Back-Reaction of Cosmological Perturbations

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1. Motivation

2. Back-Reaction Problem for cosmological perturbations

3. Application to Inflationary Cosmology

4. Speculations

N.B. Similar work for graviton back-reaction:

N. Tsamis & R. Woodard

Phys. Lett. B 301, 351 (93)

Nucl. Phys. B 474, 235 (96)

...
1. Motivation

Back Reaction of Gravitational Waves

well known: Brill & Hartle (64)

Isaacson (68) ...

- gravitational waves carry $\Phi$
  $\Rightarrow$ influence background space-time

- back reaction described by $T_{\mu\nu}^{\text{eff}}$

- cumulative effect: all modes contribute and add up

- small amplitude waves can have a large effect

Our problem: back-reaction of scalar cosmological perturbations (density perturbations)

why important:

- scalar perturbations important in inflationary cosmology

- back-reaction can influence dynamics of inflation

- back-reaction $\Rightarrow$ changes of background metric
  $\Rightarrow$ changes to global cosmol. parameters
2. Framework

\[ t \uparrow M_0 \quad x \Downarrow M \]

**Question:** How do average quantities on \( M \) evolve compared to quantities on \( M_0 \)?

\[ M: \quad \gamma_{\mu \nu} = \gamma^{(0)}_{\mu \nu} + \delta \gamma_{\mu \nu} \quad \text{metric} \]
\[ \rho = \rho^{(0)} + \delta \rho \quad \text{matter} \]

\( (\delta \gamma_{\mu \nu}, \delta \rho) \) quantum fluctuations on classical FRW background \((\gamma^{(0)}_{\mu \nu}, \rho^{(0)})\)

\[ S(\gamma_{\mu \nu}, \rho) = S(\gamma^{(0)}_{\mu \nu}, \rho^{(0)}) + S_2 (\delta \gamma_{\mu \nu}, \delta \rho / \delta \gamma_{\mu \nu}, \rho^{(0)}) + \ldots \quad \uparrow \text{canonically quantized} \]

\[ < \delta \rho > = 0 \]
\[ < (\delta \rho)^2 > \neq 0 \]
\[ \delta \rho = O(\epsilon) \]
\[ q = (\gamma_{\mu \nu}, \rho) \]
Equations:
\[
< \mathcal{E}_{\mu \nu} > = 8\pi G < T_{\mu \nu} >
\]

expanded to \( O(\epsilon^2) \)

\[
\mathcal{E}_{\mu \nu}^{(0)} + < \mathcal{E}_{\mu \nu}^{(2)} > = 8\pi G ( T_{\mu \nu}^{(0)} + < T_{\mu \nu}^{(2)} > ) + O(\epsilon^3)
\]

extract averaged quantities:

\[
\mathcal{E}_{\mu \nu}^{(0, br)} = 8\pi G ( T_{\mu \nu}^{(0)} + \mathcal{E}_{\mu \nu} )
\]

\[
\mathcal{E}_{\mu \nu} = < T_{\mu \nu}^{(2)} > - \frac{1}{8\pi G} < \mathcal{E}_{\mu \nu}^{(2)} >
\]

defines a homogeneous & isotropic metric \( \mathcal{E}_{\mu \nu}^{(0, br)} \)
which includes effects of back-reaction to \( O(\epsilon^2) \)

\( \mathcal{E}_{\mu \nu} \equiv \) effective energy-momentum tensor for cosmological perturbations
Gauge Problem

consider unperturbed space-time viewed from a different coordinate system \((\tilde{x}, \tilde{t})\)

\[\langle (\delta g)^2 \rangle \neq 0 \implies \tau_{\mu \nu} \neq 0\]

Resolution

To \(O(\epsilon^2)\) background quantities change under a coordinate transformation

\[(*) \quad \tilde{q}_0 = q_0 - \langle \chi \delta q \rangle + \frac{1}{2} \langle \chi^2 q_0 \rangle\]

\(\chi\) : generator of coordinate transformation

\(\delta\) : Lie derivative

Taking \((*)\) into account, the back-reaction equation

\[G_{\mu \nu}^{(0, \epsilon)} = 8\pi G (T_{\mu \nu}^{(0)} + \tau_{\mu \nu})\]

is well defined.
3. Application to Inflationary Cosmology

$\psi \sim 1$ : inflation ends

Inflationary dynamics:

