Neutrino mass matrix: inverted hierarchy and CP violation

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Abstract

We reconstruct the neutrino mass matrix in flavor basis, in the case of inverted mass hierarchy (ordering), using all available experimental data on neutrino masses and oscillations. We analyze the dependence of the matrix elements $m_{\alpha\beta}$ on the CP violating Dirac, $\delta$, and Majorana, $\rho$ and $\sigma$, phases, for different values of the absolute mass scale. We find that the present data admit various structures of the mass matrix: (i) hierarchical structures with a set of small (zero) elements; (ii) structures with equalities among various groups of elements: $e$-row and/or $\mu\tau$-block elements, diagonal and/or off-diagonal elements; (iii) “democratic” structure. We find values of phases for which these structures are realized. The mass matrix elements can anti-correlate with flavor: inverted partial or complete flavor alignment is possible. For various structures of the mass matrix we identify possible underlying symmetry. We find that the mass matrix can be reconstructed completely only in particular cases, provided that the absolute scale of the mass is measured. Generally, the freedom related to the Majorana phase $\sigma$ will not be removed, thus admitting various types of mass matrix.

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1 Introduction

Neutrino masses and mixing are considered to be the manifestation of physics beyond the Standard Model. The question is: how far beyond? One way to answer is to confront various models of neutrino masses with experimental results in terms of mass squared differences and mixing angles. In this approach a typical situation is that predictive models (with restricted number of free parameters) do not reproduce the data well. The introduction of additional free parameters allows to describe the data. However, in this case the predictiveness is lost.

In this connection, it is worthwhile to elaborate on the bottom-up approach: to reconstruct the underlying physics, or at least to get some hint of this physics, starting immediately from experimental data. The data include information on mass squared differences and mixing angles, which appear as independent observables. The mass matrix unifies information on the masses and the angles, as well as on possible complex phases, and therefore may give some additional insight. So, the bottom-up approach could consist of the following steps:

(i) reconstruction of the mass matrix in the flavor basis, where the charged lepton mass matrix is diagonal;
(ii) search for the symmetry basis and the energy scale at which the underlying flavor symmetry could be realized (broken);
(iii) identification of the underlying physics.

Several remarks are in order:
- complete determination of the mass matrix may be practically impossible;
- it may happen that there is no underlying symmetry at all;
- neutrino mass matrix may receive several different contributions from different processes and mass scales.

The hope is that the (at least partial) reconstruction of the mass matrix in flavor basis, the searches of its regularities and the study of dependence of the matrix structure on basis may give some hint of the mechanism of neutrino mass generation.

The neutrino mass matrices compatible with neutrino oscillation data [1] have been extensively studied in literature [2, 3, 4]. However, in most of previous works (see, e.g., [3]) the structure of the mass matrix has been analyzed in the assumption of definite CP parities of the three neutrinos, which is equivalent to absence of CP violation. Moreover, exact bi-maximal mixing has been often considered. Usual assumptions are either strongly hierarchical or completely degenerate spectra. Special attention has been given to the small elements of the mass matrix. In some recent works [5] the matrices with two independent entries exactly equal to zero have been classified.

In the paper [6], we have analyzed in detail the structure of the Majorana mass matrix of
neutrinos, for normal hierarchy (ordering) of the mass spectrum. In this case, the electron flavor is concentrated in the two light eigenstates. We have found that the structure of the mass matrix strongly depends on the CP-violating phases. New possible structures have been identified. Parameterizations of the matrix in terms of powers of a unique expansion parameter are given.

Here we will complete the analysis, studying the case of inverted hierarchy (ordering) [3, 4]. In this case the electron flavor is mainly present in the two heavy states.

In general, the spectra with normal and inverted hierarchy have different phenomenology (cosmological consequences, absolute mass scale, neutrinoless beta decay rate, oscillations). In oscillations the difference between the two spectra appears if 1-3 mixing differs from zero. At present, the only observation which could be sensitive to the mass hierarchy is the neutrino burst from SN1987A. It was shown that the data (especially, energy spectra detected by Kamiokande and IMB) can be better described in the case of normal hierarchy with Earth matter effect to be taken into account [7]. The inverted mass hierarchy is disfavored (see, however, [8]). These statements depend on the original neutrino spectra produced in the star as well on the value of 1-3 mixing. Recent calculations show that the difference of the fluxes of different neutrino types can be rather small [9], thus diminishing possible oscillation effects and therefore difference of predictions for normal and inverted hierarchy.

The goal of this paper is to present the most general study of possible structures of the mass matrix without additional assumptions. In particular, we perform comprehensive analysis of dependence of the matrix elements on CP-violating phases. We will show that the assumptions of definite CP-parities, bi-maximal mixing, strictly hierarchical or degenerate spectrum and exactly zero elements exclude a number of interesting matrix structures.

The paper is organized as follows. In section 2 we reconstruct the mass matrix in flavor basis and describe the method of our analysis. In section 3 we study possible structures of the matrix: hierarchical structures (section 3.1), structures with equalities of matrix elements (3.2), structures with inverted flavor alignment (3.3). In section 4 we describe the dependence of the matrix structure on the mass spectrum, analyzing the case of strong inverted hierarchy (section 4.1), inverted ordering (4.2) and degeneracy (4.3). In section 5 we summarize the main results of our analysis.
2 Reconstruction of the mass matrix

2.1 Parameterization and experimental input

We consider the mass and mixing pattern for three Majorana neutrinos. The flavor neutrino states are related to the mass eigenstates by the unitary mixing matrix $U$:

$$
\nu_{\alpha L} = U_{\alpha i} \nu_{i L}, \quad \alpha = e, \mu, \tau, \quad i = 1, 2, 3.
$$

In the flavor basis, the Majorana mass matrix $M$ can be written as

$$
M = U^* M^{\text{diag}} U^\dagger, \quad M^{\text{diag}} \equiv \text{diag}(m_1 e^{-2i\rho}, m_2, m_3 e^{-2i\sigma}), \quad (1)
$$

where $m_i \ (i = 1, 2, 3)$ are the moduli of the neutrino mass eigenvalues; $\rho$ and $\sigma$ are the CP violating Majorana phases, varying between 0 and $\pi$.

We use the standard parameterization for the mixing matrix $U$:

$$
U = \begin{pmatrix}
c_{13} c_{12} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_{23} - s_{23} s_{13} c_{12} e^{i\delta} & c_{23} c_{12} - s_{23} s_{13} s_{12} e^{i\delta} & s_{23} c_{13} \\
-s_{23} s_{12} - s_{13} c_{23} e^{i\delta} & -s_{23} c_{12} - s_{13} s_{12} s_{23} e^{i\delta} & c_{23} c_{13}
\end{pmatrix}, \quad (2)
$$

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$ and $\delta$ is the CP violating Dirac phase. The mixing angles vary between 0 and $\pi/2$ and $\delta$ between 0 and $2\pi$.

The choice of the parameterization in Eqs.(1,2) is convenient once all elements of mass matrix are considered [6]. In particular, dependence of the matrix on the phase $\delta$ is associated with the small parameter $s_{13}$, so that the $\delta$-dependence disappears when $s_{13} \to 0$. Furthermore, in the case of strong mass hierarchy, $m_3 \to 0$, the dependence of the matrix on the phase $\sigma$ also disappears.

We consider the mass and mixing pattern which explains the atmospheric neutrino results by $\nu_\mu - \nu_\tau$ oscillations as the dominant mode and solves the solar neutrino problem via the LMA MSW conversion. Correspondingly, the mass split between $\nu_1(\nu_2)$ and $\nu_3$ states is determined by the atmospheric neutrino mass squared difference $\Delta m^2_{\text{atm}}$ and the mass split between $\nu_1$ and $\nu_2$ ("solar" neutrino pair) by the solar mass squared difference $\Delta m^2_{\text{sol}}$. In the case of inverted mass hierarchy (ordering), the $\nu_1$ and $\nu_2$ states have masses larger than the third state and $m_2$ is the largest mass ($m_2 > m_1 > m_3$):

$$
m_1^2 \equiv m_3^2 + \Delta m^2_{\text{atm}}, \quad m_2^2 \equiv m_1^2 + \Delta m^2_{\text{sol}}. \quad (3)
$$

We use the following experimental results from neutrino oscillations [1], given at 90%
\[ \Delta m_{\text{sol}}^2 = (6.2 \pm 10^{-3}) \cdot 10^{-5} \text{ eV}^2 ; \]
\[ \Delta m_{\text{atm}}^2 = (2.5 \pm 1.4) \cdot 10^{-3} \text{ eV}^2 ; \]
\[ \tan^2 \theta_{12} = 0.41^{+0.2} \cdot 10^{-3} ; \]
\[ \tan \theta_{23} = 1^{+0.35} \cdot 0.25 ; \]
\[ s_{13} \lesssim 0.2 . \]  

The absolute mass scale and the three CP violating phases will be considered as free parameters.

