$
H = \bigoplus_p (H_p \times V_p),
$$

where $V_p \cong \mathbb{C}^{n^p}$ are irreducible $H$-modules, while in terms of operator algebras, one has

$$
B(H) \subset A \rtimes H.
$$

From the properties of the tensor product of superselection sectors it follows in particular that the representation category of $H$ must have the properties of a \textit{braided monoidal C*-category}. These properties include in particular the commutativity of certain diagrams describing the composition of isomorphisms among tensor products which are calculated in different orders. Among these, the most important ones are the pentagon and hexagon relations, whose description in terms of the quantum symmetry $H$ will be given in (17) - (19) below; in terms of commutative diagrams, they look as follows:

\begin{align*}
\text{Pentagon:} & \quad \pi_1 \times ((\pi_2 \times \pi_3) \times \pi_4) \rightarrow (\pi_1 \times (\pi_2 \times \pi_3)) \times \pi_4 \\
& \quad \downarrow \\
& \quad (\pi_1 \times \pi_2) \times (\pi_3 \times \pi_4) \rightarrow ((\pi_1 \times \pi_2) \times \pi_3) \times \pi_4
\end{align*}

\begin{align*}
\text{First hexagon:} & \quad \pi_1 \times (\pi_2 \times \pi_3) \rightarrow \pi_1 \times (\pi_3 \times \pi_2) \rightarrow (\pi_1 \times \pi_3) \times \pi_2 \\
& \quad \downarrow \\
& \quad (\pi_1 \times \pi_2) \times \pi_3 \rightarrow \pi_3 \times (\pi_1 \times \pi_2) \rightarrow (\pi_3 \times \pi_1) \times \pi_2
\end{align*}

\begin{align*}
\text{Second hexagon:} & \quad (\pi_1 \times \pi_2) \times \pi_3 \rightarrow (\pi_2 \times \pi_1) \times \pi_3 \rightarrow \pi_2 \times (\pi_1 \times \pi_3) \\
& \quad \downarrow \\
& \quad \pi_1 \times (\pi_2 \times \pi_3) \rightarrow (\pi_2 \times \pi_3) \times \pi_1 \rightarrow \pi_2 \times (\pi_3 \times \pi_1)
\end{align*}
In the sequel the main interest will be in rational field theories, i.e. theories for which the number of sectors is finite. It is then important to stress that the dimensionalities \( n_p \) of \( H \)-modules and the statistical dimensions \( d_p \) are conceptually rather different. To analyze this distinction, it is convenient to define a \textit{dimension function} for the product (5) to be a map \( D : p \mapsto D_p \in \mathbb{R}_+ \) satisfying \( D_0 = 1 \) and \( D_{p+} = D_p \) as well as

\[
D_p D_q = \sum_r N_{pq}^r D_r .
\]

For any rational field theory, there is in fact a unique dimension function \( D \); namely, the numbers \( D_p \) are just the components of the common normalized Perron-Frobenius eigenvector of the fusion matrices \( N_q \). As it turns out [9,17], these components precisely coincide with the statistical dimensions \( d_p \). Moreover, for many rational theories one easily deduces from the explicit form of the fusion rules that the components of the Perron-Frobenius vector are non-integral [17–22], from which it follows in particular that the integral dimensionalities \( n_p \) cannot coincide with the dimension function. Rather, while \( n_0 = 1 \) and \( n_{p+} = n_p \) are fulfilled, one has in general only the inequality

\[
n_p n_q \geq \sum_r N_{pq}^r n_r .
\]

Notice that the algebras whose embedding index is given by (7) are infinite-dimensional, so that it is no wonder that the statistical dimension is generically non-integral.

4 The structure of \( H \)

By the properties of the representations \( \pi_p \) of \( \mathcal{A} \) listed in section 2, the following properties of \( H \) [14] are implied by the requirement that \( H \) be a quantum symmetry.

\( \triangleright \) Because of the inclusion relation (9), \( H \) is a *-subalgebra of \( \mathcal{B}(\mathcal{H}) \), and hence is an associative *-algebra with unit \( e \).

(This also follows from a more general reasoning. Namely, assume that, in each representation, \( H \) is generated by unitary elements \( u \in \mathcal{B}(\mathcal{H}) \) (corresponding to inner automorphisms \( \text{ad}_u \)). Then the mere fact that for \( h, h' \in H \) the product \( u(h)u(h') \) is defined implies via the basic representation property that there should exist a product \( hh' \) of \( h \) and \( h' \) such that \( u(hh') = u(h)u(h') \), which is associative owing to the associativity of the product in \( \mathcal{F} \). Similarly, the existence of a unit and of a *-involution correspond to the relations \( \mathcal{F} \ni 1 = u(e) \) and \( (u(h))^* = u(h^*) \).)

