Radiation from a Moving Scalar Source

Hai Ren and Erick J. Weinberg

Physics Department
Columbia University
New York, New York 10027

Abstract

We study classical radiation and quantum bremsstrahlung effect of a moving point scalar source. Our classical analysis provides another example of resolving a well-known apparent paradox, that of whether a constantly accelerating source radiates or not. Quantum mechanically, we show that for a scalar source with arbitrary motion, the tree level emission rate of scalar particles in the inertial frame equals the sum of emission and absorption rates of zero-energy Rindler particles in the Rindler frame. We then explicitly verify this result for a source undergoing constant proper acceleration.

This work was supported in part by the US Department of Energy
1. Introduction

The problem of a uniformly accelerating electric charge gives rise to a well known apparent paradox. Since the charge is accelerating, it should radiate. However, by the principal of equivalence, the situation should be equivalent to that of a static charge in a uniform gravitational field, which certainly does not radiate. The resolution lies in the recognition that only a portion of Minkowski space-time is accessible to a uniformly accelerating observer comoving with the charge. A careful analysis at either the classical [1] or quantum [2] level then shows that it is possible for this coaccelerating observer to conclude that there is no radiation, even though a static observer sees the charge radiate.

In this paper, we consider a closely related problem, that of a uniformly accelerating source coupled to a massless scalar field $\phi$ through the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \rho \phi$$

As in the electromagnetic case, a static observer would expect the source to radiate, while a coaccelerating observer would not. Although there are number of differences from the electromagnetic cases, including the detailed form of the radiation, we show that for this case also the views of the two observers can be reconciled.

In Minkowski coordinates $(t, x, y, z)$ the source is uniformly accelerating, following the trajectory

$$x_s^\mu(s) = (a^{-1} \sinh as, 0, 0, a^{-1} \cosh as)$$

where $s$ is the proper time of the source. For describing the observations of the coaccelerating observer, it is convenient to use Rindler[3] coordinates $(\tau, x, y, \xi)$ defined by

$$t = \frac{e^{a\xi}}{a} \sinh a\tau, \quad z = \frac{e^{a\xi}}{a} \cosh a\tau$$

In terms of these flat Minkowski metric takes the form

$$ds^2 = e^{2a\xi}(-d\tau^2 + d\xi^2) + dx^2 + dy^2$$

In the Rindler coordinates the source is stationary at the point $x = y = \xi = 0$. 

1
The Rindler coordinates cover only the wedge \( z > |t| \) (region I) of Minkowski spacetime, which is the only part completely accessible to an observer comoving with the source. Region II \( (t > |z|) \) is always outside the observer’s past light cone; thus, although he can send signals to this region, he can never observe events there. Similarly, he can receive signals from region III \( (t < -|z|) \), but can never send signals there. Finally, he can have no communication with region IV \( (z < -|t|) \).

Since the source of a massless scalar field need not be conserved, its time-dependence must be specified. We take it to have constant magnitude in its rest frame:

\[
\rho = q \delta(x) \delta(y) \delta(\xi)
\]  

(1.5)

Lorentz contraction of the volume then makes the magnitude of the source time-dependent in the inertial frame:

\[
\rho = \frac{q}{a\sqrt{t^2 + a^{-2}}} \delta(x) \delta(y) \delta(z - \sqrt{t^2 + a^{-2}})
= q\sqrt{1 - v_s^2} \delta^{(3)}(x - x_s(t))
\]  

(1.6)

In Sec. II we consider the problem from a classical point of view. For the electromagnetic case, several features explain the failure of the comoving observer to observe radiation. It turns out that within region I it is never possible to clearly distinguish a radiation field distinct from the expected Coulomb field. One might instead examine the flow of energy, but this turns out to be entirely into regions inaccessible to the comoving observer. We find that these features are reproduced in the scalar case. A difference which we find is that, although the retarded potential method fails to give the correct result for the electromagnetic case [4], it does give a true solution of the field equations in the scalar case. We examine this in some detail.

In the quantum theory the radiation due to the accelerated charge appears as the emission of quanta by a bremsstrahlung effect. Thus, one might ask whether the static and the comoving observers both see emission of particles. However, this is not quite the right question. It is well known that an observer who is static in Rindler coordinates has a different definition of particle than does a static Minkowski observer, and interprets the Minkowski vacuum as a Fulling-Davies-Unruh (FDU) thermal bath of many-particle states [5-7]. The underlying reason for this difference is that modes which have positive frequency
with respect to Minkowski time are linear combinations of Rindler modes with both positive and negative frequencies. A consequence of this fact is that what a Minkowski observer calls emission of a quantum can appear to a Rindler observer as either emission or absorption of a quantum. In Sec. III we first show that for any source confined to region I the Minkowski emission rate is equal to the sum of an emission and an absorption rate calculated by the Rindler observer. We then verify by explicit calculation that the Minkowski bremsstrahlung rate due to a uniformly accelerated source is precisely equal to the the sum of the rates for emission and absorption of zero-energy quanta by a static source in the thermal bath. This calculation parallels that of Ref. 2 for the electromagnetic case.

