Optimal Renormalization-Group Improvement of the Perturbative Series for the

\[ \text{Cross-Section} \]

\[ \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \]

where \( \zeta(s) \) is the Riemann zeta function.

The above renormalization group equation (RGE) is inspired from quantum field theory in which

\[ \left[ T \cdot x \right] \left( \frac{e^{\theta} (x)}{\theta} + \frac{e^{\theta}}{\theta} \right) = \frac{\epsilon \theta}{(\theta + 1)^p} \cdot \frac{1}{p^s} \cdot \frac{1}{p^s} = 0 \]

regulating of renormalization group equations of the minimal or maximal renormalization of the perturbative series. \( \beta(s) \) is a measure of the

\[ \frac{s}{n^s} \theta \equiv (\theta) \frac{1}{p^s} \]

through powers of the logarithm.

The constant

\[ \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \]

This series is extracted from the imaginary part of the

\[ \text{Contribution to the renormalization} \]

by a perturbative (\( O(\alpha^n) \)) series (\( s \)).

For one of the \( \alpha \)-counterpart

\[ \epsilon \theta \equiv (\theta) \frac{1}{p^s} \]

we obtain the renormalization scale in consistency with the renormalization scale, the necessary improvement can only be obtained if the renormalization scale is

\[ \left[ T \cdot x \right] \left( \frac{e^{\theta} (x)}{\theta} + \frac{e^{\theta}}{\theta} \right) = \frac{\epsilon \theta}{(\theta + 1)^p} \cdot \frac{1}{p^s} \cdot \frac{1}{p^s} = 0 \]

an \( \alpha \)-counterpart of the perturbative series, \( \beta(s) \) is also included in Table 1. This, the RGE, is generally employed.
a next-to-next-to-leading (NNL) order perturbative calculation determines only the coefficients $T_{1,0}$, $T_{2,0}$, and $T_{2,1}$ of the series $\mathcal{A}$. However, the RGE $\mathcal{B}$ can be utilized to determine all coefficients $T_{k+1,k}$ and $T_{k+2,k}$ within the series $\mathcal{A}$. The contributions of this infinite set of coefficients may then be summed analytically, as described below, thereby providing an ”optimal RG-improvement” of the NNL expression.

The series $S[x, L]$, as defined in Eq. (4), may be rearranged in the following form:

$$S[x, L] = 1 + \sum_{n=1}^{\infty} x^n S_n(x L),$$

(7)

where

$$S_n(u) \equiv \sum_{n-k}^{\infty} T_{n,n-k} u^{n-k}. \quad (8)$$

Given knowledge of the $k^{th}$-order series coefficient $T_{k,0} = S_k(0)$, one can obtain $S_k(x \mu, L(\mu))$ explicitly, thereby summing over the entire set of $k^{th}$-order subleading logarithms contributing to the series $\mathcal{A}$. If we substitute the $\beta$-function series $\mathcal{A}$ into the RGE $\mathcal{B}$, we find that the aggregate coefficient of $x^{n-p} L^{n-p}$ vanishes ($n \geq p$) provided the following recursion relation is upheld:

$$0 = (n + p + 1) T_{n,n-p+1} - \sum_{\ell=0}^{p-2} (n - \ell - 1) \beta \ell T_{n-n-\ell-1,n-p}. \quad (9)$$

For example, if $p = 2$, this recursion relation $[T_{n,n-1} = \beta \ell T_{n-n-\ell-1,2}]$ relates all leading-logarithm coefficients $T_{n,n-1}$ within the series $\mathcal{A}$ to the known coefficient $T_{1,0} = 1$, thereby enabling one to sum all orders of the leading-logarithm contributions

$$x S_1(x L) = x \sum_{n=1}^{\infty} T_{n,n-1}(x L)^{n-1} = \frac{x}{1 - \beta_0 x L}. \quad (10)$$

to the series $S[x, L]$.