\( \triangleright \) According to the decomposition (8), the irreducible representations \( \pi_p \) of \( H \) are all finite-dimensional. As a consequence, for any rational field theory, \( H \) is a finite-dimensional algebra.

\( \triangleright \) On the set of \( H \)-representations, a tensor product can be defined. (This is not a trivial fact. Recall that for generic algebras the notion of a ‘product’ of representations, in contrast to the notion of Kronecker product of the associated modules, is not well-defined.) For the representation category this means that it is a monoidal category, and for \( H \) itself that it is endowed with a co-multiplication

\[
\Delta : H \rightarrow H \times H .
\]

\( H \) is a *-bi-algebra, in particular the coproduct is a *-homomorphism (but not necessarily unital).
• The product of representations is completely reducible, with fusion rule coefficients as in the decomposition (5). This means that the representation category and hence also $H$ itself must be semisimple. In particular, for a rational theory, $H$ is a finite-dimensional semisimple associative algebra, and hence a multi-matrix algebra, i.e. a direct sum of full matrix algebras,

$$H = \bigoplus_p M_{n_p} (\mathbb{C}).$$ (13)

• $H$ is endowed with a co-unit $\epsilon : H \rightarrow \mathbb{C}$ which is a unital $^*$-homomorphism. This is because as a symmetry, $H$ leaves the vacuum vector $\Omega$ invariant, or more precisely (since not the vectors $\psi \in \mathcal{H}$, but rather the rays $\lambda \psi, \lambda \in \mathbb{C}$, are the relevant quantities), it acts on $\Omega$ by scalar multiplication, $h \cdot \Omega = \Omega \epsilon(h)$. Obviously, $\epsilon(e) = 1$, and $\epsilon(a^*) = \overline{\epsilon(a)}$ for all $a \in H$. It is also immediate that the specific one-dimensional representation of $H$ carried by the vacuum sector is precisely the co-unit.

• As a representation, $\epsilon$ plays the role of the identity element in the representation ring; thus the product of any $H$-representation $\pi$ with the co-unit is isomorphic to $\pi$. This implies the existence of unitary elements $a_r$ and $a_l$ of $H$ such that $(\epsilon \times id) \circ \Delta = \text{ad}_{a_r}$, $(id \times \epsilon) \circ \Delta = \text{ad}_{a_l}$. It can be seen that a change in $a_r$ and $a_l$ changes $H$ essentially only up to isomorphism, so that it is no loss of generality to assume $a_r = \epsilon = a_l$; doing so, the latter relation simplifies to

$$(\epsilon \times id) \circ \Delta = id = (id \times \epsilon) \circ \Delta.$$ (14)

• To any $H$-representation there exists a unique conjugate representation. For $H$, this implies the existence of a linear $^*$-anti-automorphism $S : H \rightarrow H$, called the antipode, and of invertible elements $b_r, b_l \in H$ satisfying $M \circ (id \times M')(id \times S) \circ \Delta(a) \circ b_r = b_l \, \epsilon(a)$ and $M \circ (id \times M')(S \times id) \circ \Delta(a) \circ b_l = b_r \, \epsilon(a)$. Here $M$ and $M' \equiv M \circ \tau$ denote the product and ‘opposite’ product of $H$, respectively ($\tau \equiv \tau_{12}$ stands for the map that interchanges the tensor product factors).

• The tensor product of $H$-representations is commutative up to isomorphism. Therefore there is a cocommutator $R$ intertwining between $\Delta$ and the opposite coproduct $\Delta' \equiv \tau \circ \Delta$. The cocommutator must be an element of $H \times H$. The intertwining property reads

$$\Delta'(a) \cdot R = R \cdot \Delta(a)$$ (15)

for all $a \in H$. A coproduct $\Delta$ satisfying (15) is said to be quasi-cocommutative. Further, $R$ must be almost unitary in the sense that $R \cdot R^* = \Delta'(e)$, $R^* \cdot R = \Delta(e)$, and must satisfy $R \cdot R^* \cdot R = R$.

• The tensor product of $H$-representations is associative up to isomorphism. Accordingly, there is a coassociator $Q \in H \times H \times H$ intertwining between $(id \times \Delta) \circ \Delta$ and $(\Delta \times id) \circ \Delta$,

$$\left( (id \times \Delta) \circ \Delta(a) \right) \cdot Q = Q \cdot \left( (id \times \Delta) \circ \Delta(a) \right)$$ (16)

for all $a \in H$. A coproduct $\Delta$ satisfying (16) is said to be quasi-coassociative. Further, $Q$ must be almost unitary in the sense that $Q \cdot Q^* = (\Delta \times id) \circ \Delta(e)$, $Q^* \cdot Q = (\Delta \times id) \circ \Delta(e)$, and must satisfy $Q \cdot Q^* \cdot Q = Q$.