In the appendix, we study some general properties of classical radiation from a moving point scalar source.

2. Classical Radiation from a Uniformly Accelerated Scalar Source

In this section we consider the problem from a classical point of view. Our treatment parallels that of Boulware [1] for the electromagnetic case.

The first step is to determine the classical field generated by our source. We do this in Minkowski coordinates, with the source given by Eq. (1.4). Solving the wave equation $\Box \phi = -\rho$ by the retarded potential method gives

$$\phi(x) = \frac{q}{2\pi} \int_{-\infty}^{+\infty} ds \theta(t - x^0_s(s)) \delta((x - x_s(s))^2)$$

(2.1)

where $x^\mu_s(s)$ is given by Eq. (1.2). Thus

$$\phi(x) = \frac{q}{4\pi R} \theta(t + z)$$

(2.2)

where

$$R = \frac{a}{2} \left[ (X^2 - a^{-2})^2 + 4a^{-2} \rho^2 \right]^{1/2}$$

(2.3)

with $\rho^2 = x^2 + y^2$ and $X^2 = x^\mu x_\mu = \rho^2 + z^2 - t^2$. It is easy to verify that this is a solution of the field equation, even on the plane $t + z = 0$ on which its derivatives are singular.
In the electromagnetic analogue, the fields obtained from the Liénard-Wiechert potentials do not satisfy Maxwell’s equations along the surface \( t + z = 0 \), but instead differ from the actual solutions by terms proportional to \( \delta(t + z) \). These terms can be motivated by a limiting process suggested by Bondi and Gold [4]. Consider an electric charge which is at rest at \( z = 1/a \) until \( t = 0 \), and is uniformly accelerated after that. Now apply the Lorentz transformation

\[
z \rightarrow z \cosh \alpha + t \sinh \alpha, \quad t \rightarrow t \cosh \alpha + z \sinh \alpha
\]

Thus going to a frame in which the charge has a constant negative initial velocity and in which the uniform acceleration begins at \( t = -a^{-1} \sinh \alpha \). In the limit \( \alpha \to \infty \), the initial velocity of the charge approaches the speed of light, and the time at which the uniform acceleration begins goes to \(-\infty\). In this limit the Coulomb field of the initially static charge is Lorentz transformed into a delta function field along the surface \( t + z = 0 \).

Even though the retarded potential method gives a solution for the scalar case, one might wonder if this limiting procedure might lead to an additional contribution. This is readily examined. A point source which is at rest at \( z = 1/a \) until \( t = 0 \) and uniformly accelerated thereafter gives rise to a field

\[
\tilde{\phi}(x) = \frac{q}{4\pi} \left[ \frac{1}{r} \theta(r - t) + \frac{1}{R} \theta(t - r) \right]
\]

where \( r = [\rho^2 + (z - a^{-1})^2]^{1/2} \). (Note that \( t = r \) implies that \( R = r \), so that this solution is continuous everywhere.) Applying the transformation (2.4) gives

\[
\tilde{\phi}(x) = \frac{q}{4\pi} \left[ \frac{1}{r'} \theta(\lambda - t - z) + \frac{1}{R} \theta(t + z - \lambda) \right]
\]

where

\[
\lambda = e^{-\alpha} \left[ a^{-1} + \frac{\rho^2}{a^{-1} - (z - t)e^{-\alpha}} \right]
\]

\[
r' = [\rho^2 + (z \cosh \alpha + t \sinh \alpha - a^{-1})^2]^{1/2}
\]

and \( R \) is as before. Since \( \lambda \) vanishes in the limit \( \alpha \to \infty \), while \( r' \sim |z + t|e^\alpha \) for \( z + t < 0 \), we see that \( \tilde{\phi}(x) \) approaches the field (2.2) of the uniformly accelerating source for any point with \( t + z \neq 0 \). The behavior on the null plane \( t + z = 0 \) is somewhat curious. On this plane,
\[ r' = (\rho^2 + a^{-2})^{1/2} + O(e^{-\alpha}) \], while \( R = (a/2)(\rho^2 + a^{-2}) \). Consequently, the limits \( t \to -z \) and \( \alpha \to \infty \) do not commute. In fact, the field is rapidly varying in a region of width of order \( e^{-\alpha} \) about this plane, with

\[
\frac{\tilde{\phi}(z + t = \lambda)}{\phi(z + t = 0)} = \frac{2}{\sqrt{1 + a^2 \rho^2}} \quad (2.9)
\]