Important restrictions on the possible structures of the mass matrix come from the upper bound on the 1-2 mixing. The relevant quantity is the deviation of 1-2 mixing from maximal, which can be characterized by \( \cos 2\theta_{12} \). Recent analysis of the solar neutrino data [10] gives:

\[ \cos 2\theta_{12} > 0.25 \quad (0.16) , \quad 90\% \quad \text{C.L.} \quad (99\% \quad \text{C.L.}) . \]  

It is convenient to introduce dimensionless parameters:

\[ r \equiv \frac{m_3}{m_2}, \quad k \equiv \frac{m_1}{m_2} . \]

The hierarchy parameter, \( r \), is given by:

\[ r \approx \sqrt{1 - \frac{\Delta m_{\text{atm}}^2}{m_2^2}} \]  

and, for strong inverted hierarchy (ordering), \( r \ll 1 \) (\( r < 1 \)). A distinctive feature of the inverted mass spectrum is that the states \( \nu_1 \) and \( \nu_2 \) are strongly degenerate in mass, for any value of \( r \), so that \( k \) is very close to 1:

\[ k \approx 1 - \epsilon, \quad \epsilon \equiv \frac{\Delta m_{\text{sol}}^2}{2(m_3^2 + \Delta m_{\text{atm}}^2)} \leq \frac{\Delta m_{\text{sol}}^2}{2\Delta m_{\text{atm}}^2} \lesssim 10^{-2} . \]

As a consequence, the solar mass scale turns out to be very weakly imprinted into the structure of the mass matrix.

In general, the elements of the matrix are complex quantities:

\[ M_{\alpha\beta} = m_{\alpha\beta} e^{i\phi_{\alpha\beta}} . \]
linear combinations of phases are independent and have physical meaning. In what follows we will study, mainly, the absolute values of $M_{\alpha\beta}$, which give straightforward information on the matrix structure.

Due to maximal or near maximal 2-3 mixing, in the analysis it is convenient to divide the mass matrix elements in two groups: the $e$-row elements, $m_{ee}$, $m_{e\mu}$, $m_{e\tau}$, and the $\mu\tau$-block elements, $m_{\mu\mu}$, $m_{\mu\tau}$, $m_{\tau\tau}$.

2.2 The limit $s_{13} = 0$ and $m_1 = m_2$: zero order matrix

Let us introduce the matrix for $s_{13} = 0$ and $\epsilon = 0$:

$$m^0_{\alpha\beta} \equiv m_{\alpha\beta}(s_{13} = 0, \epsilon = 0),$$

which we will call the zero order matrix. This matrix gives rather precise approximation and it allows one to identify possible dominant and sub-dominant structures of the exact matrix.

It is useful to introduce

$$X \equiv xe^{i\phi_x} \equiv s_{12}^2 e^{-2i\rho} + c_{12}^2,$$

where the absolute value, $x$, and the phase, $\phi_x$, equal

$$x = \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \rho}, \quad \phi_x = -\arctan \left( \frac{\sin 2\rho}{\cot^2 \theta_{12} + \cos 2\rho} \right).$$

The zero order matrix can be written as

$$m^0 = \sqrt{\frac{\Delta m^2_{\text{atm}}}{1 - r^2}} \begin{pmatrix} x & c_{23} \sqrt{1 - x^2} & s_{23} \sqrt{1 - x^2} \\ \vdots & c_{23} x + s_{23}^2 r e^{-2i\sigma_x} & s_{23} c_{23} - x + r e^{-2i\sigma_x} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where

$$\sigma_x \equiv \sigma + \phi_x / 2$$

varies in the interval $0 \div \pi$.

Let us consider the properties of the matrix (12).

1) It depends on four independent parameters: $x = x(\rho, \theta_{12}), \sigma_x = \sigma_x(\sigma, \rho, \theta_{12}), r, \theta_{23}$. According to the experimental input (4), we find that these parameters are restricted, at 90% C.L., in the following ranges:

$$r \in [0, 1], \quad c_{23}^2 \in [0.35, 0.65], \quad x \in [\cos 2\theta_{12}, 1] = [0.25, 1], \quad \sigma_x \in [0, \pi].$$
2) Using the bounds (14), we get the following maximal and minimal values of the matrix elements:

\[ m_{ee} \in m^2[0.25, 1] , \quad m_{e\mu(\tau)} \in m^2[0, 0.6] , \quad m_{\mu\mu(\tau\tau)} \in m^2[0, 1] . \tag{15} \]

For previous studied of the allowed values of the matrix elements see [11].

3) CP is conserved only for extreme values of \( x \):

\[ x = x_{\text{min}} \equiv \cos 2\theta_{12} : \quad \rho = \pi/2 ; \]
\[ x = x_{\text{max}} \equiv 1 : \quad \rho = 0 . \]

4) The best fit value of 1-2 mixing (according to the LMA solution of the solar neutrino problem) implies

\[ x_{\text{bf}}^{\text{min}} \approx 0.42 . \tag{16} \]

The upper bound (5) on 1-2 mixing gives \( x_{\text{min}} > 0.25 \). These results have important implications for the structure of the mass matrix. In particular, \( m_{ee} \) cannot be small:

\[ m_{ee} > \cos 2\theta_{12}m_2 > 0.25m_2 > 0.25\sqrt{\Delta m^2_{\text{atm}}} . \tag{17} \]

The \( \mu\tau \)-block elements in Eq.(12) can be small only if \( r \) is equal or larger than \( x_{\text{min}} \), that is in the case of non hierarchical spectrum.

5) The six elements of matrix (12) are functions of only four parameters, so that there are two relations among the elements:

\[ m^2_{ee} + m^2_{e\mu} + m^2_{e\tau} = m^2_2 , \tag{18} \]
\[ (m^2_2 - m^2_{ee})(m^2_{\mu\mu} - m^2_{\tau\tau}) = (m^2_{e\mu} - m^2_{e\tau})(2m^2_{ee} - \Sigma_{\mu\tau}) , \tag{19} \]

where

\[ \Sigma_{\mu\tau} \equiv m^2_{\mu\mu} + m^2_{\tau\tau} + 2m^2_{\mu\tau} \tag{20} \]
is the sum of \( \mu\tau \)-block elements squared. Moreover, the four parameters are restricted to the ranges given in (14). Therefore, the matrix structure is constrained and there are correlations among the values of different elements.

There are two other useful relations for the zero order matrix elements:

\[ m^2_{e\mu} + m^2_{e\tau} = (1 - x^2)m^2_2 , \quad \Sigma_{\mu\tau} = (x^2 + r^2)m^2_2 . \tag{21} \]

The sum of all matrix elements squared is equal to the sum of mass eigenvalues squared [6]:

\[ \sum_{\alpha,\beta} m^2_{\alpha\beta} = (2 + r^2)m^2_2 . \tag{22} \]

This equality holds also when \( s_{13} \)-corrections are included.
2.3 $\mathcal{O}(s_{13})$ and $\mathcal{O}(\epsilon)$ corrections

The structure of the leading order (linear in $s_{13}$ and $\epsilon$) corrections to $m^0$ can be parameterized by the matrices $m^s$ and $m^\epsilon$:

$$m = \left| m^0 + s_{13}m^s + \epsilon m^\epsilon \right| + \mathcal{O}(s^2_{13}, \epsilon^2, s_{13}\epsilon).$$  \hfill (23)

The upper bound on $s_{13}$ (see Eq.(4)) is not very strong and there is still room for significant corrections to the matrix elements. Moreover, $s_{13}$-terms can give dominant contribution if the elements in $m^0$ are small.

The main features of the matrix $m^s$ (its explicit expression is given in the Appendix) are the following:

- $m^s_{ee} = 0$, $m^s_{e\tau}$ receives corrections only at the order $s^2_{13}$.
- In the case of maximal atmospheric mixing ($\theta_{23} = \pi/4$), also $m^s_{\mu\tau} = 0$. Moreover, corrections to $m^s_{e\mu}$ and $m^s_{e\tau}$ are equal and have opposite sign ($m^s_{e\mu} = -m^s_{e\tau}$). The same is true for $m^s_{\mu\mu}$ and $m^s_{\tau\tau}$: $m^s_{\mu\mu} = -m^s_{\tau\tau}$.
- The elements $m^s_{\alpha\beta}$ can be zero or take their maximal value depending on the value of the Dirac phase $\delta$ (see Eq.(A.6)).

Thus, the $\mathcal{O}(s_{13})$ corrections change the splitting between $m^s_{e\mu}$ and $m^s_{e\tau}$ as well as between $m^s_{\mu\mu}$ and $m^s_{\tau\tau}$ elements. Moreover, this splitting depends strongly on $\delta$.

The smallness of $s_{13}$ could be a signal of an underlying symmetry. The pattern of $s_{13}$ corrections to the mass matrix in flavor basis could suggest how this symmetry is related to the flavor of neutrinos.

For maximal possible value of $s_{13}$, the terms $s_{13}m^s_{\alpha\beta}$ can be as large as $(0.1 \div 0.2)m^2_2$. Future experiments may strengthen the upper bound on $s_{13}$, making the $s_{13}$ corrections even smaller.

Let us describe some general features of the $\epsilon$-corrections (the explicit expression for $m^\epsilon$ is given in the Appendix).