• The coassociator $Q$ and the cocommutator $R$ satisfy the compatibility conditions needed for the representation category of $H$ to constitute a braided monoidal $C^*$-category. More concretely, various specific combinations of unitality-, commutativity-, and associativity-isomorphisms of multiple tensor products lead to identical (not just up to an isomorphism)
results.

This is expressed by the following set of equations:

- the triangle identity, which with the choice $a_i = e = a_1$ reads
  $$(id \times e \times id)(Q) = \Delta(e);$$
- the square identities
  $$M_5 \circ \tau_{12} \tau_{34} b_1 \otimes (S \times id \times S)(Q) \otimes b_1 = e$$
  and
  $$M_5 \circ \tau_{12} \tau_{34} b_1 \otimes (id \times S \times id)(Q') \otimes b_1 = e;$$
  (Here $M_\ell$ denotes multiple products, defined inductively by
  $M_2 = M$, $M_\ell = M \circ (M_{\ell - 1} \times id).$)
- the pentagon identity
  $$(\Delta \times id \times id)(Q) \cdot (id \times id \times \Delta)(Q) = (Q \otimes e) \cdot (id \times \Delta \times id)(Q) \cdot (e \otimes Q);$$
- and the hexagon identities
  $$Q_{231} \cdot (\Delta \times id)(R) \cdot Q_{123} = R_{13} \cdot Q_{132} \cdot R_{23},$$
  $$Q_{312} \cdot (id \times \Delta)(R) \cdot Q_{123} = R_{13} \cdot Q_{213} \cdot R_{12}.$$
  (Here I employ the common notation to write $Q_{231} \simeq Q^{(2)} \otimes Q^{(3)} \otimes Q^{(1)}$, $R_{13} \simeq R^{(1)} \otimes e \otimes R^{(2)}$, etc. for $Q \equiv Q_{123} \simeq Q^{(1)} \otimes Q^{(2)} \otimes Q^{(3)}$ etc.)

- The tensor product is modular in the sense that the so-called monodromy matrix which
  is an element of $M_{\Sigma_n}(H)$ (see (36) below) must be invertible.

To summarize: The quantum symmetry $H$ of a rational field theory is a finite-dimensional
unital associative *-algebra having a coproduct, co-unit and antipode; the coproduct is quasi-
commutative and quasi-coassociative, the representation category of $H$ is a braided monoidal C*-category,
and the monodromy matrix is invertible. Such structures have been given the name rational Hopf algebras;
they were first considered by Vecsernyés [14] and Szlachányi [15,23], modifying and extending earlier ideas of Mack and Schomerus [24,25].

As the coproduct is not necessarily coassociative and unit preserving, rational Hopf algebras
are generically not genuine Hopf algebras. But they share a lot of the properties of quasitriangular
quasi Hopf algebras [26] and weak quasi Hopf algebras [24]. The main distinctive features are the *-properties and the further restriction that the monodromy matrix be invertible. It is worth mentioning that in early treatments [27,28] of quantum symmetries in
conformal field theory it was shown that the symmetry algebra is endowed with a coproduct.
In addition, however, it was assumed, for no particular reason, that this coproduct be coassociative,
and consequently the results of these investigations do not describe the most general case.

If $\Delta$ is coassociative, i.e. $Q = e \otimes e \otimes e$, then the relation (14) implies that $\Delta$ is unital. On
the other hand $\Delta$ is necessarily non-unital, i.e. $\Delta(e) \neq e \otimes e$, if an integer-valued dimension
function does not exist. For an algebraist, the non-unitality is certainly not the most natural
property, but it is the only way in which in the absence of an integer-valued dimension function
the axioms of a rational Hopf algebra can be satisfied. Namely, just evaluate $e$ in the product
representation $\varpi_p \times \varpi_q$, either directly which yields $(\varpi_p \otimes \varpi_q)(\Delta(e))$, or after application
of the tensor product decomposition which yields

$$\oplus_r N_{pq}^r \varpi_r(e) = \oplus_r N_{pq}^r 1_{n_r}. \tag{20}$$

If $\Delta$ is unital, the first expression evaluates to

$$(\varpi_p \times \varpi_q)(\Delta(e)) = \varpi_p(e) \otimes \varpi_q(e) = 1_{n_p} \otimes 1_{n_q}, \tag{21}$$

and hence equality of the expressions implies that the set $\{n_p\}$ of integers satisfies the defining
relation (10) of a dimension function.
5 Four space-time dimensions