In the electromagnetic case, this limiting process leads to a delta function contribution to the field strengths which is not obtained by the Liénard-Wiechart method and which is needed to satisfy Maxwell’s equations on the plane. In the scalar case, the fields obtained by the two methods differ at most by a finite amount on the \( t + z = 0 \) plane. The differences seem a bit more significant when we consider the energy-momentum tensor \( T^{\mu \nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu \nu} \partial_{\sigma} \phi \partial^\sigma \phi \). With the retarded potential solution (2.2), the step functions give rise to singular contributions (proportional to squares of delta functions) to \( T^{\mu \nu} \). While the limiting method gives a way of defining these singular contributions more precisely, it is not at all clear that in the limit \( \alpha \to \infty \) they agree with any reasonable definition of the former case. This is not surprising. Within the region \( 0 \leq t + z < \lambda \sim e^{-\alpha} \) (and thus on the null plane, in the limit) it is possible to distinguish a source which was initially moving with constant velocity from one which has always been uniformly accelerating. The essential point is that the comoving observer’s analysis of the situation depends only on the values of the field within region I, but not on its boundary, so that the ambiguity in defining \( T^{\mu \nu} \) on the boundary is immaterial to the problem of reconciling the views of the static and the comoving observers.

We can also examine the effects of this limiting procedure on the gradients of the field \( \partial^\mu \phi(x) \), which constitute the energy-momentum tensor. One can do this by applying the limiting procedure directly to \( \partial^\mu \phi \), or alternatively, by simply differentiating Eq. (2.6)

\[
\partial^\mu \tilde{\phi}(x) = \frac{q}{4\pi} \left[ \theta(\lambda - t - z) \partial^\mu \left( \frac{1}{r'} \right) + \theta(t + z - \lambda) \partial^\mu \left( \frac{1}{R} \right) \right] \quad (2.10)
\]

In the limit \( \alpha \to \infty \), we get a \( \delta \)-function contribution from the Lorentz-transformed Coulomb field. These \( \delta \)-function terms are the same as those one would get by directly differentiating the retarded potential solution (2.2).

We now explicitly calculate the energy-momentum tensor for the case of the uniformly accelerating source. Since we will need it only for \( z + t > 0 \), we ignore the singular contri-
butions. In this region a straightforward calculation in Minkowski coordinates gives

\[
T^{\mu\nu} = \frac{q^2 a^2}{16\pi^2} \left\{ \frac{1}{R^4} \left( x^\mu x^\nu - \frac{1}{2} \eta^{\mu\nu} X^2 \right) + \frac{1}{R^2} \left[ a^{-2} \rho^\mu \rho^\nu + \frac{1}{2} (X^2 - a^{-2}) (x^\mu \rho^\nu + \rho^\mu x^\nu) - \rho^2 x^\mu x^\nu \right] \right\}
\]  

(2.11)

where \( \rho^\mu \equiv (0, x, y, 0) \) and the \( \theta(t + z) \) factors have been suppressed.

It is also straightforward to obtain the components of \( T^{\mu\nu} \) with respect to the Rindler coordinates. In particular, the components \( T^{\tau j} \), which correspond to the energy flux seen by a comoving observer, vanish everywhere. This follows from the fact that the field (2.2) is static when written in terms of Rindler coordinates. Alternatively, one can simply transform from the Minkowski result, using the formulas

\[
T_{\tau j} = \frac{\partial x^\mu}{\partial \tau} T_{\mu j} = 0
\]  

(2.12)

for \( j = 1 \) or \( 2 \) and

\[
T_{\tau \xi} = \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \xi} T_{\mu \nu} = 0
\]  

(2.13)

with indices \( \mu \) and \( \nu \) referring to Minkowski components.

While the comoving observer sees no flow of energy, and thus no radiation, matters are not so simple for the static observer. Thus, let us calculate the power radiated by the source, as seen in Minkowski coordinates. This quantity is Lorentz invariant, and is most easily calculated in a frame where the source is instantaneously at rest. Furthermore, because the acceleration of the source is uniform, the radiation should be the same at all points along the world-line of the source. We therefore look along the forward light-cone of the point \( x = y = t = 0, z = a^{-1} \) at which the source is at rest. On this light-cone, the components of the energy flux are

\[
T^{ij} = \frac{q^2}{16\pi^2} \left[ \frac{a^2 \cos^2 \theta}{r^2} + \frac{a \cos \theta}{r^3} \right] \hat{r}^j
\]  

(2.14)

where \( \hat{r} \) is a unit three-vector from the point \( x = y = 0, z = a^{-1} \) to the field point, \( r \) is the three-dimensional distance between these two points, and \( \theta \) is the angle between \( \hat{r} \) and the \( z \)-axis. By integrating over a sphere of radius \( r \), one finds that the energy flux along this
light cone at time $t = r$ is

$$
\int dS_j T^j = \frac{q^2}{16 \pi^2} \int d\Omega \left[ a^2 \cos^2 \theta + \frac{a \cos \theta}{r} \right] = \frac{q^2 a^2}{12 \pi} \tag{2.15}
$$

We show in the Appendix that Eq. (2.15) is exactly the result expected for the power radiated by an uniformly accelerating source.