More generally, the recursion relation $\mathcal{A}$ may be utilized to obtain a succession of first-order inhomogeneous linear differential equations for the functions $S_k(u)$ within Eq. (4). If one multiplies Eq. (4) by $u^{n-p}$ and then sums over $n$ from $n = p$ to $\infty$, one finds from the definition $\mathcal{A}$ of $S_k(u)$ that

$$0 = \frac{d S_{n-1}}{du} - \sum_{\ell=0}^{p-2} \beta \ell \frac{d S_{n-\ell-1}}{du} - \sum_{\ell=0}^{p-2} (p - \ell - 1) \beta_\ell S_{p-\ell-1}, \quad (11)$$

which can be trivially rearranged ($k = p - 1$) into a set of first-order linear differential equations

$$\frac{d S_k}{du} - \frac{k \beta_0}{(1 - \beta_0 u)} S_k = \frac{1}{(1 - \beta_0 u)} \sum_{\ell=0}^{k-1} \beta_\ell \left( u \frac{d}{du} + k - \ell \right) S_{k-\ell}. \quad (12)$$

<table>
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<th>$n_f$</th>
<th>$T_{1,0}$</th>
<th>$T_{2,0}$</th>
<th>$T_{2,1}$</th>
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TABLE 1: Coefficients for the imaginary part of the four-loop-order vector-current correlation function, as well as coefficients for the four-loop order MS $\beta$-function, are listed for three, four, and five quark flavors.
with initial conditions $S_0(0) = T_{k, 0}$.

Noting that $S_0(u) = 1$ and that $T_{1,0} = 1$ regardless of the number of active flavors, we find the first four solutions of $[\mathfrak{I}]$ to be

\[
S_1(xL) = \frac{1}{1 - \beta_0 xL},
\]

\[
S_2(xL) = \frac{T_{2,0} - \frac{\beta_2}{\beta_0} \log(1 - \beta_0 xL)}{(1 - \beta_0 xL)^2},
\]

\[
S_3(xL) = \left( \frac{\beta_3^2}{\beta_0^2} - \frac{\beta_3}{\beta_0} \right) \frac{1}{(1 - \beta_0 xL)^2} + \left( \frac{2T_{3,0} + \beta_3}{\beta_0} \right) \frac{2}{(1 - \beta_0 xL)^3} \log(1 - \beta_0 xL) + \left( \frac{2T_{2,0} + \beta_2}{\beta_0} \right) \frac{2}{(1 - \beta_0 xL)^3} \log^2(1 - \beta_0 xL) \right),
\]

\[
S_4(xL) = -\frac{1}{2} \beta_1 \beta_0 \frac{1}{(1 - \beta_0 xL)^2} + \left( \frac{2T_4,0 + \beta_4}{\beta_0} \right) \frac{2}{(1 - \beta_0 xL)^3} \log(1 - \beta_0 xL) + \left( \frac{2T_3,0 + \beta_3}{\beta_0} \right) \frac{2}{(1 - \beta_0 xL)^3} \log^2(1 - \beta_0 xL) + \left( \frac{2T_2,0 + \beta_2}{\beta_0} \right) \frac{2}{(1 - \beta_0 xL)^3} \log^3(1 - \beta_0 xL).
\]

To explore the near-infrared regime of perturbative QCD, we specialize to the case of three active flavors. Using Table 1, $n_f = 3$ values within Eqs. (16)–(20), we find that the version of the series $[\mathfrak{I}]$ which incorporates full summation of leading and two subsequent subleading orders of logarithms is given by

\[
S^{(\Sigma)}[x(\mu), L(\mu)] = 1 + x S_1(xL) + x^2 S_2(xL) + x^3 S_3(xL),
\]

where $(u = xL)$

\[
S_1(u) = \frac{1}{1 - 9u/4},
\]

\[
S_2(u) = \frac{1.63982 - 16 \log(1 - 9u/4)}{(1 - 9u/4)^2},
\]

\[
S_3(u) = \frac{16 \log(1 - 9u/4) + 3.16049 \log^2(1 - 9u/4)}{(1 - 9u/4)^3}. \tag{20} \]