Because of the importance of local observable algebras, the different topological nature of causal complements suggests that there is a major distinction between $D \geq 4$ and $D < 4$ space-time dimensions. This is indeed the case. While in low-dimensional space-time, the superselection sectors generically carry representations of the braid groups $B_n$, for $D \geq 4$ one is dealing exclusively with representations of the permutation groups $S_n$, so that in particular the statistical dimensions are integral. Further, it turns out [29] that in space-time dimension $\geq 4$ the integral statistical dimension $d_\tau$ plays simultaneously also the role of the dimension $n_\tau$ of the representation $\varpi_\tau$ of the quantum symmetry and of the multiplicity of the corresponding sector $\mathcal{H}_\tau$ in the Hilbert space $\mathcal{H}$, that is

$$d_\tau = n_\tau.$$  \hspace{0.5cm} (22)

In the framework of Doplicher-Haag-Roberts sectors, it is even possible to reconstruct the field algebra $\mathcal{F}$ from the observable algebra and its superselection structure, in such a manner that the charged fields generate the sectors by action on vectors in the vacuum sector. Let me mention some details of this Doplicher-Roberts [29,30] construction. One can prove that for space-time dimension $\geq 4$ the following holds:

1. First, with $f^i_p$ standing for a field in the sector labelled by $p$, in sloppy notation the relation

$$f^i_p f^j_q = \sum_{k=1}^{n_\tau} \sum_{l=1}^{n_\tau} (R_{(p,q)})_{ij} f^k_l f^i_p,$$  \hspace{0.5cm} (23)

holds with numerical $n_p n_q \times n_p n_q$-matrices $R_{(p,q)}$.

2. The sector labels $p$ correspond to the finite-dimensional irreducible representations of a compact group $G$, and the labels $i$ correspond to a basis of the associated $G$-modules; in particular, the statistical dimensions $d_p$ are given by the dimensionalities $n_p$ of these irreducible representations. The quantum symmetry $H$ is the group algebra of $G$, $H \cong \mathbb{C}G$. $G$ plays the role of a global gauge group; if it is a Lie group, then in a Lagrangian formulation its Lie algebra can be interpreted as the algebra of conserved Noether charges with respect to the symmetry.

3. The tensor product of $G$-representations is isomorphic to the product of sectors.

4. Up to an overall plus or minus sign, corresponding to the distinction between bosons and fermions, the matrices $R_{(p,q)}$ are products of Clebsch-Gordan coefficients which correspond to the composition of intertwining maps $\varpi_p \times \varpi_q \rightarrow \mathbb{1}_r \varpi_r \leftarrow \varpi_q \times \varpi_p$.

In short, the category of DHR representations is isomorphic to the representation category of some compact group $G$, and apart from the multiplicities corresponding to these $G$-representations, the sectors are either bosonic or fermionic. In terms of particles, the Doplicher-Roberts result means that parabosons and parafermions are equivalent to ordinary bosons and fermions, respectively, that carry a nontrivial representation of a compact internal symmetry group; in simple Lagrangian models of parabosons or -fermions, this equivalence is realized in terms of a Klein transformation of the fields. Note that according to the construction, the number of sectors is necessarily infinite. This does not imply, however, that there is an infinite number of particles which are elementary in the sense that in a path integral framework they correspond to elementary fields in the Lagrangian; rather, in this framework the sectors would typically correspond to multi-particle states or bound states.

In the proof of the Doplicher-Roberts results, use is made of the Tannaka-Krein reconstruction theorem which states that to any symmetric monoidal category (i.e., monoidal category for
which the commutativity isomorphism squares to the identity) for which a functor to the category of finite-dimensional vector spaces exists, one can construct a group whose representation category is equivalent to the original category. To perform this Tannaka-Krein reconstruction, one must know the intertwiners between isomorphic representations (in particular for the decomposition of tensor products) rather explicitly. Therefore it is not directly applicable to algebraic field theory, where only the reference endomorphisms \( \rho_p \) but not the representations \( \pi_p \) are known well enough. Rather, the appropriate version of the reconstruction theorem is the one due to Deligne, where the requirement of the existence of the functor to finite-dimensional vector spaces is replaced by the axiom that a certain number, the so-called rank of an object of the category, is always integral. In the Doplicher-Roberts construction, this rank is given by the statistical dimension \( d_p \); in particular, the map that associates \( d_p \) to the sector \( p \) is an integer-valued dimension function.