Although this result is suggestive of radiation, the real test is whether the energy in the field changes over time. This can be addressed by calculating the net energy flux through a closed three-dimensional hypersurface. In particular, let us consider the region $[1]$ given by $z > |t| + \epsilon$ (with the limit $\epsilon \to 0$ understood) and $z_1 < z < z_2$, with $z_1 < a^{-1} < z_2$, and calculate the net energy flux out of this region, as seen by a static observer using Minkowski coordinates. Because $T^{tz}$ is an odd function of $t$, the total flux through either of the surfaces $z = z_1$ or $z = z_2$ vanishes. The flux through the surfaces $z = \pm t + \epsilon$ is

$$
\int d^2 \rho \int_{z_1}^{z_2} dz (T^{tt} + T^{tz})(t = \pm t, x, y, z) = \frac{q^2 a^6}{2 \pi^2} (z_2 - z_1) \int d^2 \rho \frac{\rho^2}{(1 + a^2 \rho^2)^4} = \frac{q^2 a^2}{12 \pi} (z_2 - z_1) \tag{2.16}
$$

Thus, the energy flowing in through the surface $z = -t + \epsilon$ is exactly equal to that flowing out through the surface $z = t + \epsilon$. Hence, the static observer, like the comoving observer, will conclude that there is no net energy production in the region.

In fact, the total radiation flowing out of any closed three-dimensional hypersurface symmetric in $t$ and confined to region I is zero. To see this, let $\mathcal{V}$ be a four-dimensional spacetime volume with three-dimensional boundary $\partial \mathcal{V}$, whose outward normal we denote as $n_\mu$. The total flux through $\partial \mathcal{V}$ is

$$
\oint_{\partial \mathcal{V}} d^3 x T^{t \mu} n_\mu = \int d^4 x \partial_\mu T^{t \mu} = \int d^4 x (\partial^t \phi(x)) (\square \phi(x))
$$

$$
= -q \int d^4 x \sqrt{1 - \mathbf{v}_c^2} \delta^{(3)}(x - x_c(t)) \partial^t \phi(x) \tag{2.17}
$$

$$
= -q \int d \tau \sqrt{1 - \mathbf{v}_c^2} \partial^t \phi(x) \bigg|_{x = x_c(t)} = 0
$$

since $\sqrt{1 - \mathbf{v}_c^2}$ is even in $t$ while $\partial^t \phi$ is odd in $t$. 

5
As an alternative to looking for energy flow as evidence of radiation, one might also examine the field (or, more precisely, its gradient) to see whether it is possible to distinguish separate Coulomb and radiation components. Along the forward light-cone of the source in its instantaneous rest frame, the spatial gradients of the field are

$$\partial_j \phi = \frac{q}{2\pi} \left[ \frac{1}{r^2} + \frac{a \cos \theta}{r} \right] r^j$$  \hspace{1cm} (2.18)

where the notation is the same as in Eqs. (2.14) and (2.15). Using their $r$-dependence to identify the two terms in brackets as Coulomb and radiation components, respectively, we see that the latter dominates when $|ar \cos \theta| \gg 1$. However, in region I the condition $z > |t|$ implies that

$$\frac{\text{radiation field}}{\text{Coulomb field}} = ar \cos \theta < \frac{\cos \theta}{1 - \cos \theta}$$  \hspace{1cm} (2.19)

Hence, in the region accessible to the comoving observer the radiation field can be dominant only for $\theta$ very close to zero. Even in that small region the issue is confused. Let $l$ be the distance from a given point in region I to the world line of the source, measured to the point on the world-line where the spacelike separation is greatest. For $\cos \theta$ near unity one finds that $l^2 \approx 2r/a$, so that the radiation component in Eq. (2.18) takes on the Coulomb form, but with $l$ taking the place of $r$.

3. Scalar Bremsstrahlung and the FDU Thermal Bath

In the quantum theory radiation is not continuous, but rather is a series of discrete events — the emission of discrete quanta — each corresponding to a change in the state of the quantum field. The static and the uniformly accelerating observers do not agree on the initial state of the field — the Rindler observer interprets the Minkowski vacuum as a thermal bath of many-particle states — but they should agree on whether the state changes. To show how this works, we first show that for an arbitrary source confined to region I, the emission rate seen by the static observer is equal to the sum of the emission and absorption rates measured by the accelerating observer. We then specialize to the uniformly accelerating source of Eq. (1.5), and verify this result by explicit calculations.