Moreover, we note from Eq. (16) that

\[
S_4(xL) = -\frac{5.35589}{(1 - \frac{9}{4} xL)^2} + \left[ -6.62811 + 4.65981 \log(1 - \frac{9}{4} xL) \right] \frac{1}{(1 - \frac{9}{4} xL)^3}
\]

\[
+ \left[ T_{4,0} + 11.9840 + 31.8738 \log(1 - \frac{9}{4} xL) + 29.5046 \log^2(1 - \frac{9}{4} xL) - 5.61866 \log^3(1 - \frac{9}{4} xL) \right],
\]

thereby providing for inclusion of the $x^4 S_4(xL)$ contribution to the series $[\mathfrak{I}]$ for three active flavors. The series coefficient $T_{4,0}$ appearing in Eq. (21) has not yet been calculated perturbatively, which is why we have not included the $x^4 S_4(xL)$ contribution to $S(x, L)$ in Eq. (11). (An asymptotic Padé approximant estimate $T_{4,0} \simeq 1.90$ for the $n_f = 3$ case is presented in ref. 11.)

To examine whether the summation of leading and subsequent subleading logarithm factors decreases dependence on the unphysical renormalization-scale parameter $\mu$, we compare the $\mu$-dependence of Eq. (11) for a fixed value of $s$ to that of the $n_f = 3$ version of the series $[\mathfrak{I}]$ truncated after four-loop-order (4$\ell$) contributions to the vector-current correlation function:

\[
S^{(4\ell)}[x(\mu), L(\mu)] = 1 + x + (1.63982 + 9L/4)x^2 + (-10.2839 + 11.3792L + 81L^2/16)x^3. \tag{22}
\]
Such $\mu$-dependence enters Eqs. (13) and (14) both through $L = \log(\mu^2/s)$ and through $x = x(\mu)$, which is assumed to evolve via Eq. (15) (with $n_f = 3$ choices for $\beta_{0-3}$) from an initial value choice $x(m_r) = \alpha_s(m_r)/\pi = 0.33/\pi$ \cite{16,17}.

Figure 1 displays a comparison of the $\mu$-dependence of Eqs. (13) and (14) at fixed $s = 1.5$ GeV$^2$. Although both expressions exhibit little variation with $\mu$ over the 1.3 GeV $\leq \mu \leq$ 3 GeV range, we see that Eq. (13) exhibits much less variation with $\mu$ in the near-infrared regime below 1.3 GeV. These results clearly indicate that renormalization-scale-invariance is more effectively upheld via the summations of leading and subsequent subleading orders of logarithms that occur within Eq. (13). We emphasize that Eqs. (13) and (14) both follow from “RG-improvement” of the same calculational information [the coefficients $T_{1,0}$, $T_{2,0}$, and $T_{3,0}$]; simply put, such RG-improvement is more effectively implemented in Eq. (13) than in Eq. (14).

The usual prescription for obtaining the purely-perturbative (non-power-law) QCD contributions to $R(s)$ at four-loop order \cite{3,4} is to set $\mu = \sqrt{s}$ within Eq. (13), and then to substitute the resulting series,

$$S^{(4)}(x(\sqrt{s}), L(\sqrt{s})) = 1 + x(\sqrt{s}) + 1.63982 \, x^3(\sqrt{s}) - 10.2839 \, x^5(\sqrt{s}),$$

(23)

into Eq. (13). [Note from Eq. (13) that $L(\sqrt{s}) = 0$.] This prescription follows from the presumed renormalization-scale invariance of the truncated series \cite{4}, thereby leading to an expression that depends only on the physical scale $s$ for the electron-positron annihilation process. Note that all $s$-dependence of Eq. (13) resides entirely in the variable $L$. Let us first fix $\mu = m_r$ within Eq. (13) so as to incorporate the benchmark couplant value $x(m_r) = 0.33/\pi$ everywhere $x$ appears in Eqs. (13)–(14). With this choice, the following summation-of-logarithms series can be substituted into Eq. (13) to obtain $R(s)$:

$$S^{(4)}[x(m_r), L(m_r)] = 1 + \frac{0.33}{\pi} \, S_1 \left( \frac{0.33}{\pi} \log \left( \frac{m^2_r}{s} \right) \right) + \frac{0.33}{\pi} \, S_3 \left( \frac{0.33}{\pi} \log \left( \frac{m^2_r}{s} \right) \right).$$