- All matrix elements receive corrections proportional to $\Delta m^2_{sud}$: $m^\epsilon_{\alpha\beta} = 0$ for particular values of the phases $\rho$ and $\sigma$ only.
- In the case of maximal atmospheric mixing ($\theta_{23} = \pi/4$), corrections to $m^\epsilon_{e\mu}$ and $m^\epsilon_{e\tau}$ are equal to each other: $m^\epsilon_{e\mu} = m^\epsilon_{e\tau}$. The same is true for the corrections to $m^\epsilon_{\mu\mu}$ and $m^\epsilon_{\tau\tau}$: $m^\epsilon_{\mu\mu} = m^\epsilon_{\tau\tau}$. 


From Eq.(7) we get for the best fit values of $\Delta m^2$, $\epsilon \lesssim 10^{-2}$, therefore the corrections $m'_{\alpha\beta}$ are about 1% (using for $\Delta m^2$ the ranges in (4), we get $\epsilon \lesssim 0.05$). These corrections, however, have crucial phenomenological consequence: they break the degeneracy between $m_1$ and $m_2$ and thus explain the solar neutrino conversion. The value of the solar mass difference emerges from minor details of the mass matrix (this is not the case for normal hierarchy [6]). The pattern of $O(\epsilon)$ corrections could give some information about the origin of the small parameter $\Delta m^2_{sol}/\Delta m^2_{atm}$.

2.4 $\rho - \sigma$ plots

The dependence of the mass matrix on $s_{13}$ and, consequently, on the phase $\delta$, is rather weak and the mass matrix is mainly determined by $x = x(\rho)$, $r$ and $\sigma_x = \sigma_x(\rho, \sigma)$. Therefore, to perform a complete scanning of possible structures of the matrix, it is convenient to use the $\rho - \sigma$ plots which show lines of constant masses $m_{\alpha\beta}$ in the plane of Majorana phases $\rho$ and $\sigma$ [6]. In the Figs.1-6, we show the $\rho - \sigma$ plots for different values of the hierarchy parameter $r$.

We have taken non-zero values for $\Delta m^2_{sol}$ and $s_{13}$, so that one can identify their effects in the diagrams, as deformations of the zero order form (12) of the mass matrix. For example, in Figs.1-4 and in Fig.6 we take $\theta_{23} = \pi/4$, which implies $m^0_{e\mu} = m^0_{e\tau}$ and $m^0_{\mu\mu} = m^0_{\tau\tau}$. The differences between the plots of these pairs of elements are due to $s_{13}$ terms.

Each point in the $\rho - \sigma$ plots [6] corresponds to physically different mass matrix (obviously, the same point should be taken for all elements). The $\rho - \sigma$ plots allow one immediately to see (i) ranges in which a given matrix element can change, (ii) ranges of phases in which a given element can be zero (small), (iii) correlations among values of different elements.

According to Figs.1-6, a large class of structures is allowed by the present data. Using $\rho - \sigma$ plots, one can immediately identify the regions of parameters for which the matrix has:

- hierarchical structure: in which some elements are very small (white regions) and others are large (dark regions);
- non-hierarchical structure: where all matrix elements are of the same order - have gray color; one can find structures with certain ordering of elements.
- democratic structure: where all elements take the same or nearly the same value.

3 Structures of the mass matrix

In what follows, we will study possible structures of the mass matrix using, first, the zero order approximation (12) and, then, evaluating the possible role of order $s_{13}$ corrections.
3.1 Hierarchical structures and zeros

We will refer to the mass matrix structure as to a hierarchical one if some elements are smaller than others by a factor $0.1 \div 0.2$. Large elements (they should be of order $m_2$) belong to the dominant structure, other elements to the sub-dominant structure. Also further structuring is possible within the dominant and sub-dominant blocks.

The hierarchical structures of the mass matrix may testify for the existence of certain symmetries.

The hierarchical matrices can be found by searching for zeroes (small values) of some elements. They can be identified as white regions in the $\rho - \sigma$ plots. In the analytical treatment, we use the zero order matrix (12).

1) The element $m_{ee}$ cannot be zero (see Eq.(17)). It belongs to the dominant structure of the matrix. The hierarchical structures with $m_{ee} \approx 0$, widely discussed in the literature [3, 4], are strongly disfavored now.

2) The elements $m_{e\mu}$ and $m_{e\tau}$ are simultaneously zero (small) when $x = 1$. This corresponds to $\rho = 0$ and, consequently, $\phi_x = 0$ and $\sigma_x = \sigma$. For $x = 1$, the zero order matrix (12) becomes:

$$m^0 = m_2 \begin{pmatrix} 1 & 0 & 0 \\ ... & |c_{23}^2 + s_{23}^2 re^{-2i\sigma}| & s_{23} c_{23} \right| - 1 + re^{-2i\sigma} \\ ... & |s_{23}^2 + c_{23}^2 re^{-2i\sigma}| \end{pmatrix} .$$

(24)

The $s_{13}$ and $\epsilon$ terms (section 2.3) can give the main contributions to the elements $m_{e\mu}$ and $m_{e\tau}$.

3) Using Eq.(12), we find conditions at which one of the elements of the $\mu\tau$-block is zero:

$$m_{\mu\mu} = 0 : \cos 2\sigma_x = -1, \quad x = r \tan^2 \theta_{23} ;$$
$$m_{\tau\tau} = 0 : \cos 2\sigma_x = -1, \quad r = x \tan^2 \theta_{23} ;$$
$$m_{\mu\tau} = 0 : \cos 2\sigma_x = 1, \quad r = x .$$

(25)

These analytic expressions describe the position of white regions in the $\rho - \sigma$ plots, in first approximation. Some additional shift of these regions is due to $s_{13}$ corrections (that is, zeros can be realized for changed values of $\rho$ and $\sigma$); $\epsilon$ corrections are negligible.

As follows from (25), almost maximal atmospheric mixing implies that the $\mu\tau$-block elements can be zero only for $r \approx x$, that is for non-hierarchical mass spectrum. The exact equality $x = r$ implies:

$$\sin^2 \rho = \frac{1 - r^2}{\sin^2 2\theta_{12}} , \quad r \geq \cos 2\theta_{12} .$$

(26)

Therefore, only for $r$ large enough there is the possibility of hierarchical structures of the mass matrix with very small elements in the $\mu\tau$-block. Moreover, (26) shows that, for
increasing $r$, the regions of small $\mu\tau$-block elements move from $\rho \approx \pi/2$ to $\rho \approx 0, \pi$, as one can see comparing Figs. 1-6.

Taking $\delta = 0 (\pi)$, one can check that a white region appears in the plot of $m_{\mu\mu}$ ($m_{\tau\tau}$) already for $r \approx 0.1$. This is a case in which $s_{13}$ corrections are important: deviation of $\theta_{23}$ from maximal value and relatively large $s_{13}$ can add coherently, increasing the difference between $m_{\mu\mu}$ and $m_{\tau\tau}$.

In the case $m_{\mu\mu} = 0$, the matrix has the form

$$m^0 = m_2 \begin{pmatrix} x \sqrt{\frac{r(1-x^2)}{r+x}} & \sqrt{\frac{x(1-x^2)}{r+x}} \\ \vdots & 0 & \sqrt{r}x \\ \vdots & \vdots & \vdots & |x-r| \end{pmatrix},$$

where $r = x \cot^2 \theta_{23}$. Since $x - r = r(\tan^2 \theta_{23} - 1)$, the element $m_{\tau\tau}$ is proportional to the deviation of the atmospheric mixing from maximal one. In the case $m_{\tau\tau} = 0$, the matrix has an analogous form, but with the interchanges $r \leftrightarrow x$ and $m_{\mu\mu} \leftrightarrow m_{\tau\tau}$. The structure (27) is realized in Fig.5, for $\rho \approx \sigma \approx \pi/2$.

Both diagonal elements of the $\mu\tau$-block can be zero at maximal 2-3 mixing and $x = r$, so that

$$m^0 = m_2 \begin{pmatrix} r & \sqrt{\frac{1-r^2}{2}} & \sqrt{\frac{1-r^2}{2}} \\ \vdots & 0 & r \\ \vdots & \vdots & 0 \end{pmatrix},$$

This structure is realized in Fig.2, in the regions $\rho \approx \sigma \approx \pi/2$ and in Fig.3, for $\rho \approx \pi/4, 3\pi/4$ and $\sigma \approx \pi/2$.

In the case $m_{\mu\tau} = 0$, the matrix (12) has the form:

$$m^0 = m_2 \begin{pmatrix} r & c_{23}\sqrt{1-r^2} & s_{23}\sqrt{1-r^2} \\ \vdots & r & 0 \\ \vdots & \vdots & r \end{pmatrix}.$$  

Notice that the three diagonal elements are necessarily equal. This structure is shown in Fig.2, for $\sigma \approx 0, \pi$ and $\rho \approx \pi/2$ and in Fig.3, for $\sigma \approx 0, \rho \approx \pi/4$ and also for $\sigma \approx \pi, \rho \approx 3\pi/4$.

4) The conditions for zero values of the $e$-row elements ($x = 1$) and $\mu\tau$-block elements are consistent with each other, so that one may have any combination of zeros in both blocks. In particular, taking $x = 1$ in Eq.(27), we get

$$m^0 = m_2 \begin{pmatrix} 1 & 0 & 0 \\ \vdots & 0 & \sqrt{r} \\ \vdots & \vdots & |1-r| \end{pmatrix},$$
or the same with $m_{\mu\mu} \leftrightarrow m_{\tau\tau}$, if $m_{\tau\tau} = 0$. This case is realized in Fig.5, for $\rho \approx 0, \pi$ and $\sigma \approx \pi/2$.

