Closed String Amplitudes from Gauge Fixed String Field Theory

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I. INTRODUCTION

Over the past few years there has been a resurgence of interest in string field theory as a tool in studying non-perturbative effects in string theory. It was proposed by Sen [1] that condensing the open string tachyon in bosonic string theory will lead to the annihilation of the brane and to the closed string vacuum. In string field theory this should be described by a classical solution whose action is minus the tension of the annihilated brane. \[ \tilde{\phi} \]

This classical solution was studied numerically using level truncation techniques [2, 3, 4] in Witten’s cubic string field theory [5], but to date the analytical solution was not found. Given a classical solution one can re-expand the action around the new vacuum which will result in a new kinetic term, but the same cubic interaction vertex. Instead of finding the classical solution one can guess a form for the kinetic operator in the closed string vacuum and start from there.

Gaiotto et al. [6] proposed a pure ghost midpoint insertion as the new kinetic operator. That led to some beautiful results, but is rather singular. Therefore it makes sense to look for other forms of the kinetic operator in the closed string vacuum.

As a guide to finding this kinetic operator we look at the Feynman rules and find kinetic operators that reproduce closed string diagrams. The Feynman rules were derived from the singular ghost insertion in Ref. 6 using some regularization. The propagator we propose here will be less singular and will not suffer some of the problems encountered there.

In Feynman-Siegel gauge the kinetic operator for the open string is \( e_0(L_0 - 1) \), where \( e_0 \) is the ghost zero mode and \( L_0 - 1 \) the open string Hamiltonian. The propagator is

\[
G = g^2 b_0 \int dt \ e^{-t(L_0-1)},
\]

where \( b_0 \) is the antighost zero mode. The expression \( \exp(-t(L_0 - 1)) \) can be visualized as generating an open worldsheet of length \( t \). The variables \( t \), which are integrated over, become the Feynman parameters of the string graphs and parameterize the moduli space of Riemann surfaces.

This propagator is clearly unsuited for closed strings, since it adds boundaries to the worldsheet. If instead of \( L_0 \) one uses

\[
\tilde{L}_0 = \frac{1}{\pi} \int d\sigma \sin \sigma (T + \tilde{T}),
\]

(2)
this operator will generate a worldsheet without adding any boundary. The reason is that 
the energy momentum tensor will not act at $\sigma = 0$ and at $\sigma = \pi$. This is a realization of an 
idea presented in Ref. 7 that closed strings should arise from boundaries shrinking to points.

In the next section the Feynman rules are derived using this kinetic operator and it is 
demonstrated how they lead to all closed string diagrams.

In section 3 we propose a larger family of kinetic operators, which are all appropriate for 
generating closed surfaces. They are all related to each other by conformal transformations.
In particular, we recover the singular propagator of Ref. 6.

We end with some comments.

II. FEYNMAN RULES

We wish to derive the Feynman rules from the cubic string field theory action with the 
kinetic operator replaced by (2). Using similar definitions for ghost and antighost operators

$$\hat{c}_0 = \frac{1}{\pi} \int d\sigma \sin \sigma (c + \tau), \quad \hat{b}_0 = \frac{1}{\pi} \int d\sigma \sin \sigma (b + \tilde{b}),$$

the action is

$$S = \frac{1}{g_s^2} \int \frac{1}{2} \Phi \ast \hat{c}_0 \hat{L}_0 \Phi + \frac{1}{3} \Phi \ast \Phi \ast \Phi.$$  \hspace{1cm} (4)

Here $\Phi$ is a string field, and the star and integral are defined as usual on string fields [5]. The only difference compared to the standard gauge fixed form is the kinetic operator, involving $\hat{c}_0$ and $\hat{L}_0$, whose mode expansion is

$$\hat{L}_0 = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1 - 4n^2} L_{2n}.$$  \hspace{1cm} (5)

To avoid having too many fields in the theory, one must impose a gauge condition of the 
fields. The appropriate one seems to be $\hat{b} \Phi = 0$.

The propagator derived from this action is

$$G = g_s^2 \hat{b}_0 \int dt \ e^{-iL_0}.$$ \hspace{1cm} (6)

Ignoring the $\hat{b}$ insertion for now, the propagator for given $t$ generates a worldsheet that is a 
segment of a sphere. The points $\sigma = 0, \pi$ are the north and south pole, and are not moved 
by this propagator. The equator at $\sigma = \pi/2$ is moved forward a distance $t$. 

FIG. 1: A schematic depiction of the tree level Feynman diagram contributing to the three point function of closed string vertices. The three lines $op$ make up the interaction vertex, and attached to them are three propagators. At $a$, $b$ and $c$ closed string vertex operators are inserted, and the two halves of the strings labeled by $pa$, by $pb$ and by $pc$ are identified with each other.