In terms of the representation theory of von Neumann algebras, the Doplicher-Roberts result is highly non-trivial: it shows that the multiplication law for physical representations of these complicated algebras just mimicks that for compact groups, a fact that certainly would not be expected a priori. This non-triviality becomes particularly obvious when one tries to generalize the construction to low space-time dimensions. In that case, the statistical dimensions are typically no longer integral, so that the Doplicher-Roberts arguments break down already at an early stage. It is thus an open question whether there exists an analogue of the construction for the case of rational Hopf algebras.

6 Truncated quantum groups

Let me return now to the case of rational field theories, and hence to low space-time dimensionality. As already mentioned, an alternative approach to quantum symmetry is via weak quasi Hopf algebras [24]. These, in turn, can often be obtained by a suitable truncation procedure from an associated quantum group \( \mathcal{U} = U_q(g) \), where \( g \) is a semisimple Lie algebra and the deformation parameter \( q \) is a primitive root of unity.

The representation theory of \( \mathcal{U} \) for \( q \) a root of unity differs significantly from the one for generic \( q \) (where it is isomorphic to the representation theory of \( g \)). As a consequence, one must truncate the representation category [31, 28] by restricting the set of allowed representations to the physical, i.e. unitarizable ones, and removing all non-unitarizable sub-representations from the tensor products of unitarizable ones (this is consistent because the non-unitarizable representations generate an ideal of the representation category). But for a field theoretic description, this not yet enough; rather, to avoid inconsistencies one must [32] perform the truncation already at the level of the algebra, namely by defining the quantum symmetry as the quotient \( \mathcal{U}_t := \mathcal{U} / \mathcal{J} \) of \( \mathcal{U} \) by the ideal

\[
\mathcal{J} = \{ u \in \mathcal{U} \mid \Pi_{\text{phys}}(u) = 0 \},
\]

(24)

where \( \Pi_{\text{phys}} \) denotes the direct sum of all physical representations. The quotient \( \mathcal{U}_t \) is a weak quasi Hopf algebra, and hence satisfies most of the defining properties of a rational Hopf algebra; whether \( \mathcal{U}_t \) or some similar object can be endowed with the full structure of a rational Hopf algebra, including in particular all properties related to the \("\)-involution, is not known.

It must be emphasized that, while the quantum group \( \mathcal{U} \) serves as a starting point of the construction, in the truncation process actually most of its structure gets lost. In particular, typically various distinct quantum groups lead, by truncation, to equivalent weak quasi Hopf algebras [25]. The equivalence of these algebras is however not provided by ordinary isomorphisms, as typically the dimensionalities of different such algebras do not coincide. This happens because by construction the numbers \( n_p \) are just given by with the dimensionalities
of the modules \( L_p(g) \) of the underlying semisimple Lie algebra \( g \); for instance, \( n_p = p + 1 \) for \( p = 0, 1, \ldots, k \) (with \( k \) the level of the associated untwisted affine Lie algebra) if \( g = \text{sl}_2 \).

On the other hand, an advantage of this construction is that it allows for employing the well-developed (see e.g., [17, 33, 34]) theory of unitarizable representations of quantum groups at roots of unity. For example, the coproduct of \( \mathcal{U}_q \) is of the form

\[
\Delta([u]) = [P \cdot \Delta_{x\ell}(u)],
\]

(25)

where \( \Delta_{x\ell} \) is the coproduct of \( \mathcal{U} \) and \( P \) a projector in \( \mathcal{U} \times \mathcal{U} \). Moreover, whenever no fusion coefficient \( N_{pq}^r \) is larger than 1 (and hence e.g., for \( g = \text{sl}_2 \) [24] and for arbitrary \( g \) at level one [25]), there is a simple formula for \( P \), namely

\[
P = \sum_{p \neq q} N_{pq}^r (P_p \otimes P_q) \cdot \Delta(P_r),
\]

(26)

with \( P_q \in \mathcal{U} \) a set of independent projectors spanning the center of \( \mathcal{U} \) (in \( \Pi_{\text{phys}} \)).