If $r = 1$ (degenerate spectrum), in (28) only $m_{\epsilon\epsilon}$ and $m_{\mu\tau}$ differ from zero (and equal to 1). This hierarchical structure is shown in Fig.6, for $\rho \approx \pi/2$ and $\rho \approx 0, \pi$.

For $r = 1$, in (29) also $m_{\epsilon\mu} = m_{\epsilon\tau} = 0$ and the matrix becomes the identity. This structure appears in Fig.6, for $\rho \approx \sigma \approx 0, \pi$.

The discussed mass matrices with zero elements are shown in Table 1. In the Table we give also the intervals of $r$ and $\tan \theta_{23}$ for which the structures can be realized. These intervals are computed requiring non-zero elements to be quite large ($> 0.3m_2$), in order to clearly distinguish between dominant and sub-dominant blocks. So, the matrices we have found have hierarchical structure.

One can check, using analytic relations given in Eqs. (11 - 13), that all matrices with zeros can be obtained using $\rho, \sigma = 0, \pi/2$, that is definite CP-parities.

Notice that the “zero” elements are zeroes up to $O(s_{13}, \epsilon)$ corrections. In general, corrections are small ($\sim 10\%$) and could be very small ($\sim 1\%$), if the upper bound on $s_{13}$ becomes more stringent. Moreover, in all the hierarchical structures with $m_{\epsilon\mu} = m_{\epsilon\tau} = 0$ ($x = 1$), the $\mu\tau$-block elements have no $O(s_{13})$ corrections (see Eq.(A.6)).

In Fig.3 and Fig.4 different values of $\delta$ are used ($\pi/2$ and 0). Therefore, the relative phase of zero order matrix elements and $O(s_{13})$ terms is different in the two figures. One can see, in particular, how this changes the values of $\rho$ and $\sigma$ corresponding to very small matrix elements (white regions).

Even for non-zero values of $s_{13}$ and $\epsilon$ one may have exact zeros in the matrix. In the case of inverted mass spectrum, there are five possibilities of two (and only two) exact zero elements [5]:

- The four cases $m_{\epsilon\tau} = m_{\mu\mu(\tau\tau)} = 0$ or $m_{\epsilon\mu} = m_{\mu\mu(\tau\tau)} = 0$. The elements $m_{\epsilon\mu}$ and $m_{\epsilon\tau}$ cannot be both zero exactly, otherwise there is no solar mixing. However, our analysis shows that, if one is small, also the other is, because they cannot be zero separately in the limit $s_{13} = \epsilon = 0$.

- The case $m_{\mu\mu} = m_{\tau\tau} = 0$. This possibility is present also in the limit $s_{13} = \epsilon = 0$ (Eq.(28)).

As far as study of possible matrix structures is concerned, the requirement of exact zero values of some elements can be misleading. Indeed, some elements can be small or very small but non-zero. The smallness of an element could be explained by some flavor symmetry. However, the flavor symmetry is broken anyway: it is difficult to expect exact zeros. Moreover, zeros which exist at tree level can be unstable under radiative corrections.
<table>
<thead>
<tr>
<th>I</th>
<th>$m_{e\mu}, m_{e\tau}$</th>
<th>$0 \leq r \leq 1$</th>
<th>$0.75 \div 1.35$</th>
<th>$(1 \ 0 \ 0)$</th>
<th>[10 0 \ldots \ast ]</th>
<th>[\ldots \ast \ast ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$m_{\mu\mu}$</td>
<td>$0.2 \lesssim r \leq 1$</td>
<td>$0.75 \div 0.85$</td>
<td>$1.15 \div 1.35$</td>
<td>$(\ast \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
<tr>
<td>III</td>
<td>$m_{\tau\tau}$</td>
<td>$0.2 \lesssim r \leq 1$</td>
<td>$0.75 \div 0.85$</td>
<td>$1.15 \div 1.35$</td>
<td>$(\ast \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
<tr>
<td>IV</td>
<td>$m_{\mu\mu}, m_{\tau\tau}$</td>
<td>$0.4 \lesssim r \leq 0.8$</td>
<td>$0.95 \div 1.05$</td>
<td>$(\ast \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
<tr>
<td>V</td>
<td>$m_{\mu\tau}$</td>
<td>$0.4 \lesssim r \leq 0.8$</td>
<td>$0.75 \div 1.35$</td>
<td>$(\ast \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
<tr>
<td>VI</td>
<td>$m_{e\mu}, m_{e\tau}, m_{\mu\mu}$</td>
<td>$0.6 \lesssim r \leq 0.8$</td>
<td>$1.15 \div 1.35$</td>
<td>$(1 \ 0 \ 0)$</td>
<td>$(\ldots \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
<tr>
<td>VII</td>
<td>$m_{e\mu}, m_{e\tau}, m_{\tau\tau}$</td>
<td>$0.6 \lesssim r \leq 0.8$</td>
<td>$0.75 \div 0.85$</td>
<td>$(1 \ 0 \ 0)$</td>
<td>$(\ldots \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
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<td>VIII</td>
<td>$m_{e\mu}, m_{e\tau}, m_{\mu\mu}, m_{\tau\tau}$</td>
<td>$r \approx 1$</td>
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<td>$(1 \ 0 \ 0)$</td>
<td>$(\ldots \ast \ast)$</td>
<td>$(\ldots \ast \ast)$</td>
</tr>
</tbody>
</table>

Table 1: Hierarchical structures of the mass matrix. The classification is based on the very small ($O(s_{13}, \epsilon)$) entries of the matrix, which are listed in the second column. In the third and fourth columns the corresponding allowed ranges for $r$ and $\tan \theta_{23}$ are given. The matrices $m^0$, shown in the last column, are simplified forms of the structures given in Eqs.(24, 27-30). Parameters can be chosen in such a way that the elements denoted with “∗” belong to the range $0.3 - 0.7$. The $O(s_{13})$ corrections can be as large as 0.2 for the elements $e\mu$, $e\tau$, $\mu\mu$ and $\tau\tau$ (see Eq.(A.6)).
3.2 Equalities of matrix elements

Equalities of some matrix elements can be considered as the signature of certain symmetry or certain origin of the neutrino masses.

1) “Democratic” matrix. The zero order matrix (12) can have all six equal elements,

\[ m^0 = \frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \\ ... & 1 & 1 \\ \\ ... & ... & 1 \end{pmatrix} , \quad (31) \]

if and only if

\[ \theta_{23} = \frac{\pi}{4}, \quad x = \sqrt{\frac{1}{3}}, \quad r = 1, \quad \cos 2\sigma_x = 0 . \quad (32) \]

That is, the “democratic” matrix corresponds to degenerate mass spectrum, maximal 2-3 mixing and large CP-violating phases. According to Eq.(11), the condition \( x = \sqrt{1/3} \) gives

\[ \sin^2 \rho = \frac{2}{3\sin^2 2\theta_{12}} . \quad (33) \]

For the best fit value of 1-2 mixing we have \( \sin \rho \approx 0.9 \). Then, the condition \( \cos 2\sigma_x = 0 \) implies (see Eqs.(11,13)) \( \sin \sigma \approx 0.56 \) or \( \approx 0.83 \). The present solar neutrino data admit \( \rho = \pi/2 \) (\( \sigma = \pi/4,3\pi/4 \)). Non-zero \( \rho \) and \( \sigma \) lead to non-zero phases \( \phi_{\alpha\beta} \) of the matrix elements.

Notice that the values of parameters (32) correspond to the structure (31) only for \( s_{13} = 0 \). Substituting (32) in Eq.(A.6) and taking \( s_{13} \approx 0.2 \), we find that \( m_{e\mu} \) and \( m_{e\tau} \) can receive corrections (with opposite sign) as large as 35\% of their zero order value. Also \( m_{\mu\mu} \) and \( m_{\tau\tau} \) can be shifted by 30\% in opposite directions. The magnitude of corrections depends strongly on \( \delta \). In principle, for \( s_{13} \neq 0 \), parameters can be readjusted in order to recover the structure (31). In particular, this requires a deviation from maximal 2-3 mixing. If quite large \( s_{13} \) and exactly maximal 2-3 mixing were found in future experiments, the democratic structure would be excluded.

2) Equal \( e \)-row elements. All the \( e \)-row elements in Eq.(12) are equal, \( m_{ee} = m_{e\mu} = m_{e\tau} \), if and only if

\[ \theta_{23} = \frac{\pi}{4}, \quad x = \sqrt{\frac{1}{3}} . \quad (34) \]
In this case, one has also $m_{\mu\mu} = m_{\tau\tau}$. Under the condition (34), we get

$$m^0 = \frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & a(r) \\ \ldots & 1 & \ldots \\ \ldots & \ldots & 1 \end{pmatrix}, \quad \frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \ldots & a(r) & \ldots \\ \ldots & \ldots & a(r) \end{pmatrix},$$

(35)

The ratio $r$ is not restricted, so that equality of the $e$-row elements can be realized for any type of spectrum (hierarchical, non-hierarchical, degenerate).