Quantum fluctuations $\rightarrow$ perturbations of $(\nabla \psi, \psi)$

Questions:

- What is Equation of State of back-reaction?
- When is back-reaction important?

\[ ds^2 = (1 + 2\psi) \, dt^2 - a^2(t) \, (1 - 2\psi) \delta_{ij} \, dx^i \, dx^j \]

$\psi = \psi_0 + \delta \psi$
Effective EMT for Cosmological Perturbations

\[ \Gamma^{(1)}_{\mu
u} = 8\pi G (T^{(1)}_{\mu\nu} + \Sigma_{\mu\nu}) \]  
back reaction problem

\[ \phi \] metric perturbation

\[ \delta \psi \] matter " "

\[ \Sigma_{00} = \frac{1}{8\pi G} \left[ 12 H \langle \dot{\phi} \phi \rangle - 3 \langle \dot{\phi}^2 \rangle + 3 a^{-2} \langle \phi^2 \rangle \right] 
+ \frac{1}{2} \langle \delta \psi^2 \rangle + \frac{1}{2} \alpha^{-2} \langle \nabla \delta \psi \rangle^2 
+ \frac{1}{2} \nabla''(y_0) \langle \delta \psi^2 \rangle + 2 \nabla'(y_0) \dot{\phi} \delta \psi \]

\[ \Sigma_{ij} = a^2 \delta_{ij} \left\{ - \frac{1}{8\pi G} \left[ \left( 24 H^2 + 6 \dot{H} \right) \langle \phi^2 \rangle + 24 H \langle \dot{\phi}^2 \rangle + 4 \langle \phi \dot{\phi} \rangle - \frac{4}{3} a^{-2} \langle \nabla \phi \rangle^2 \right] 
+ 4 \dot{y}_0^2 \langle \phi^2 \rangle + \frac{1}{2} \langle \delta \psi^2 \rangle - \frac{1}{\epsilon} a^{-2} \langle \nabla \delta \psi \rangle^2 
- 4 \dot{y}_0 \delta \psi \phi \right\} 
- \frac{1}{2} \nabla''(y_0) \langle \delta \psi^2 \rangle + 2 \nabla'(y_0) \dot{\phi} \delta \psi \right\} \}
Evaluation of $<\phi^2>$:

$$V(y) = \frac{1}{2} m^2 y^2$$

$$<\phi^2> = \int dk \frac{1}{k} |\delta_k|^2$$

$$S_k(t) = \frac{m}{2\sqrt{6} \pi} \left\{ \begin{array}{ll}
1 & k > H(t)a(t) \\
\left[ 1 - \frac{4}{y_0^2(t)} \log \frac{k}{H(t)a(t)} \right] & H(t)a(t) < k < H(t)a(t_i) \\
0 & k < H(t)a(t_i) \end{array} \right.$$  

$$<\phi^2>(t) = \frac{2}{9\pi^2} m^2 \frac{N^3(t_i,t)}{y_0^4(t)} + \frac{1}{24\pi^2} m^2 \log \left[ \frac{\varepsilon}{H(t)a(t)} \right]$$

$N(t_i,t)$: # of e-foldings between $t_i$ and $t$

$t_i$: beginning of inflation

$k_i = H(t_i)a(t_i)$: IR cutoff

$\varepsilon$: UV cutoff

Long wavelength contribution: dominates at late times
Evaluation: long wave limit in inflationary (slow roll) cosmology:

solution of linear gravitational perturb. eqs.: \[ \phi(x,t) = C(x) \]

constraint (Einstein) equation: \[ \dot{\phi} + H \phi = 4\pi G \rho_0 \delta \phi \Rightarrow \delta \rho(\phi, \rho_0) \]

slow rolling: \[ \dot{\rho}_0 \approx -\frac{V'}{3H} \]