7 Reconstruction

As it turns out, it is possible to reconstruct from the quantum symmetry \( H \) various aspects of the field theory. In particular there is a construction analogous to the one that associates the statistics parameter, and hence the numbers \( d_p \) and \( n_p \), to a representation \( \pi_p \) of the observable algebra \( \mathcal{A} \). From the list of properties of \( H \) presented in section 4, it is far from obvious what this construction could be in the finite-dimensional setting of the quantum symmetry of a rational field theory. The clue to the construction is the use of so-called amplimorphisms, i.e. \(^\ast\)-monomorphisms \( \mu_p \) from \( H \) to matrix algebras over \( \mathbb{H} \),

\[
\mu_p : \ H \to M_{n_p}(\mathbb{H}),
\]

(27)

instead of representations of \( H \), which are \(^\ast\)-algebra homomorphisms \( \varpi : \ H \to M_n(\mathbb{C}) \). Note that to be able to produce non-integral statistical dimensions \( d_p \), one has to employ amplimorphisms of \( H \) rather than just endomorphisms. It is a remarkable result that one can indeed arrive at non-integral statistical dimensions, and even non-integral embedding indices \( a_p^2 \), by finite-dimensional constructions.

Before treating the amplimorphisms in some detail, let me say a few words about the representations \( \varpi \). They are unitarizable and completely reducible; the defining irreducible representations \( \varpi_p \) read \( H \ni a \equiv \sum_p \sum_j a_p^j e_p^{ij} \mapsto (\varpi_p(a))^{ij} = a_p^{ij} \). Here \( e_p^{ij} \), \( i, j = 1, 2, \ldots, n_p \), denote the matrix units which provide a basis of \( H \). In this basis the \(^\ast\)-involution reads simply \( (e_p^{ij})^\ast = e_p^{ji} \).

With the help of the coproduct, one defines the product of representations as \( (\varpi_1 \times \varpi_2)(a) := (\varpi_1 \times \varpi_2)(\Delta(a)) \). The properties (15), (16) etc. of \( R \) and \( Q \) ensure the commutativity and associativity of the product of representations up to unitary equivalence. In the basis of matrix units, the co-unit obeys \( \epsilon(e_p^{ij}) = 1 \) for \( e_p^{ij} = e_n \), \( \epsilon(e_p^{ij}) = 0 \) else; as a consequence, products of representations with the co-unit \( \epsilon \) lead to representations that are equivalent to the original ones.

The antipode can be chosen such that \( S(e_p^{ij}) = e_p^{ji} \). Then the conjugate \( \varpi^* \) of a representation \( \varpi \) satisfies \( (\varpi^*(a))^{ij} := (\varpi_p(S(a)))^{ij} = (\varpi^+_p(a))^{ij} \) for all \( a \in H \). The properties of \( S \) ensure that \( \varpi^* : H \to M_n(H) \) is a \(^\ast\)-homomorphism, with \( n \) the dimension of \( \varpi \).

Amplimorphisms contain the full information about representations. Namely, on one hand, any amplimorphism \( \mu : H \to M_n(H) \) induces a representation by composing it with the co-unit,

\[
\varpi_\mu := \epsilon \circ \mu : H \to M_n(\mathbb{C}), \quad \varpi_\mu^{ij}(a) := \epsilon(\mu^{ij}(a))
\]

(28)
for \( a \in H \) and \( i,j = 1,2,\ldots,m \). On the other hand, any (non-zero) representation \( \varpi \) of \( H \) defines an amplimorphism \( \mu_\varpi : H \to M_m(\mathcal{H}) \), with \( m \) the dimension of \( \varpi \), through

\[
\mu_\varpi := (id \times \varpi) \circ \Delta ;
\]

(29)

amplimorphisms of this type are called a special amplimorphisms. Similarly, an amplimorphism \( \nu : H \to M_n(\mathcal{H}) \) is called natural if \( \nu \sim \mu_\varpi \), i.e. if there is an equivalence \( T \in (\mu_\varpi) \) for some representation \( \varpi : H \to M_n(\mathcal{H}) \). (Amplimorphisms \( \mu \) and \( \nu \) are called equivalent, \( \mu \sim \nu \), if there is a \( T \in (\mu(\nu)) \) with \( TT^* = \mu(e), T^* T = \nu(e) \). The space \( (\mu(\nu)) \) of intertwiners between \( \mu : H \to M_m(\mathcal{H}) \) and \( \nu : H \to M_n(\mathcal{H}) \) consists of all \( T \in M_{m \times n}(\mathcal{H}) \) such that \( \mu(a)T = T\nu(a) \) for all \( a \in H \), and \( \mu(e)T = T = T\nu(e) \).

One can define subobjects, direct sums and an associative product of amplimorphisms; the latter reads \( (\mu \times \nu)^{ij,:}_{ij,:}(a) = \mu^{ij,:}(\nu^{ij,:}(a)) \).