Using the free parameters $r$ and $\sigma_x$, one can produce further structures or reach equalities in the $\mu\tau$-block. If

$$\cos 2\sigma_x = \pm \frac{\sqrt{3}(1 - r^2)}{2r},$$

the diagonal elements (sign plus) or off-diagonal elements (sign minus) of the $\mu\tau$-block are equal to $e$-row elements and we arrive at the zero order matrices:

$$\frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & a(r) \\ \ldots & 1 & \ldots \\ \ldots & \ldots & 1 \end{pmatrix}, \quad \frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \ldots & a(r) & \ldots \\ \ldots & \ldots & a(r) \end{pmatrix},$$

(36)

where

$$a(r) \equiv \sqrt{\frac{3r^2 - 1}{2}}.$$

Notice that $m_{\mu\mu} = m_{\tau\tau} \propto a(r)$ in the first case and $m_{\mu\tau} \propto a(r)$ in the second one can be zero, if $r^2 = 1/3$, but they can also be equal to the other elements, as in (31), if $r = 1$ (degenerate spectrum).

All elements of the $\mu\tau$-block are equal to each other if $r \cdot \cos 2\sigma_x = 0$. This condition can be realized for $r = 0$ (and arbitrary $\sigma_x$), which corresponds to strong mass hierarchy, or for $\sigma_x = \pi/4, 3\pi/4$ and arbitrary spectrum. In the latter case we get the matrix:

$$\frac{m_2}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \ldots & b(r) & b(r) \\ \ldots & \ldots & b(r) \end{pmatrix},$$

(37)

where

$$b(r) = \frac{m(\mu\tau - \text{block})}{m(e - \text{row})} = \frac{1}{2} \sqrt{3r^2 + 1}.$$

The ratio $b(r)$ changes from 1/2, for strong mass hierarchy, to 1, for degenerate spectrum. The structure (37) suggests that the magnitude of matrix element can be connected to the electron flavor.

3) **Equal $\mu\tau$-block elements.** The $\mu\tau$-block elements can be equal, $m_{\mu\mu} = m_{\mu\tau} = m_{\tau\tau}$, in three cases:
\( a \) \( \theta_{23} = \frac{\pi}{4}, \quad \cos 2\sigma_x = 0. \)

The matrix has the following form:

\[
m^0 = m_2 \begin{pmatrix} x & 1 - \frac{r^2}{2} \sqrt{1 - x^2 + r^2} & 1 - \frac{r^2}{2} \sqrt{1 - x^2 + r^2} \\ \ldots & \left(1 - \frac{r^2}{2}\right) \sqrt{1 - x^2 + r^2} & \left(1 - \frac{r^2}{2}\right) \sqrt{1 - x^2 + r^2} \\ \ldots & \ldots & \ldots \end{pmatrix}.
\] (38)

There are several interesting particular cases. The \( \mu \tau \)-block elements are equal to \( m_{\mu \mu} \) if \( r^2 = 3x^2 \):

\[
m^0 = \frac{m_2}{3} \begin{pmatrix} r & \sqrt{3} - r^2 & \sqrt{3} - r^2 \\ \ldots & r & r \\ \ldots & \ldots & r \end{pmatrix}.
\] (39)

The elements \( m_{e\mu} \) and \( m_{e\tau} \) can be larger than the other elements for relatively small \( r \) (notice that \( r > \sqrt{3x_{\min}} \)) or equal to them (\( r = 1 \)). The case \( r = 0 \), discussed in the literature \([3]\), requires \( x = 0 \) and therefore maximal solar mixing, which is now excluded by data.

If \( r^2 = 2 - 3x^2 \), the matrix (38) becomes:

\[
m^0 = m_2 \sqrt{\frac{1 + r^2}{6}} \begin{pmatrix} \sqrt{\frac{2}{1 + r^2}} & 1 & 1 \\ \ldots & 1 & 1 \\ \ldots & \ldots & 1 \end{pmatrix}.
\] (40)

All elements but \( m_{ee} \) are equal. For \( r = 0 \), \( m_{ee} \) is two times larger than the other elements; for \( r = 1 \), all the elements are equal.

\( b \) \( \theta_{23} = \frac{\pi}{4}, \quad r = 0. \)

This case corresponds to strong inverted hierarchy. The zero order mass matrix can be immediately obtained from (38) taking the limit \( r \to 0 \). The properties of this matrix will be discussed in detail in section 4.1.

\( c \) \( x = r, \quad \cos 2\sigma_x = -\cot^2 2\theta_{23}. \)

In this case the \( \mu \tau \)-block elements are all equal to \( m_3/\sqrt{2} \) and \( m_{ee} = m_3 \). Also the element \( m_{e\mu} \) (\( m_{e\tau} \)) is equal to \( m_3/\sqrt{2} \) if \( r^2/2(1-r^2) = c_{23}^2 \) \((= s_{23}^2)\). According to Eq.(14), this implies \( 0.45 \lesssim r \lesssim 0.65 \). The mass matrix takes the form:

\[
m^0 = \frac{m_3}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 1 & \frac{1}{2}\sqrt{2 - 3r^2} \\ \ldots & 1 & 1 \\ \ldots & \ldots & 1 \end{pmatrix}.
\] (41)

(or the same with \( m_{e\mu} \leftrightarrow m_{e\tau} \)).
The equality of $\mu \tau$-block elements can be an indication of a flavor symmetry with the same charge assigned to $\nu_\mu$ and $\nu_\tau$, such as $L_e - L_\mu - L_\tau$. This symmetry would imply also $m_{e\mu} = m_{e\tau}$. Notice that this equality holds in the cases (a) and (b).

4) Equal diagonal elements. There are two possibilities for $m_{ee} = m_{\mu\mu} = m_{\tau\tau}$:

(a) $\theta_{23} = \frac{\pi}{4}$, $\cos 2\sigma_x = \frac{3x^2 - r^2}{2xr}$.

In this case, $x \leq r \leq 3x$. The mass matrix has the form

$$ m = m_2 \begin{pmatrix} x & \sqrt{\frac{1-x^2}{2}} & \sqrt{\frac{1-x^2}{2}} \\ \ldots & x & \sqrt{\frac{r^2-x^2}{2}} \\ \ldots & \ldots & x \end{pmatrix}. \quad (42) $$

Imposing equalities of diagonal elements also with $m_{e\mu}$ (with $m_{e\tau}$) we reproduce the first matrix in Eq.(36) (Eq.(39)). Notice that the diagonal elements can be much larger than off-diagonal elements only in the limit $x \rightarrow 1$, $r \rightarrow 1$, which corresponds to degenerate spectrum.

(b) $x = r$, $\cos 2\sigma_x = 1$.

The matrix is given by Eq.(29). If $r = c_{23}/\sqrt{1+c_{23}^2}$ (0.35 $\lesssim r \lesssim 0.65$), also $m_{e\mu}$ is equal to the diagonal elements and the matrix becomes

$$ m = m_3 \begin{pmatrix} 1 & 1 & \frac{1}{r}\sqrt{1-2r^2} \\ \ldots & 1 & 0 \\ \ldots & \ldots & 1 \end{pmatrix}. \quad (43) $$

The matrix with $m_{e\tau}$ equal to the diagonal elements can be found by substituting $c_{23} \leftrightarrow s_{23}$ and $m_{e\mu} \leftrightarrow m_{e\tau}$ in (43).

5) Equal off-diagonal elements. The conditions for the equality $m_{e\mu} = m_{e\tau} = m_{\mu\tau}$ can be found from Eq.(12):

$$ \theta_{23} = \frac{\pi}{4}, \quad \cos 2\sigma_x = \frac{3x^2 + r^2 - 2}{2xr}. $$

In this case the mass matrix has the following form:

$$ m^0 = \frac{m_3}{\sqrt{2}} \begin{pmatrix} \sqrt{2}x & \sqrt{1-x^2} & \sqrt{1-x^2} \\ \ldots & \sqrt{2x^2 + r^2 - 1} & \sqrt{1-x^2} \\ \ldots & \ldots & \sqrt{2x^2 + r^2 - 1} \end{pmatrix}. \quad (44) $$
Imposing equalities also with $m_{ee}$ ($m_{\mu\mu}$), we get the second matrix in Eq.(36) (Eq.(40)).

There are two possibilities to have, at the same time, equal diagonal and off-diagonal elements: the identity matrix and the democratic matrix. This kind of equality suggests a permutation symmetry $S_3$ of the flavor neutrinos [12].

### 3.3 Ordering structures and flavor alignment

As one can see in the $\rho - \sigma$ plane, there are regions where all the matrix elements are of the same order (intermediate gray in the $\rho - \sigma$ plots). In these regions the matrix may have certain “ordering” structures.

Do masses correlate with flavors? That is, are there any correlations between charged lepton and neutrino masses? We will call such a correlation the *flavor alignment*.

One criterion of alignment (motivated by possible horizontal symmetry) can be introduced prescribing different lepton charges, $q_\alpha$, for different flavor neutrino states, $\alpha = e, \mu, \tau$. Suppose that neutrino masses equal

$$m_{\alpha\beta} = \lambda q_\alpha + q_\beta, \quad \lambda < 1.$$  \hspace{1cm} (45)

Then the alignment exists if $q_e > q_\mu > q_\tau$. The smaller $\lambda$ or/and the larger the difference of charges $q_\alpha$, the stronger is the alignment. In the case $q_\mu = q_\tau$ (which might be indicated by maximal $\mu - \tau$ mixing), one can speak of partial alignment, associated to the lepton number $L_e$. In this case all $\mu\tau$-block elements are equal.