We begin by expanding the quantum field in terms of normal modes. To do this, we need a set of solutions of the scalar wave equation $\Box \phi = 0$ which form a complete set on a spacelike
slice through space-time. A convenient choice for such a slice is the hypersurface given by $t = 0$ in Minkowski coordinates. This slice lies partly in region I (where it is specified by $\tau = 0$) and partly in region IV. For Rindler coordinates, a full set of modes comprises a complete set in the Rindler coordinates for region I, supplemented by a similar complete set in terms of the analogous coordinates for region IV.

Consider first the decomposition appropriate to Rindler space. Let

$$\phi(x) = \phi_R(x) + \phi_L(x)$$

(3.1)

where $\phi_L$ ($\phi_R$) vanishes if $x$ lies in region I (region IV). Because the Rindler metric is independent of $x$, $y$, and $\tau$, the modes in region I can be chosen to be of the form

$$f_{k_x, k_y, \omega} = e^{ik_x x + ik_y y - i\omega \tau} h_{k_x, k_y, \omega}(\xi)$$

(3.2)

The field in region I can then be expanded as

$$\phi_R(x) = \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_0^{+\infty} d\omega \left[ a_{k_x, k_y, \omega} f_{k_x, k_y, \omega}(x) + h.c. \right]$$

(3.3)

where, as usual, the positive and negative frequency modes have been separated. A similar decomposition for $\phi_L(x)$ can be made in region IV.

The normalization of the $f_{k_x, k_y, \omega}(\xi)$ can be fixed by requiring that

$$(f_{k_x, k_y, \omega}, f'_{k'_x, k'_y, \omega'})_{Rind} = F(\omega) \delta(\omega - \omega') \delta(k_x - k'_x) \delta(k_y - k'_y)$$

(3.4)

where for any two functions $f(x)$ and $g(x)$ we define

$$(f, g)_{Rind} \equiv i \int d^3 x \sqrt{h} n^\mu \left[ f^*(x) \partial_\mu g(x) \right]$$

(3.5)

Here the integration is over the region I hypersurface $\tau = 0$, with $n^\mu = (e^{-a\xi}, 0, 0, 0)$ being the normal to that hypersurface and $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ the induced three-dimensional metric. If $f$ and $g$ are solutions of the scalar field equation, then $(f, g)$ is independent of the choice of the spacelike hypersurface. The choice for the function $F(\omega)$ determines the commutation relations of the creation and annihilation operators $a$ and $a^\dagger$ and, through these, the density of one-particle states; physical results are insensitive to the particular choice made.
In Minkowski coordinates the field is usually expanded in plane waves. For the present purposes it is more convenient to choose the modes to have definite transverse momenta \( k_x \) and \( k_y \), but not definite \( k_z \) or frequency. Although the modes need not each have a single frequency, they should be chosen so that their Fourier components (with respect to the Minkowski time \( t \)) are either all positive frequency or all negative frequency. This ensures that the associated operators in the mode expansion of the field have simple interpretations as particle creation and annihilation operators. Specifically, we choose for the Minkowski modes the linear combinations of Rindler modes

\[
g^{(1)}_{k_z, k_y, \omega}(x) = A(\omega) \left[ f^{(R)}_{k_z, k_y, \omega}(x) + e^{-\frac{\pi \omega}{a}} \left( f^{(L)}_{-k_z, -k_y, \omega}(x) \right)^* \right]
\]

and

\[
g^{(2)}_{k_z, k_y, \omega}(x) = B(\omega) \left[ f^{(L)}_{k_z, k_y, \omega}(x) + e^{-\frac{\pi \omega}{a}} \left( f^{(R)}_{-k_z, -k_y, \omega}(x) \right)^* \right]
\]

which were shown by Unruh [7] to have only positive frequency Fourier components; here superscripts \( R \) and \( L \) refer to the modes defined in region I and region IV, respectively.

The normalization of these can be fixed by requiring

\[
\langle g^{(i)}_{k_z, k_y, \omega}, g^{(j)}_{k'_z, k'_y, \omega'}, f \rangle_{\text{Mink}} = F(\omega) \delta_{ij} \delta(\omega - \omega') \delta(k_x - k'_x) \delta(k_y - k'_y)
\]

where

\[
(f, g)_{\text{Mink}} \equiv i \int d^3x f^*(x) \overline{\omega} \delta_t g(x)
\]

with the integration is over the hypersurface \( t = 0 \). (Again, for solutions of the field equation, \( (f, g) \) does not depend on the choice of the constant time surface.) Note that if the same choice of \( F(\omega) \) is made in Eqs. (3.4) and (3.8), then for a pair of functions with support only in region I (or only in region IV) \( (f, g)_{\text{Mink}} = (f, g)_{\text{Rind}} \), while if \( f \) has support in region I and \( g \) in region IV, \( (f, g)_{\text{Mink}} = 0 \). Note also that \( (f^*, g^*) = -(f, g) \). It follows that if the Rindler modes (3.2) are properly normalized, then the Minkowski modes (3.6) and (3.7) will be normalized if

\[
A(\omega) = B(\omega) = [2 \sinh(\pi \omega/a)]^{-1/2}
\]