To see the relation with the braid group, combine the identity \( R_{12} \cdot (\Delta \times id)(R) = (\Delta' \times id)(R) \cdot R_{12} \) with the first hexagon equation. This leads to the quasi Yang–Baxter equation

\[
R_{12}Q_{23}Q_{132}R_{23}Q^* = Q_{323}Q_{23}R_{132}Q_{232}R_{12} .
\]

(30)

Now define the maps

\[
\sigma_i := \begin{cases} 
\hat{R}_{i,i+1} & \text{for } i \in \mathbb{Z}_n, \\
\hat{Q}_{i+1,i+1} \circ \hat{R}_{i,i+1} \circ \hat{Q}_{i+2,i+1}^* & \text{for } i \in \mathbb{Z}_n + 1,
\end{cases}
\]

acting on the infinite tensor product \( H \times H \times H \times \ldots \), where \( Q_{ijk} \) stands for multiplication with \( Q_{ijk} \) from the right, and \( \hat{R}_{ij} \) for multiplication with \( R_{ij} \) followed by transposition \( \tau_{ij} \) of the indicated tensor factors. These maps \( \sigma_i \) satisfy the defining relations of the infinite braid group \( B_\infty \) and hence furnish a representation of \( B_\infty \) on \( H \times H \times H \times \ldots \).

Further information about these braid group representations is obtained with the help of the amplimorphisms of \( H \). The braiding properties of amplimorphisms are encoded in the statistics operators \( \varepsilon \) which are intertwiners between \( \mu \times \nu \) and \( \nu \times \mu \). For special amplimorphisms \( \mu_\varpi, \mu_\psi \) corresponding to representations \( \varpi_\varpi \) and \( \varpi_\psi \), \( \varepsilon \) is defined by

\[
\varepsilon(\mu_\varpi;\mu_\psi) = [(id \times \varpi_\varpi \times \varpi_\psi)(Q)] \cdot [(id \times \tau) \circ (id \times \varpi_\psi \times \varpi_\varpi)(R)] \cdot [(id \times \varpi_\varpi \times \varpi_\psi)(Q^*)].
\]

(32)

\( \varepsilon \) is unitary in the sense that \( \varepsilon(\mu_\varpi;\mu_\psi) \cdot \varepsilon(\mu_\psi;\mu_\varpi)^* = (\mu_\varpi \times \mu_\psi)(e) \) and \( \varepsilon(\mu_\varpi;\mu_\psi)^* \cdot \varepsilon(\mu_\psi;\mu_\varpi) = (\mu_\psi \times \mu_\varpi)(e) \), and upon multiplication the statistics operators give rise to a colouration of braids.

A left inverse \( \Phi_\varpi : M_n(\mathcal{H}) \to H \) of an amplimorphism \( \nu : H \to M_n(\mathcal{H}) \) is a positive linear map which satisfies \( \Phi_\varpi(1_n) = 1 \) and \( \Phi_\varpi(\nu(a) \cdot B \cdot \nu(c)) = a \cdot \Phi_\varpi(B) \cdot c \) for all \( a,c \in H \) and all \( B \in M_n(\mathcal{H}) \). For special amplimorphisms \( \mu_\varpi \), it can be defined as

\[
\Phi_\varpi(A) := P^{\varpi}_p \cdot \bar{\rho}_p(A) \cdot P^{\varpi}_p \quad \text{for all } A \in M_n(\mathcal{H}),
\]

(33)

where

\[
P^{ij,:}_p = [\text{tr } \varpi_p(b_i b_j^*)]^{-1/2} Q^{(1)}(\varpi_p)^{ij,:} S(Q^{(2)}(\tau^*)^*)
\]

(34)

for \( i,j = 1,2,\ldots,n_p \), which is a partial isometry in \( (\mu_\varpi \times \mu_\varpi)[id] \).

Given the statistics operators and left inverses, one can finally compute the statistical parameter \( \lambda_\varpi \in H \) of a special amplimorphism \( \mu_\varpi \) as

\[
\lambda_\varpi = \Phi_\varpi \circ \Phi_\varpi(\varepsilon(\mu_\varpi;\mu_\varpi)).
\]

(35)

Similarly, the monodromy matrix \( Y \) is

\[
Y_{rs} = d_s d_r \cdot \Phi_\varpi \circ \Phi_\varpi(\varepsilon(\mu_r;\mu_s) \cdot \varepsilon(\mu_s;\mu_r)).
\]

(36)
It can be shown that

$$\lambda_p = (\eta_p / d_p) \cdot \epsilon$$  \hspace{1cm} (37)

with $\eta_p$ the statistics phase and $d_p$ the statistical dimension of the superselection sector $\pi_p$, and that $Y_{rs} = y_{rs} \cdot \epsilon$ with $y_{rs} \in \mathbb{C}$. Further, if $Y$ is invertible, then by defining

$$V(S)_{rs} := |\sigma|^{-1} \cdot y_{rs}, \quad V(T)_{rs} := (\sigma / |\sigma|)^{1/3} \cdot \delta_{rs} \eta_s$$  \hspace{1cm} (38)

with $\sigma = \sum_p d_p^2 / \omega_p$, one obtains a unitary representation $V$ of the double cover $\text{SL}(2, \mathbb{Z})$ of the modular group. In the case of conformal field theories, this representation fixes the value of the Virasoro central charge modulo 8.