Let us consider first the possibility of partial alignment of the zero order matrix in the limit of maximal 2-3 mixing. Using the matrix (38), we find that $m_{ee} < m_{e\mu}$ provided that $x^2 < 1/3$ (which is consistent with Eq.(16)). Then, the condition $m_{e\tau} > m_{e\mu}$ would require $r > 1$, which is impossible for inverted mass spectrum. Thus, even partial alignment can not be achieved. At best one can get “democratic” structure, $m_{ee} = m_{e\mu} = m_{\tau\tau}$, if $x^2 = 1/3$ and $r = 1$.

If 2-3 mixing deviates from maximal, a split appears between $m_{e\mu}$ and $m_{e\tau}$ as well as $\mu\tau$-block elements. However, the same consideration as above holds for averaged values of the $e$-row and $\mu\tau$-block elements: $(m_{e\mu}^2 + m_{e\tau}^2)/2$ and $\Sigma_{\mu\tau}/4$ (see Eq.(21)). The $s_{13}$ corrections can split $m_{\mu\mu}$ and $m_{\tau\tau}$ elements, but no improvement of alignment can be obtained.

In the case of inverted mass spectrum, *inverted alignment*, $q_e < q_\mu < q_\tau$, is possible. The alignment can be partial ($q_e < q_\mu = q_\tau$). Taking maximal 2-3 mixing and $r = 0$ (which corresponds to maximal split of the elements), we find from (38):

$$m_{ee} : m_{e\mu} : m_{\tau\tau} = 1 : \sqrt{(1 - x^2)/(2x^2)} : \frac{1}{2}.$$
The inequality $m_{ee} > m_{e\mu} > m_{\tau\tau}$ is satisfied for $1/3 < x^2 < 2/3$. In particular, one can get the matrix of the form

$$m^0 = N \begin{pmatrix} 1 & \lambda & \lambda \\ \vdots & \lambda^2 & \lambda^2 \\ \vdots & \vdots & \lambda^2 \end{pmatrix},$$

(46)

with $N/m_2 = \lambda = \sqrt{1/2}$.

Also for $r \neq 0$ the matrix can be reduced to the form (46). Taking $r^2 = (1 - 2x^2)/x^2$ in (38) (this is possible if $1/3 \leq x^2 \leq 1/2$), one gets $N/m_2 = x$ and $\lambda^2 = (1 - x^2)/(2x^2)$, which in turn implies $1/2 \leq \lambda^2 \leq 1$.

A complete inverted alignment can be achieved for non-maximal 2-3 mixing. Inserting $r = 0$, $x^2 = 1/2$ and $c_{23}^2 = s_{23}$ into (12), we get

$$m^0 = \frac{m_2}{\sqrt{2}} \begin{pmatrix} 1 & \lambda & \lambda^2 \\ \vdots & \lambda^2 & \lambda^3 \\ \vdots & \vdots & \lambda^4 \end{pmatrix},$$

(47)

with $\lambda = \tan \theta_{23} \approx 0.79$.

The role of $\mu$ and $\tau$ flavors are interchanged if $c_{23} = s_{23}^2$. In this case a structure analogous to (47) is realized, with $\lambda = \cot \theta_{23} \approx 0.79$.

Structures in which $\mu$ and $\tau$ flavors are associated with substantially different mass scales are excluded. Indeed, the difference $m_{\mu\mu}^2 - m_{\tau\tau}^2$ is proportional to the small parameter $\cos 2\theta_{23}$. Moreover, if there is a strong ordering between $m_{\mu\mu}$ and $m_{\tau\tau}$, the element $m_{\mu\tau}$ is larger than both of them (see Eqs.(27,30)), while flavor alignment would require an intermediate value.

Notice that free parameters $r$, $\rho$, $\sigma$, etc. can be found for which no correlation of the masses and lepton charges of the matrix elements exist at all. This possibility can be called flavor disorder.

4 Dependence of the matrix structure on the type of mass spectrum

Let us analyze how the matrix structure depends on $r$. We also consider perspectives to reconstruct the mass matrix in flavor basis in future experiments.

As follows from Eq.(12), in the limit $s_{13} = 0$, $k = 1$, the $e$-row elements do not depend on $r$, so only the structure of the $\mu\tau$-block depends on the type of mass spectrum.

We split our discussion in three parts: (i) strong inverted hierarchy: $r \approx 0$, practically $0 \leq r \lesssim 0.2$; (ii) inverted ordering: $0.2 \lesssim r \lesssim 0.8$; (iii) degeneracy: $r \approx 1$, practically $0.8 \lesssim r \leq 1$. 

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4.1 Strong inverted hierarchy

Taking $r = 0$ ($m_3 = 0$) in (12), we get the zero order matrix

$$ m^0 = \sqrt{\Delta m^2_{\text{atm}}} x \begin{pmatrix} 1 & c_{23} \sqrt{1/x^2 - 1} & s_{23} \sqrt{1/x^2 - 1} \\ \vdots & c_{23}^2 & s_{23} c_{23} \\ \vdots & \vdots & s_{23}^2 \end{pmatrix} $$

(48)

which depends on two parameters only, $x$ and $\theta_{23}$. The dependence of $m^0$ on the Majorana phase $\sigma$, associated with $m_3$, disappears. Furthermore, once $\theta_{12}$ and $\theta_{23}$ are fixed, the matrix (48) depends only on $x = x(\rho)$. In Fig. 7, we show the absolute values of the matrix elements as functions of $\rho$. The only freedom (associated to variations of $\rho (x)$) is reduced to change the relative size of two groups of elements: $m_{ee}$ plus $\mu \tau$-block elements on one side and $m_{e\mu}, m_{e\tau}$ on the other.

In the strong hierarchy case, we have (see Eq.(22)):

$$ \sum_{\alpha, \beta} m^2_{\alpha\beta} = 2m^2_2 = 2\Delta m^2_{\text{atm}} . $$

(49)

The sums of $e$–row and $\mu \tau$–block elements are (see Eq.(21)):

$$ m^2_{ee} + 2(m^2_{e\mu} + m^2_{e\tau}) = m^2_2(2 - x^2) , \quad \Sigma_{\mu\tau} = m^2_2 x^2 = m^2_{ee} . $$

(50)

So, the $e$-row elements dominate over the $\mu \tau$–block elements. They are comparable for $x = 1$, which corresponds to $m_{ee} = m_2$. The second equality in (50) quantifies the dominance of the $ee$–element.

The mass matrix could be determined completely if direct measurements of neutrino masses were sensitive to $m \approx \sqrt{\Delta m^2_{\text{atm}}}$. This can be checked by future cosmological measurements, which will be sensitive to sum of neutrino masses of the order of 0.1 eV [13]. Then, $x$ parameter can be found if $m_{ee}$ is measured in the neutrinoless double beta decay.

In the matrix (48) $m_{ee} = \sqrt{\Delta m^2_{\text{atm}}} x$ is non-zero. Furthermore, the $\mu \tau$-block elements vary in restricted ranges:

$$ m^0_{\mu\mu, \tau\tau} \in m_2[0.1, 0.65] , \quad m^0_{\mu\tau} \in m_2[0.1, 0.5] , $$

so that they cannot be zero either. Therefore, the only hierarchical structure which appears in the case of inverted hierarchy corresponds to $m_{e\mu} \approx m_{e\tau} \approx 0$. These masses are small, simultaneously, for $x \approx 1$ ($\rho \approx 0, \pi$). Substituting $x = 1$ in Eq.(48), we get

$$ m^0 = \Delta m^2_{\text{atm}} \begin{pmatrix} 1 & 0 & 0 \\ \vdots & c_{23}^2 & s_{23} c_{23} \\ \vdots & \vdots & s_{23}^2 \end{pmatrix} , $$

(51)
which is a particular case of the hierarchical structure (24). In the limit $s_{13} = m_3 = 0$ and $x = 1$, CP violation is absent ($\delta$ and $\sigma$ are irrelevant and $\rho = 0$).

The hierarchical structure with $m_{ee} \approx m(\mu\tau \text{ block}) \approx 0$, widely discussed in literature [3, 4] in connection to $L_e - L_\mu - L_\tau$ symmetry, is strongly disfavored now. For the allowed values of $x$, the symmetry has to be strongly broken, or realized in a basis which differs from the flavor one [14].