Closed strings do not exist as external states in open string field theory. Instead, one can use the open-to-closed string vertices of [8, 9, 10], which can be regarded as the gauge invariant observables of the theory [6, 11]. For any closed string vertex operator $V = c\bar{\tau}V_m$, where $V_m$ is a dimension (1,1) primary in the matter CFT, one defines the gauge invariant operator

$$O_V = \int V(\bar{\tau})\Phi.$$  

(7)

In Feynman diagrams the closed string vertex $V$ will be inserted at the midpoint at the end of a propagator and the right and left parts of the string will be identified by the integral.

Closed string amplitudes are the correlators of any number of those gauge invariant operators. We will concentrate on the three-point function

$$\langle O_1 O_2 O_3 \rangle.$$  

(8)

At tree level it is given by one cubic interaction vertex, three propagators extending from it, with a closed string vertex at the end of each. This is depicted schematically in Fig. 1.

To better visualize the worldsheet it is useful to cut it along the midpoint of the three propagators. Since the propagators were segments of a sphere, we find now six segments of a hemisphere glued to each other, forming one segment of the hemisphere. If the lengths of
FIG. 2: Another depiction of the same diagram as in Fig. 1. Here we cut the propagators along the midpoints $oa$, $ob$ and $oc$, and glued the ends of the propagators marked by $p_b p$ and $p_c p$. The result is a segment of the southern hemisphere, with a deficit angle (where we should glue the two lines marked $ap$).

The three propagators are $t_1, t_2, t_3$, the total angle around the hemisphere is $2(t_1 + t_2 + t_3)$. Since the two ends of the segment are identified, there could be a conical singularity at the pole. This is shown in Fig. 2.

If we fix the total length of the three propagators $t = t_1 + t_2 + t_3$, the integration over the relative lengths covers the moduli space of spheres with four marked points (the three vertices and the pole). This fact will be proven in the next subsection. The three $b$ insertions can be represented as contour integrals on the Riemann surface. Two of them should be associated with the two moduli giving the correct measure over the moduli space. The third, associated with the integration over the total angle, should give some measure for that integral.

So at tree level the correlator of the three closed string vertices in string field theory (8) will be given by a conformal field theory calculation on the sphere with the closed string vertices and one extra marked point. This marked point is the remnant of the shrunken boundary, and we should take care to see what happens there. If, for example, we started with all Neumann conditions we will find that no momentum flows through this marked point. This looks like a zero momentum vertex insertion, and may be a soft dilaton as proposed in [6].

Instead, if we had Dirichlet boundary conditions on the open strings, we will be left with the constraint that this point on the worldsheet is mapped to a fixed point in space. To
remove this constraint we have to integrate over all locations of this D-instanton. Alternatively, one may impose periodic boundary conditions, which will not restrict the position of this point, or the momentum flowing through it, but this is not very natural for open strings.

The result of the conformal field theory calculation is multiplied by a power of the open string coupling $g_s^4$ and by the integral over the extra parameter $t$, call it $T$ (which possibly diverges). At higher order in perturbation theory there are graphs with more marked points. Those come from open string diagrams with more than one boundary, each shrinking to a point. The graph with $m$ shrunken boundaries will have a factor of $g_s^{2+2m}$, and the angle around each marked point is still arbitrary, giving something like $T^m$. Therefore we get the final result

$$\langle O_1 O_2 O_3 \rangle \sim g_s^2 \sum_m (g_s^2 T)^m \langle V_1 V_2 V_3 \rangle_{\text{sphere}, m},$$

where the last correlation function is the conformal field theory result for the sphere with three vertex operators and the $m$ marked points (treated in one of the ways proposed above, or differently).

Other graphs will give similar expressions for the torus and higher genus contributions to this amplitudes. The structure will be very similar, with a series in $g_s^2 T$ multiplying the correct power of the coupling. The same is true for correlators of more closed string vertices.

A. Covering of moduli

To complete the discussion on the closed string diagrams it is necessary to prove that integrating over the Feynman parameters in the string diagrams cover the moduli space of Riemann surfaces. There are two standard ways of demonstrating this, one is by checking that the limits of integration do not leave holes in the moduli space, so that all corners of moduli space can be recovered from the Feynman parameters.

The other method is to find a minimal area problem that leads to the same surfaces as the string diagrams. For every point in moduli space there will be a minimal surface that will correspond to some string diagram [12, 13, 14].

One can use the first method directly. If we represent the surface like in Fig. 2. as a hemisphere with a conical singularity at the pole. This is similar to the open string diagram presented in Fig. 4 of Ref. 12, where instead of a hemisphere there is a cylinder. There too
FIG. 3: By a conformal map it is possible to map the hemisphere in Fig. 2 to a semi-infinite cylinder of circumference $2t$. The point $p$, which is the remnant of the boundary was pushed to infinity.