To lowest order in perturbation theory, the amplitude that a source \( \rho(x) \) leads to the
creation from the vacuum of a quantum in the state $|n\rangle$ is

$$A = \langle n| i \int d^4x \sqrt{g} \rho(x) \phi(x)|\text{vac}\rangle$$

(3.11)

The total emission probability is obtained by squaring the amplitudes and summing over all one-particle final states. After expanding the fields in terms of the modes (3.6) and (3.7), one finds that the probability for emission of a Minkowski quantum with transverse momentum $(k_x, k_y)$, assuming that the field was initially in the Minkowski vacuum, is

$$dP_{k_x, k_y}^{\text{Mink}} = \sum_j \int_0^\infty d\omega F^{-1}(\omega) \left| \int d^4x \sqrt{g} \rho(x) g^{(j)}_{k_x, k_y, \omega}(x) \right|^2$$

(3.12)

Now let us assume that $\rho(x)$ vanishes outside of region I. The contribution from the $g^{(1)}$ modes is then

$$dP_{k_x, k_y}^{\text{Mink}, 1} = \int_0^\infty d\omega F^{-1}(\omega) |A(\omega)|^2 \left| \int d^4x \sqrt{g} \rho(x) J^{(R)}_{k_x, k_y, \omega}(x) \right|^2$$

(3.13)

$$= \int_0^\infty d\omega \left[ \frac{1}{e^{2\pi \omega/a} - 1} + 1 \right] \frac{dp_{k_x, k_y}^{\text{Rind}}(\omega)}{d\omega}$$

where $dp_{k_x, k_y}^{\text{Rind}}/d\omega$ is the emission probability per unit frequency range in the Rindler vacuum.

The factor multiplying $dp_{k_x, k_y}^{\text{Rind}}/d\omega$ converts this vacuum emission probability to the sum of the spontaneous and the induced emission probabilities in a thermal state with temperature $a/2\pi$, which is how the Rindler observer interprets the Minkowski vacuum, Similarly, the contribution from the $g^{(2)}$ modes is

$$dP_{k_x, k_y}^{\text{Mink}, 2} = \int_0^\infty d\omega F^{-1}(\omega)|B(\omega)|^2e^{-2\pi \omega/a} \left| \int d^4x \sqrt{g} \rho(x)(f^{(R)}_{-k_x, -k_y, \omega}(x))^* \right|^2$$

(3.14)

$$= \int_0^\infty d\omega \left[ \frac{1}{e^{2\pi \omega/a} - 1} \right] \frac{dp_{-k_x, -k_y}^{\text{Rind}}(\omega)}{d\omega}$$

which is the probability for absorption of a quantum with transverse momentum $(-k_x, -k_y)$ in the same thermal state. Adding Eqs. (3.13) and (3.14), we obtain the desired result.

Let us now verify this result for the special case of the constantly accelerating source. The total transition probability is infinite, since the source is present for all times. We therefore calculate the emission and absorption rates, defined as transition probabilities per unit proper time of the source.
We start with the Minkowski calculation. Instead of the modes \((3.6)\) and \((3.7)\), we work with plane wave modes. A standard calculation then gives the emission rate

\[
dW_{k_z,k_y}^{Mink} = \frac{1}{(2\pi)^3} \int \frac{dk_z}{(2k_0)} \left| \int d^4x \rho(x) e^{i(k_0 t - \mathbf{k} \cdot \mathbf{x})} \right|^2
\]

\[
= \frac{q^2}{(2\pi)^3} \int \frac{dk_z}{(2k_0)} \int d\tau' \int d\tau'' \times \exp \left[ -i \frac{k_z}{a} (\cosh a\tau' - \cosh a\tau'') + i \frac{k_0}{a} (\sinh a\tau' - \sinh a\tau'') \right]
\]

where \(k_0 \equiv |\mathbf{k}| \equiv (k_z^2 + k_\perp^2)^{1/2}\) and \(T\) represents the (infinite) total proper time along the trajectory of the source. If we write \(\tau \equiv (\tau' + \tau'')/2\) and \(\sigma \equiv (\tau' - \tau'')/2\) and define

\[
\eta = \cosh^{-1} \left[ \frac{k_\perp}{k_\perp} \cosh a\tau - \frac{k_z}{k_\perp} \sinh a\tau \right]
\]

this expression can be rewritten as \([8]\)

\[
dW_{k_z,k_y}^{Mink} = \frac{q^2}{(2\pi)^3} \int \frac{dk_z}{(2k_0)} \int d\tau \int d\eta \int d\sigma \exp \left[ \frac{2ik_\perp \cosh \eta}{a} \sinh a\sigma \right]
\]

\[
= \frac{q^2}{4\pi^3a} \int d\eta K_0 \left( \frac{2k_\perp \cosh \eta}{a} \right)
\]

\[
= \frac{q^2}{4\pi^3a} \left| K_0 \left( \frac{k_\perp}{a} \right) \right|^2
\]

Here we have cancelled the factor of \(T\) by the integral \(\int_{-\infty}^{+\infty} d\tau\).