For maximal atmospheric mixing, the mass matrix (48) takes the form:

$$m^0 = \sqrt{\Delta m_{atm}^2} x \begin{pmatrix} 1 & \sqrt{1-x^2/2} & \sqrt{1-x^2/2} \\ \ldots & 1/2 & 1/2 \\ \ldots & \ldots & 1/2 \end{pmatrix}. \quad (52)$$

Depending on $x$, the ratio between $m_{e\mu}$ ($m_{e\tau}$) and the other matrix elements can strongly change, as shown in Fig.7. Three interesting cases,

$$\begin{pmatrix} 1 & 0 & 0 \\ \ldots & 1/2 & 1/2 \\ \ldots & \ldots & 1/2 \end{pmatrix}, \quad \sqrt{2/3} \begin{pmatrix} 1 & 1/2 & 1/2 \\ \ldots & 1/2 & 1/2 \\ \ldots & \ldots & 1/2 \end{pmatrix}, \quad \sqrt{1/3} \begin{pmatrix} 1 & 1 & 1 \\ \ldots & 1/2 & 1/2 \\ \ldots & \ldots & 1/2 \end{pmatrix},$$

are realized for $x = 1$, $x = \sqrt{2/3}$ and $x = \sqrt{1/3}$, respectively. Only the first of these three structures, which corresponds to CP conservation ($\rho = 0$), has been considered before [3]. It is a particular case of (51). The second and the third matrices are particular cases of (40) and (37), respectively.

As follows from Eq.(A.6), for $r = 0$ and $\theta_{23} = \pi/4$, the $O(s_{13})$ corrections to the elements of the matrix (52) have very simple form:

$$m_{ee}^s = m_{e\mu}^s = 0,$$
$$m_{e\tau}^s = -m_{e\mu}^s = m_2 x \cos \varphi_1/\sqrt{2},$$
$$m_{\tau\tau}^s = -m_{e\mu}^s = m_2 \sqrt{1 - x^2} \cos(\varphi_2 - \phi_x),$$

where $\varphi_1$ and $\varphi_2$ are defined in Appendix. Taking into account these corrections, one can explain the details of Fig.7. In particular, for $\rho = 0$ ($x = 1$, first matrix in Eq.(53)), the corrections to the $\mu\tau$-block elements disappear ($m_{\tau\tau}^s = 0$), and $s_{13}$—terms give dominant contributions to the $e$-row elements ($m_{e\tau}^s = m_2 \cos \delta/\sqrt{2}$).

When $m_{e\mu}^0 \approx m_{e\tau}^0 \approx 0$ ($x \approx 1$), $O(s_{13})$ corrections can be used to get the inequality $m_{e\mu} \gg m_{e\tau}$ (or vice versa). Indeed, introducing the small parameter $\gamma \equiv \sqrt{1 - x^2}$, we find:

$$m_{e\mu} \approx \frac{m_2}{\sqrt{2}} \left| \gamma - s_{13} \sqrt{1 - \gamma^2} \cos \varphi_1 \right|, \quad m_{e\tau} \approx \frac{m_2}{\sqrt{2}} \left| \gamma + s_{13} \sqrt{1 - \gamma^2} \cos \varphi_1 \right|.$$  

Choosing $\delta$ such that $\cos \varphi_1 = -1$, one gets $m_{e\mu} \gg m_{e\tau}$ for $\gamma \approx s_{13} \sqrt{1 - \gamma^2}$.  

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4.2 Inverted ordering

This is a rather generic case, in which almost all structures discussed in section 3 can appear. In particular, all hierarchical structures but VIII and IX can be realized (see Table 1). One or two $\mu\tau$-block elements can be zero. As far as equalities of matrix elements are concerned, only the exact “democratic” structure is excluded. On the other hand, only in the inverted ordering case certain correlations of masses and lepton charges appear.

For a small value of $r$, the $\mu\tau$-block elements cannot be very small and their dependence on $\sigma$ is weak, therefore the unique possible hierarchical structure is I (Eq.(24)), as one can see in Fig.1 ($r = 0.1$).

For larger values of $r$, the modifications in the $\mu\tau$-block elements can be strong. We have seen, in section 3.1, that $\mu\tau$-block elements can be very small only for $x \approx r$. According to Eq.(25), one can get $m_{\mu\mu}$ or $m_{\tau\tau}$ equal to zero for values of $r$ as small as $\sim x_{\min}/2$, because of non-maximal 2-3 mixing. Therefore, the structures II and III can be realized for $r \geq 0.2$. Instead, the equalities $m_{\mu\tau} = 0$ or $m_{\mu\mu} = m_{\tau\tau} = 0$ can be realized only for $r$ as large as $\sim x_{\min} \geq 0.4$ (structures IV and V).

If one requires that $m_{e\mu}$ and $m_{e\tau}$ are small together with some $\mu\tau$-block elements, the condition $x \approx 1$ enforces the minimal value of $r$ to be larger: $r_{\min} \sim 1/2$ for the structures VI and VII; $r_{\min} \sim 1$ for VIII and IX. These considerations lead to the lower bounds for $r$ given in the third column of Table 1.

In the non-hierarchical case $m_2 > \sqrt{\Delta m^2_{atm}}$. The absolute mass scale increases with $r$, e.g., for $r \sim 0.8$ we get $m_2 \sim 0.1$ eV and $m_{ee} \sim (0.03 \div 0.1)$ eV. Measuring $m_2$ and $m_{ee}$ we can immediately determine $x$, provided that $s_{13}$ is further restricted by experiment. Then $m_2$ and $\Delta m^2_{atm}$ determine $r$ according to (6). The only unknown parameter in the (zero order) matrix will be $\sigma$. Its variations can strongly change the structure of the $\mu\tau$-block.

4.3 Degenerate spectrum

In the case of degenerate spectrum, the mass matrix coincides practically with that for normal mass ordering [6]. The information about the type of mass ordering is imprinted in small, $O(\Delta m^2_{atm}/m_2^2)$, deviation of $r$ from 1 (for inverted ordering $r < 1$).

A number of various mass structures are allowed in this case. The hierarchical structures I - III as well as VIII, IX can be realized. Moreover, the matrices VIII and IX appear only in the degenerate case, taking the limit $r \to 1$ in Eq.(28) and Eq.(29), respectively. When the matrix approaches the identity (IX), also the atmospheric mixing $\theta_{23}$ is generated by the small corrections to the dominant structure.

The structures IV and V can not be realized: they require $r = x$ and, consequently,
$x = 1$. In turn, the latter implies $m^0_{e\mu} = m^0_{e\tau} = 0$. Also the structures VI and VII are forbidden if $r \approx 1$, because they require $x \approx 1$: $x = r = 1$ implies $m^0_{\mu\mu} = m^0_{\tau\tau}$.

In the limit $r \to 1$, the zero order mass matrix depends on two unknown parameters: $x$ and $\sigma_x$. The first one can be determined directly from kinematic measurements of the absolute mass scale and detection of the neutrinoless 2$\beta$ decay: $x \approx m_{ee}/m_2$. The only free parameter to which we have no access is the phase $\sigma$. Dependence of the $\mu\tau$-block elements on this phase is even stronger than in the inverted ordering case.

5 Discussion and conclusions

We have analyzed the structure of the Majorana mass matrix of the three flavor neutrinos, in the case of inverted mass hierarchy (ordering).

The structure of the mass matrix strongly depends on the Majorana phases $\rho$ and $\sigma$. We find that $e$-row elements strongly depend on $\rho$ and very weakly on $\sigma$. In contrast, the $\mu\tau$-block elements depend both on $\rho$ and $\sigma$, moreover the dependence on $\sigma$ becomes stronger with increase of degeneracy. The dependence of the matrix on the Dirac phase $\delta$ is weak, because it is associated with the small parameter $s_{13}$.

The dominant structures of the mass matrix are determined, essentially, by four parameters: $r = m_3/m_2$, $\theta_{23}$, $x = x(\rho, \theta_{12})$ and $\sigma$.

We find that present data allow for a large variety of different mass matrix structures.

1) The hierarchical structures have a set of small or zero elements and a set of large elements. Any element but $m_{ee}$ can be zero. The elements $m_{e\mu}$ and $m_{e\tau}$ can be very small for any type of the mass spectrum. In addition, one or two elements of the $\mu\tau$-block can be very small for inverted ordering and degenerate spectra.

All the hierarchical structures can be realized for definite CP-parities of the mass eigenstates. The structures II,III and VI,VII (see Table 1) are allowed only if there is a deviation from maximal atmospheric mixing. In the case of strong inverted hierarchy, only the structure I is possible. The structures IV-VII are allowed only for inverted ordering spectrum, while VIII and IX only in the case of degeneracy.

The unique hierarchical structure which is present in the whole range $0 \leq r \leq 1$ is I (in fact it is present also for normal ordering of the mass spectrum, $1 \leq r \lesssim 3$, but not for normal hierarchy, $r > 3$). This means that this structure is stable under modifications of the neutrino mass scale (in fact also under inversion of the ordering).

2) Various equalities between matrix elements are possible. In particular, equalities of the $e$-row elements or/and $\mu\tau$-block elements, or diagonal elements or/and off-diagonal elements can be achieved. The “democratic” mass matrix is also allowed in the flavor
basis. Some equalities of elements (in contrast to zeros) can be realized for non-trivial phases only.

3) We have studied correlations between masses and flavors. We find that flavor alignment is impossible. However, one can reach inverted flavor alignment for rather large values of the expansion (ordering) parameter: $\lambda = 0.5 - 0.8$. Also flavor “disorder” is not excluded.

4) We have shown that $O(s_{13})$ and $O(\Delta m_{sol}^2/\Delta m_{atm}^2)$ terms can be as large as $(0.1 \div 0.2)m_2$ and $(0.01 \div 0.02)m_2$, respectively. Terms proportional to $s_{13}$ depend on the CP violating Dirac phase $\delta$. The values of $s_{13}$, $\delta$ and $\Delta m_{sol}^2$ are related to small details of the matrix structure. In the degenerate case, also $\Delta m_{atm}^2$ is very weakly imprinted in the structure of the mass matrix.