One has to glue segments of the top of the cylinder, and the moduli space is the same, except for the modulus associated with the size of the boundary, which now is the fixed angle $2t$. The other two parameters represent the relative size of the six segments at the equator (they are pair-wise equal). The corners of moduli space are when two vertex operators approach each other, which clearly can be done.

Like in the open string case, the same is true for arbitrary diagrams. At higher loops one can get disconnected diagrams with several hemispheres, and one has to sum over all different ways of gluing a fixed number of segments on them.

Instead of making this argument more rigorous, we can use the second method of minimal surfaces. This technique cannot be applied directly, since minimal surfaces are flat, with all the curvature concentrated at singular points. Our surfaces have constant positive curvature, apart for the singular points. The trick is to map the hemisphere to a semi-infinite cylinder using $u = \ln \tan(\sigma/2)$. This is a conformal transformation, so we can study all the surfaces with this flat metric, instead of the curved one. This can be seen in Fig. 3.

The original surfaces had in general $n$ closed string punctures, and $m$ shrunken boundaries with angles $2t_i$. The new surfaces will be made up of $m$ semi-infinite tubes of circumferences $2t_i$ and with $n$ punctures where they are glued. Those surfaces solve a minimal area problem very similar to the one discussed in Ref. 6, and the proof carries over. Given a Riemann surface with $n + m$ punctures and numbers $t_i$ associated to $m$ of them, find the minimal area metric
such that all curves homotopic to the puncture \( i \) are at least of length \( 2t_i \).

Since all curves homotopic to the \( m \) punctures have a finite size, the punctures are pushed away to infinity, and we are left with \( m \) semi-infinite cylinders glued together and with the \( n \) punctures at the end where the cylinders are glued.

### III. GENERALIZATIONS

The kinetic operator \( \hat{L}_0 \) is not unique in generating closed surfaces. Instead of the function \( \sin \sigma \) appearing in the definition (2) we can use any other function \( f(\sigma) \) subject to the constraints that it is positive, except for a simple zero at \( f(0) = f(\pi) = 0 \), and that \( f(\pi - \sigma) = f(\sigma) \). The first condition is required so the propagator will not generate a boundary and the second one so gluing the two halves of the string would not generate a line singularity.

The factor of \( \sin \sigma \) in the definition of \( \hat{L}_0 \) corresponds to the standard metric on the sphere \( ds^2 = d\sigma^2 + \sin^2 \sigma dt^2 \). For a general function the propagator will generate segments of a deformed sphere, with the metric \( ds^2 = d\sigma^2 + f(\sigma)^2 dt^2 \).

One special case is to take \( f(\sigma) = \lim_{\epsilon \to 0} \epsilon \), which gives the sphere squeezed into a very thin cylinder. This is very close to the regularized propagator considered in Ref. 6. To be more rigorous, one should take a function that is equal to \( \epsilon \) almost everywhere, and approaches zero at the boundaries. If one rescales the infinitesimal cylinder to finite size it will give the Feynman graphs considered at the end of the last section, and depicted in Fig. 3.

It is easy to prove the equivalence of all those propagators, a simple coordinate transformation maps the sphere with the funny metric to the usual sphere

\[
\ln \tan \frac{\sigma}{2} = \int \frac{d\sigma'}{f(\sigma')}.
\]

This is a conformal transformation, so all those graphs are equivalent. Clearly the two simplest choices are the round sphere and the infinite cylinder.

### IV. CONCLUSIONS

We have constructed a family of gauge fixed kinetic operators on open string fields that give closed string diagrams when used to calculate the correlators of closed string vertices.
This construction could serve as a guide to finding the gauge invariant kinetic operator at the vacuum of string field theory. This would be analogous to the way the BRST operator was identified as the gauge invariant kinetic operator for open string field theory [15, 16, 17, 18].

It would be interesting to check these operators in a real calculation. One way to do that is doing the explicit conformal field theory calculations, mapping the surfaces to the sphere like was done for open strings in Ref. 19. One may also use an algebraic approach, like in Ref. 20, but in the standard oscillator basis the kinetic operators that vanish on the boundary (5) are very non-diagonal. The calculation might be simpler in a different basis.

Another problem is the nature of the series in equation (9). This kind of sum should be expected in any formalism of closed strings in terms of open strings. Starting with a Riemann surface with no handles, but any number of boundaries and shrinking them to points will give the sphere with extra marked points. So many open string diagrams will lead to the same closed string diagram. In particular it would be nice if $T \sim 1/g_s$, and the series converged to some number.

This kind of kinetic operator can be used also for the superstring, again generating closed surfaces. But finding the interaction terms in the action at the closed string vacuum might be a difficult problem.

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