8 Classification

Having identified rational Hopf algebras as the quantum symmetries of rational field theories, there remain two basic tasks: the classification of such algebras, and the identification of the rational Hopf algebras relevant for specific theories.

The coproduct $\Delta$ of $H$ can be viewed as an embedding of $H$ into $H \times H$. As such embeddings are relevant only up to inner unitary automorphisms of $H \times H$, there are equivalences with appropriate elements $U \in H \times H$. This ‘gauge freedom’ can be exploited to present a rational Hopf algebra in a canonical form; in particular, with a suitable choice of $U$ one can put $a_r = \epsilon = a_1$, as was already used above, and $b_r$ and $b_t$ can be taken as invertible central elements of $H$ one of which is positive. As might have been expected, even after fixing this gauge freedom a complete classification of rational Hopf algebras is extremely difficult; actually at present it is beyond reach. The classification involves in particular the classification of solutions to the pentagon and hexagon equations (17) – (19).

However [14, 35], for any given (finite) number of sectors and any fixed set of fusion rules among them, it is in principle straightforward to write down the most general compatible coproduct, solve the pentagon and hexagon identities, and compute the statistics operators, left inverses, statistics parameters, and the monodromy matrix. In particular, after suitable choice of ‘coordinates’ in the space of multi-matrix algebras, one can encode the necessary manipulations in a computer program. At least for small numbers of sectors, the solutions can then also be obtained in practice [35]. That a complete solution of the pentagon and hexagon equations, which constitute a huge system of coupled nonlinear equations in the relevant coordinates, is indeed possible for non-trivial fusion rules, is related to the fact that this system of equations is actually highly over-constrained.

What can so far not be obtained in this approach to the classification are the integers $n_p$, and hence the dimensionalities of the rational Hopf algebras. Rather, in the algorithm just mentioned an important ingredient is that the coordinates are chosen in such a manner that they do not depend on these integers at all; to any solution to the remaining constraints there is then associated an infinity of possible choices of the $n_p$, which is only restricted by the conditions $n_0 = 1$ and $n_{p+} = n_p$ and by the inequality (11). For any set of prescribed fusion rules, there exists a minimal choice of these integers, but at present no physical argument is known which would exclude any non-minimal choice. For the minimal choice, one has in particular $n_p = 1$ whenever $d_p = 1$, and moreover $d_p = d_q$ if $\pi_p \cong \pi_q \times \pi_r$ for some $r$ with $d_r = 1$ (in conformal field theory terms, this means that $n_p = 1$ for simple currents, and $d_p = d_q$ if the sectors $p$ and $q$ lie on the same simple current orbit). In the case of truncated quantum groups, the dimensionalities $n_p$ are always non-minimal; in particular $n_p$ is larger than 1 for all simple sectors different from the vacuum sector.

It is also straightforward to associate to a rational Hopf algebra obtained via this classification a (unitary) conformal field theory; namely, one just has to search the set of known
conformal field theories for those which have the correct number of primary fields and the correct fusion rules, conformal dimensions and Virasoro central charge. For example, for the Ising fusion rules [14], one obtains the Ising model as well as the level 2 $A_1$ and $E_8$ and the level 1 $B_r$ Wess-Zumino-Witten theories. Thereby the classification of rational Hopf algebras corresponds to a partial classification of unitary rational conformal field theories, namely, the possible fusion rules, the conformal dimensions modulo integers, and the Virasoro central charge modulo 8. It is not yet clear whether different conformal field theories in this classification that share the same rational Hopf algebra (e.g. in the example of the Ising fusion rules, the level 1 $B_r$ theories with $r$ differing by an integer multiple of 8) are to be considered as distinct quantum field theories.

To conclude, let me mention that for all rational Hopf algebras analyzed so far [35] it turned out to be possible to identify associated unitary rational conformal field theories. If this is a generic feature, it implies that as far as it is only the rational Hopf algebra that matters, any two-dimensional rational quantum field theory can be viewed as being equivalent to a rational conformal field theory.

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