One interesting possibility, proposed in a recent paper [15], is to generate radiatively $s_{13}$ and $\Delta m_{sol}^2$, starting from a leading order matrix at high energy in which they are zero.

If $s_{13}$ stays at the present upper bound ($\sim 0.2$), $O(s_{13})$ corrections can modify significantly the matrix structure, because they shift in opposite directions the elements $m_{e\mu}$ and $m_{e\tau}$, $m_{\mu\mu}$ and $m_{\tau\tau}$.

In general, the normal hierarchy spectrum [6] corresponds to a mass matrix with dominant $\mu\tau$-block. Flavor alignment is possible. Vice versa, in the case of inverted hierarchy, there is dominance of the $e$-row elements or, at least, of the $ee$-element. Inverted flavor alignment is possible. If the absolute mass scale increases, the spectrum becomes closer to the degenerate one and the difference between matrices which correspond to normal and inverted spectrum practically disappears. The democratic mass matrix is possible only for exactly degenerate spectrum.

We have considered the possibility to determine the mass matrix in future neutrino experiments. We find that the matrix can be reconstructed completely in the case of inverted mass hierarchy, provided that (i) the sensitivity to the absolute mass scale will reach $\sqrt{\Delta m_{atm}^2}$, (ii) the neutrinoless double beta decay will be discovered, (iii) stronger upper bounds on $s_{13}$ will be obtained.

In the case of inverted ordering or degenerate spectrum the phase $\sigma$ becomes important. This phase cannot be determined, thus leaving large uncertainty in the structure of the $\mu\tau$-block.

**Appendix : general formulae**

We present explicit analytical expressions for the matrix elements.

The smallness of the parameter $s_{13}$ is very important for the analysis of the matrix
elements. Defining

\[ X \equiv xe^{i\phi_x} \equiv s_{23}^2 ke^{-2i\rho} + c_{12}^2, \]
\[ Y \equiv ye^{i\phi_y} \equiv s_{12}c_{12}(1 - ke^{-2i\rho}), \]
\[ Z \equiv ze^{i\phi_z} \equiv c_{13}^2 ke^{-2i\rho} + s_{12}^2, \]  \hspace{1cm} (A.1)

one can write the elements as series of powers of \( s_{13} \):

\[ M_{ee}/m_2 = Z - s_{13}^2 Z', \]
\[ M_{e\mu}/m_2 = c_{13}(c_{23} Y - s_{13} s_{23} e^{-i\delta} Z'), \]
\[ M_{e\tau}/m_2 = c_{13}(-s_{23} Y - s_{13} c_{23} e^{-i\delta} Z'), \]
\[ M_{\mu\mu}/m_2 = c_{23}^2 X + s_{23} r e^{-2i\sigma} - s_{13} \sin 2\theta_{23} e^{-i\delta} Y + s_{13}^2 s_{23} e^{-2i\delta} Z', \]
\[ M_{\mu\tau}/m_2 = s_{23} c_{23}(-X + r e^{-2i\sigma}) - s_{13} \cos 2\theta_{23} e^{-i\delta} Y - s_{13}^2 s_{23} c_{23} e^{-2i\delta} Z', \]
\[ M_{\tau\tau}/m_2 = s_{23} c_{23}(X + r e^{-2i\sigma}) - s_{13} \cos 2\theta_{23} e^{-i\delta} Y - s_{13}^2 s_{23} c_{23} e^{-2i\delta} Z', \]  \hspace{1cm} (A.2)

where

\[ Z' \equiv z' e^{i\phi_{z'}} \equiv Z - r e^{2i(\delta - \sigma)}. \]  \hspace{1cm} (A.3)

We are interested in the limit \( k \to 1 \). For \( k = 1 \), it follows from Eq.(A.1) that

\[ x = z = \sqrt{1 - y^2}, \]  \hspace{1cm} (A.4)

where \( x \) is given in Eq.(11). Taking into account that \( c_{12} > s_{12} \), it is easy to compute also the phases:

\[ -\phi_x = \phi_z + 2\rho, \quad \phi_y = \frac{\pi}{2} - \rho, \]  \hspace{1cm} (A.5)

where \( \phi_x \) is given in Eq. (11).

Let us write explicitly the matrix \( m^s \), introduced in Eq.(23). Defining

\[ \varphi_1 = \phi_z - \phi_y - \delta, \quad \varphi_2 = \phi_y - \delta, \quad \varphi_{\alpha\beta} = \text{arg} \ M_{\alpha\beta}^0, \]

we get, using Eq.(A.2),

\[ m^s = m_2 \begin{pmatrix} 0 & -s_{23} z' \cos \varphi_1 & c_{23} z' \cos \varphi_1 \\ \ldots & -\sin 2\theta_{23} \sqrt{1 - x^2} \cos(\varphi_2 - \varphi_{\mu\mu}) & -\cos 2\theta_{23} \sqrt{1 - x^2} \cos(\varphi_2 - \varphi_{\mu\tau}) \\ \ldots & \ldots & \sin 2\theta_{23} \sqrt{1 - x^2} \cos(\varphi_2 - \varphi_{\tau\tau}) \end{pmatrix}. \]  \hspace{1cm} (A.6)

The maximal values of these corrections can be easily computed: using Eqs.(A.3,A.4), one finds

\[ z'^2 = x^2 + r^2 + 2rx \cos(2\delta - 2\sigma - \phi_2) \in [(x - r)^2, (x + r)^2]. \]
Finally, we give the explicit expression of the matrix $m^e$, introduced in Eq.(23):

$$m^e = m_2 \left( \begin{array}{cccc}
-c_{12}^2 \cos(2\rho + \phi_z) & c_{23}c_{12}s_{12} \cos(2\rho + \phi_y) & s_{23}c_{12}s_{12} \cos(2\rho + \phi_y) \\
\ldots & -c_{23}^2 s_{12}^2 \cos(2\rho + \phi_{\mu\nu}) & c_{23}^2 s_{23} s_{12}^2 \cos(2\rho + \phi_{\mu\tau}) & -s_{23}^2 s_{12}^2 \cos(2\rho + \phi_{\tau\tau}) \\
\ldots & \ldots & \ldots & \ldots \\
-s_{23}^2 s_{12}^2 \cos(2\rho + \phi_{\tau\tau}) & \ldots & \ldots & \ldots \\
\end{array} \right). \quad (A.7)$$

The cosines in Eq.(A.7) take always the values $\pm 1$ for $\rho, \sigma = 0, \pi/2$. We have seen that very small matrix elements usually appear for $\rho, \sigma \approx 0, \pi/2, \pi$ (see section 3.1).

References


Figure 1: The $\rho - \sigma$ plots for inverted hierarchical spectrum, with $r = 0.1$. Contours are shown of constant mass (iso-mass) $m = (0.1, 0.2, \ldots, 0.9)m^{\text{max}}$, where $m^{\text{max}} = 0.05$ eV is the maximal value that the matrix elements can have, so that the white regions correspond to the mass interval $(0 - 0.005)$ eV and the darkest ones to $(0.045 - 0.05)$ eV. The contour $m = 0.5m^{\text{max}}$ is dashed. We take $\Delta m_{\text{sol}}^2 = 6 \cdot 10^{-5} \text{eV}^2$, $\Delta m_{\text{atm}}^2 = 2.5 \cdot 10^{-3} \text{eV}^2$ and $\tan^2 \theta_{12} = 0.4$, $\tan \theta_{23} = 1$, $s_{13} = 0.1$, $\delta = \pi/2$. 

\[ \Delta m_{\text{sol}}^2 = 6 \cdot 10^{-5} \text{eV}^2, \Delta m_{\text{atm}}^2 = 2.5 \cdot 10^{-3} \text{eV}^2 \]
Figure 2: The same as in Fig.1, but for $r = 0.4$. In this case $m^{\text{max}} = 0.055 \text{ eV}$.
Figure 3: The same as in Fig.1, but for $r = 0.7$. In this case $m^{\text{max}} = 0.07 \text{ eV}$. 
Figure 4: The same as in Fig.1, but for $r = 0.7$ and $\delta = 0$. 
Figure 5: The same as in Fig.1, but for $r = 0.7$ and $\tan \theta_{23} = 0.75$. 
Figure 6: The same as in Fig.1, but for $r = 0.99$: the spectrum is degenerate. In this case $m^{\text{max}} = 0.36 \text{ eV}$, so that the white regions correspond to the mass interval $(0 - 0.036) \text{ eV}$ and the darkest ones to $(0.324 - 0.36) \text{ eV}$. 
Figure 7: Dependence of the absolute value of mass matrix elements (in eV) on $\rho$, in the case of mass spectrum with strong inverted hierarchy ($r = 0$). We take $\Delta m^2_{sol} = 6 \cdot 10^{-5}$eV$^2$, $\Delta m^2_{atm} = 2.5 \cdot 10^{-3}$eV$^2$ and $\tan^2 \theta_{12} = 0.4$, $\tan \theta_{23} = 1$, $s_{13} = 0.1$, $\delta = \pi/2$. 