FRW & inflation: \[ H^2 = \frac{8\pi G}{3} \rho \]

\[ \downarrow \text{combination} \]

\[ \delta \rho = -\frac{2V}{V'} \phi \]

\[ \downarrow \dot{\rho}, \delta \rho, \delta \phi \text{ negligible} \]

\[ \begin{align*}
    \rho_{(2)}^{(2)} &= 2V' \langle \phi \delta \phi \rangle + \frac{1}{2} V'' \langle \delta \phi^2 \rangle \\
    \rho_{(2)} &= 2V' \langle \phi \delta \phi \rangle - \frac{1}{2} V'' \langle \delta \phi^2 \rangle + \frac{3}{2} \langle \phi^2 \rangle \rho \end{align*} \]

\[ \downarrow \]

\[ \begin{align*}
    \rho_{(2)}^{(1)} &= \left( 2 \frac{V''V^2}{V_{12}^2} - 4V \right) \langle \phi^2 \rangle \\
    \rho_{(2)}^{(2)} &= \left( -2 \frac{V''V^2}{V_{12}^2} - 4V + 8V \right) \langle \phi^2 \rangle = -\rho_{(2)}^{(2)}
\end{align*} \]
Back Reaction applied to Chaotic Inflation

Model: \( V(y) = \frac{1}{2} m^2 y^2 \)

Cosmology:

- Hubble radius \( a_H(t) \)
- Fixed comoving scales

Intuition: \( t \rightarrow \Rightarrow \# \text{ of modes with } \lambda > a_H(t) \rightarrow \Rightarrow \)

\[ R_{\text{uv}} \equiv \int^{(2)} \frac{dk}{k_{H}(t)} \]

Calculation: \[
\left< \phi^2(t) \right>_{\text{W}} = \int \frac{dk}{k} \left| \Phi_k \right|^2 \\
= \frac{m^2 \lambda^2}{32 \pi^4 y_0^4(t)} \int \frac{dk}{k} \left( \ln \frac{H(t) \lambda(t)}{k} \right)^2 \\
= \frac{2}{3} \frac{m^2 \lambda^2}{32 \pi^4 y_0^4(t)} \left( \frac{y_0 \lambda(t)}{\lambda_h^4} \right)^3 \\
\Rightarrow \left[ \frac{2}{3} \int \frac{dk}{k} \right] \sim \frac{3}{4 \pi} \frac{m^2 \lambda^2}{H_p^4} \left( \frac{y_0 \lambda(t)}{\lambda_h^4} \right)^4
\]

Implications: \( \rightarrow \)
\[ y \sim m^{-1} \text{ inflation ends} \]
\[ y \sim m^{-1} \text{ Planck density} \]
\[ y \in [m^{-1/2}, m^{-1}] \text{ stochastic terms important} \]

Our result: If \( y_0(t_*) > m^{-1/3} \) then back-reaction will become important before the "end" of inflation.
4. Speculations

i) \[ \Lambda_{\text{eff}} (t) = \Lambda + \int \sigma_{(2)} (t) \, dt < \Lambda \]

\[ \text{during inflation: } \left| \sigma_{(2)} (t) \right| \to \infty \text{ as } t \to \]

\[ \Rightarrow \Lambda_{\text{eff}} (t) \to \infty \text{ as } t \to \]

\[ \Rightarrow \text{ Dynamical relaxation mechanism for } \Lambda_{\text{eff}} \]

ii) Will \( \Lambda_{\text{eff}} (t) \) stabilize?

Once \( \Lambda_{\text{eff}} (t) < \sigma_{m} (t) \)

\[ \Rightarrow \text{inflation stops} \]

\[ \Rightarrow \text{back-reaction stops increasing (but } \sigma_{m} (t) \to \)]

\[ \Rightarrow \Lambda_{\text{eff}} (t) > \sigma_{m} (t) \]

\[ \Rightarrow \text{scaling solution } \Lambda_{\text{eff}} (t) \sim \sigma_{m} (t) \text{ } \forall t \]

i.e. \( \sigma_{m} (t) \sim 1 \)
Conclusions

1. Cosmological perturbations have an effect on the background space-time (back-reaction) described in terms of
   \[ \tilde{T}_{\mu\nu} = \langle \mathcal{T}^{(2)}_{\mu\nu} - \frac{1}{8\pi G} \mathcal{F}^{(2)}_{\mu\nu} \rangle \]

2. Back-reaction is gauge-invariant

3. Back-reaction effects cumulative, can be important even if the amplitude of the linear perturbations is small. (large phase space)

4. Application to inflationary cosmology:
   \[ \mathcal{F}^{(2)}_{\mu\nu} : \begin{cases} P^{(2)} = -f^{(2)} \\ f^{(2)} < 0 \end{cases} \]
   like a negative cosmological constant!

5. Implication for chaotic inflation with \( V(y) = \frac{1}{2} m^2 y^2 \):
   back-reaction effects become very important if \( y_{\text{olt}} \gg m^{-1/3} \).