(24)

In Figure 2, we compare the $s$-dependence of this series to that of Eq. (13), for which all $s$ dependence resides in the evolution of $x(\sqrt{s})$. To make this comparison, such evolution is anchored to the initial value $x(m_r) = 0.33/\pi$ via the differential equation \cite{3} with $n_f = 3$ values for $\beta_{0-3}$ \cite{16}. This initial value ensures that the series \cite{16} and \cite{17} coincide when $\sqrt{s} = m_r$. Figure 2 shows that both series continue to coincide over the range 750 MeV $\leq \sqrt{s} \leq m_r$. For values of $\sqrt{s}$ less than 750 MeV, however, the truncated series \cite{16} drops off quite suddenly at $\sqrt{s} \approx 650$ MeV, a consequence of the large negative coefficient of $x^3(\sqrt{s})$, whereas the series \cite{17} as obtained from the full summation-of-logarithms series \cite{16} continues to probe the infrared domain of $R(s)$ even for values of $\sqrt{s} \approx 400$ MeV. In short, the summation of all leading and subsequent two subleading logarithms within the perturbative series \cite{17} serves to
extend the domain of the \( R(s) \) series further into the infrared. This property, as well as the reduced renormalization-scale dependence evident in Figure 1 suggests that such summation is particularly appropriate for the near-infrared region characterizing sum-rule applications of purely-perturbative QCD corrections to current-correlation functions.

Fig. 2: The center-of-mass squared-energy \((s)\) dependence of the four-loop \((4L)\) truncated series \(\{4\}\) is compared to that of the summation-of-logarithms \((\Sigma)\) series \(\{5\}\), as described in the text. In both series, \(\alpha_s(m_t)\) is taken to be 0.33 so that the series equilibrate at \(\sqrt{s} = m_t\).

It is also evident from Figure 1 that the domain of the summation-of-logarithms series \(\{5\}\) manifests a singularity below \(\sqrt{s} = 400\) MeV, despite the fact that the \(s\)-dependence of the series \(\{4\}\) is decoupled entirely from any infrared behavior of the couplant \(x\), which is held constant at \(x(m_t)\). To understand this restriction on the domain of \(R(s)\), we first note that each summation \(\{3\} - \{1\}\) becomes singular when \(1 - \beta_0 x L \to 0\). Such resummation singularities have also been observed to occur in completely different contexts, including deep inelastic structure functions \(\{8\}\).

The singularity property of eqs. \(\{3\} - \{1\}\) is upheld for all summations \(S_k(x L)\). The solution to the differential equation \(\{24\}\) is necessarily of the form

\[
S_k(x L) = \frac{T_{k,0}}{(1 - \beta_0 x L)^k} + (\text{Particular solution depending on } \{S_{k-1}, S_{k-2}, \ldots, S_1\}).
\]

Since the coefficients \(T_{k,0}\) are results of \(k\)-th order Feynman diagram calculations, the \(k\)-th order pole in \(\{24\}\) at \(1 - \beta_0 x L = 0\) is genuine and will not be canceled by particular-solution contributions that are sensitive to at most \((k - 1)\)-th order Feynman-diagramatic coefficients \(\{T_{1,0}, T_{2,0}, \ldots, T_{k-1,0}\}\). For a given choice of renormalization scale \(\mu\), this singularity implies [via Eqs. \(\{23\}\) and \(\{25\}\)] that each summation \(S_k(x L)\) within the full series \(\{21\}\) becomes singular for a sufficiently small value of \(s\):

\[
1 - \beta_0 \frac{\alpha_s(\mu)}{\pi} \log \left( \frac{\mu^2}{s_{\text{min}}} \right) = 0 \quad \Rightarrow \quad s_{\text{min}} = \mu^2 \exp \left( \frac{-\pi}{\beta_0 \alpha_s(\mu)} \right).
\]