We want to compare this result with the Rindler emission and absorption rates. There is a problem here because the source is static in Rindler coordinates. Normally, one would then conclude that there was also no induced emission. However, because the density of quanta in the FDU thermal bath diverges as the frequency goes to zero, matters are more subtle. We adopt the approach of Ref. 2 and replace the static source of Eq. \((1.5)\) by the time-dependent source

\[
\rho = \sqrt{2q} \cos E\tau \delta(\xi) \delta(x) \delta(y)
\]

The limit \(E \to 0\) will be taken at the end of the calculation.
To proceed further we need an explicit expression for the Rindler modes. If the choice $F(\omega) = 1$ is made in Eq. (3.4), then the appropriately normalized modes are [5]

$$f^{(R)}_{k_x, k_y; \omega} = \frac{1}{2\pi^2} \left[ \sinh(\pi \omega / a) \right]^\frac{1}{2} K_i \left( \frac{k_1}{a} \right) e^{i k_x x + i k_y y - i \omega \tau}$$  \hspace{1cm} (3.19)

where $K_i(z)$ is the Bessel function of imaginary argument. Using this expression together with Eq. (3.18) for the source, and comparing with Eq. (3.13), we see that the emission rate per unit frequency range in the Rindler vacuum is

$$\frac{d\omega^{Rind}_{k_x, k_y} (\omega)}{d\omega} = \frac{q^2}{2\pi^2 a T} \sinh(\pi \omega / a) \left[ K_i \left( \frac{k_1}{a} \right) \right]^2 \left[ \int_{-\infty}^{+\infty} d\tau e^{-i \omega \tau} \cos E \tau \right]^2$$  \hspace{1cm} (3.20)

where $T$ represents the length of the total time interval. The integrals over $\tau$ each give factors of $\pi \delta(E - \omega)$. (There are also terms involving $\delta(E + \omega)$; we omit these, since $E$ and $\omega$ are both positive.) Writing $2\pi \delta(0) = T$ allows us to cancel the factor of $1/T$, and leaves us with

$$\frac{d\omega^{Rind}_{k_x, k_y} (\omega)}{d\omega} = \frac{q^2}{4\pi^2 a} \sinh(\pi \omega / a) \left[ K_i \left( \frac{k_1}{a} \right) \right]^2 \delta(E - \omega)$$  \hspace{1cm} (3.21)

If the limit $E \to 0$ were taken at this point, we would obtain zero emission, which would be the correct result for a static source in the vacuum. Because we are interested in the FDU thermal bath corresponding to the Minkowski vacuum, we first add the induced emission rate to the spontaneous emission rate of Eq. (3.21), and then integrate over $\omega$, to obtain the total emission rate in the thermal bath

$$dW^{cm;therm}_{k_x, k_y} = \frac{q^2}{4\pi^3 a} \sinh(\pi E / a) \left[ \frac{1}{e^{2\pi E / a} - 1} + 1 \right] \left| K_i \left( \frac{k_1}{a} \right) \right|^2$$  \hspace{1cm} (3.22)

If we now take the limit $E \to 0$, only the contribution from the induced emission survives, giving

$$dW^{cm;therm}_{k_x, k_y} = \frac{q^2}{8\pi^3 a} \left| K_0 \left( \frac{k_1}{a} \right) \right|^2$$  \hspace{1cm} (3.23)

The absorption rate in the thermal bath is equal to the induced emission rate. Because both rates are independent of the sign of $k_x$ and $k_y$, we can simply double the above result to obtain

$$dW^{At;therm}_{k_x, k_y} = \frac{q^2}{4\pi^3 a} \left| K_0 \left( \frac{k_1}{a} \right) \right|^2$$  \hspace{1cm} (3.24)

which is indeed equal to the Minkowski result (3.17) for the emission rate.
APPENDIX