For example, if the renormalization scale \(\mu\) is chosen (as in Figure 2) to be \(m_t\), a choice for which \(\alpha_s(m_t) (= 0.33 \pm 0.02\) \(\{8\}\) is phenomenologically accessible, then each term in the series \(\{21\}\) is seen to become progressively more singular as \(s\) approaches \(m_t^2 \exp[-4\pi/(9 \times 0.33)] = (215\) MeV)\(^2\) from above. Furthermore, Figure 2 shows that the low-energy behaviour of the resummation expression \(\{25\}\) is only weakly dependent on \(\mu\) in the region \(\sqrt{s} > 600\) MeV for choices of \(\mu\) between \(0.6 m_t\) and \(1.6 m_t\), with the appearance of singular points corresponding to \(195\) MeV \(< s_{\text{min}} < 250\) MeV from \(\{24\}\).

It is to be emphasized that this infrared boundary on the physical scale \(s\) entering term-by-term within the series \(\{21\}\) is not a manifestation of any infrared boundary \(\{23\}\) on the evolution of the QCD couplant \(x(\mu)\). Even if the higher order contributions to the \(\beta\)-function \(\{24\}\) were somehow to conspire to allow the couplant to be well-behaved in the infrared region [e.g., to have infrared-stable fixed point behavior], the restriction \(\{24\}\) would still apply upon making a specific choice of the renormalization-scale parameter \(\mu\) and its corresponding value of \(\alpha_s(\mu)\). Curiously, though, this infrared restriction on \(s\) can be easily shown to coincide with the “infrared-slavery” Landau singularity \(\Lambda\) associated with naive evolution of the QCD couplant via a one-loop \(\beta\)-function. The one loop version of Eq. \(\{24\}\),

\[
\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta_0 x^2, \quad x = \frac{\alpha_s}{\pi}
\]
FIG. 3: The $s$ dependence of the four-loop truncated series \(^1\) for equally-spaced values of the renormalization scale in the range $0.6m_\tau \leq \mu \leq 1.6m_\tau$. The couplant $x(\mu)$ is obtained from three-flavour four-loop evolution via \(^2\) from the initial condition $x(m_\tau) = 0.33/\pi$ using Table \(^3\) values for the $\beta$ function coefficients $\beta_0\beta_2$.

is satisfied by the relation

$$\frac{\alpha_s}{\pi} = \frac{1}{\beta_0 \log(\mu^2/\Lambda^2)}$$

which is equivalent to Eq. \(^4\) provided $s_{\text{min}}$ is identified with $\Lambda^2$. Indeed, in a one-loop world $[x^{(1)}(\mu) = 1/\beta_0 \log(\mu^2/\Lambda^2)]$ where $\Lambda$ serves as a universal infrared boundary, the one-loop analogues of the summation-of-logarithms series \(^5\) and the truncated series \(^6\) are necessarily equivalent:

$$S_{1\tau}^{(2)} = 1 + x^{(1)}(\mu) S_1 \left[ x^{(1)}(\mu) \log \left( \mu^2/s \right) \right] = 1 + \frac{1}{\beta_0 \log \left( \frac{\mu^2}{\Lambda^2} \right)} \left\{ \frac{1}{1 - \beta_0 \log \left( \frac{\mu^2}{\Lambda^2} \right)} \log \left( \frac{\mu^2}{\Lambda^2} \right) \right\}$$

$$= 1 + \frac{1}{\beta_0 \log \left( \frac{\mu^2}{\Lambda^2} \right)} = 1 + x^{(1)} \left( \sqrt{\mu} \right)$$

We find it remarkable that $\Lambda$, the one-loop couplant’s Landau pole, persists as an infrared boundary on the domain of each summation contributing to Eq. \(^6\), the summation-of-logarithms formulation of the perturbative series within $R(s)$. Consequently, $s = \Lambda^2$ serves as an infrared boundary for any approximation to $R(s)$ involving the truncation of the series \(^6\), such as the expression \(^7\) obtained via optimal RG-improvement of the four-loop vector-current correlation function’s imaginary part.

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