In this appendix, we study some general properties of classical scalar radiation. First, we consider a system of scalar charged point particles interacting with the scalar field $\phi(x)$ which they produce. The total action is

$$
S = \int d^4x \sqrt{-g} \frac{1}{2} (\nabla \phi)^2 + \sum_n \int d\lambda \left[ -g_{\mu\nu}(x_\lambda) \left( \frac{dx_\mu(\lambda)}{d\lambda} \frac{dx_\nu(\lambda)}{d\lambda} \right) \right]^{1/2} + \sum_n \int d\lambda \phi(x_\lambda) \left[ -g_{\mu\nu}(x_\lambda) \left( \frac{dx_\mu(\lambda)}{d\lambda} \frac{dx_\nu(\lambda)}{d\lambda} \right) \right]^{1/2}
$$

(A.1)

Varying the action with respect to metric gives the energy-momentum tensor (in Minkowski space)

$$
T^{\mu\nu}_{\text{tot}}(x) = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \sum_n \int d\tau_n [m_n + q_n \phi(x_\lambda)] \frac{dx_\mu}{d\tau_n} \frac{dx_\nu}{d\tau_n} \delta^{(4)}(x - x_n)
$$

(A.2)

$$
\equiv T^{\mu\nu} + T^{\mu\nu}_{\text{matter}}
$$

The conservation of $T^{\mu\nu}_{\text{tot}}$ can be explicitly verified using the equations of motion.

Next we study the radiation field and power of a single point scalar charge moving along the path $\mathbf{x}_s(t)$. The Liénard-Wiechert potential is

$$
\phi(\mathbf{x}, t) = \frac{q}{4\pi} \sqrt{1 - \mathbf{v}^2} \frac{1}{r - \mathbf{v} \cdot \mathbf{r}}
$$

(A.3)

where $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_s(t')$, $\mathbf{v} \equiv \mathbf{v}(t')$, $t' \equiv t - r$. We have

$$
\frac{\partial t'}{\partial t} = \frac{1}{1 - \mathbf{v} \cdot \mathbf{r}}, \quad \nabla t' = -\frac{\dot{\mathbf{r}}}{1 - \mathbf{v} \cdot \mathbf{r}}
$$

(A.4)

Introducing $s \equiv 1 - \mathbf{v} \cdot \dot{\mathbf{r}}$ and keeping only the leading $1/r$ dependence, we obtain the radiation fields

$$
\partial_t \phi(\mathbf{x}, t) = \frac{\partial \phi}{\partial t'} \frac{\partial t'}{\partial t} = \frac{q}{4\pi} \frac{1}{rs^3} \left[ (\dot{\mathbf{v}} \cdot \dot{\mathbf{r}}) \sqrt{1 - \mathbf{v}^2} - s \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{\sqrt{1 - \mathbf{v}^2}} \right]
$$

(A.5)

$$
\nabla \phi(\mathbf{x}, t) = \frac{\partial \phi}{\partial t'} \nabla t' = -\frac{q}{4\pi} \frac{1}{rs^3} \left[ (\dot{\mathbf{v}} \cdot \dot{\mathbf{r}}) \sqrt{1 - \mathbf{v}^2} - s \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{\sqrt{1 - \mathbf{v}^2}} \right]
$$

(A.6)

where all quantities on the right-hand side are to be evaluated at $t' = t - r$. The total power
radiated is

\[
P(t') = \int r^2 d\Omega \dot{T}_{rt}(\mathbf{x}, t) \frac{dt}{dt'} = \int r^2 d\Omega \dot{\mathbf{r}} \cdot \partial^t \phi \frac{dt}{dt'}
\]

\[
= \left( \frac{q}{4\pi} \right)^2 \int d\Omega \frac{1}{s^2} \left[ (\dot{\mathbf{v}} \cdot \dot{\mathbf{r}}) \sqrt{1 - \mathbf{v}^2} - s \mathbf{v} \cdot \dot{\mathbf{v}} \frac{\dot{\mathbf{v}}}{\sqrt{1 - \mathbf{v}^2}} \right]^2
\]

where the integration is over a large sphere along the forward light-cone of the source and centered at \(\mathbf{x}_s(t')\). The integration can be carried out by introducing a coordinate system such that \(\mathbf{v} \cdot \dot{\mathbf{r}} = v \cos \theta, \dot{\mathbf{v}} \cdot \mathbf{v} = v |\dot{\mathbf{v}}| \cos \alpha, \mathbf{v} \cdot \dot{\mathbf{r}} = |\dot{\mathbf{v}}|(\cos \theta \cos \alpha + \sin \theta \sin \alpha \cos \varphi)\). One obtains

\[
P = \frac{q^2}{4\pi^3} \frac{1}{3} \frac{d^2 x^\mu}{d\tau^2} \frac{d^2 x_\mu}{d\tau^2} = \frac{q^2}{4\pi^3} \left[ \gamma^2 \dot{\mathbf{v}}^2 + \gamma^6 (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \right]
\]

This is half the electromagnetic value. The result can also be obtained by studying the non-relativistic limit of Eq.(A.7) and then using Lorentz invariance.

**REFERENCES**


