The purpose of this report is to investigate the details of the fast decay process in the transition period between the Gaussian and exponential periods. In particular, we explore the strength of the confinement potential barrier and how it affects the $\hbar \approx 0$. We choose $B > \hbar > 0$ in this sense. For the potential barrier $V(x)$, we assume

\[ V(x) = \begin{cases} 0 & \text{for } x > 1, \\ \hbar & \text{for } 0 < x < 1, \\ 1 + \hbar & \text{for } -1 < x < 0. \end{cases} \]

The wave function $\psi(x)$ in the transition period between the Gaussian and exponential periods is determined by the time-dependent Schrödinger equation:

\[ \frac{\partial \psi(x,t)}{\partial t} = i \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \psi(x,t), \]

subject to the initial condition $\psi(x,0) = \psi_0$. We solve the Schrödinger equation numerically using the finite difference method. For the range of $x$, we take $[0,1]$. In order to avoid the singularity at $x = 0$, we use a logarithmic mesh for the range $[0,1]$. To solve the initial value problem, we take the initial wave function $\psi(x,0)$ as the Dirichlet boundary condition $\psi(x,0) = \psi_0$. The boundary condition is $\psi(x,0) = 0$ for $x < 0$ and $\psi(x,0) = \psi_0$ for $x > 0$.

The goal is to study the decay process of the particle in the presence of a potential barrier. In the transition period, the decay process is dominated by tunneling through the potential barrier. The rate of decay is given by the exponential function $e^{-\lambda t}$, where $\lambda$ is the decay rate. The decay rate is determined by the height of the potential barrier and the wavelength of the particle. The decay rate is smaller for higher potential barriers and for shorter wavelengths of the particle. In the limit of high potential barriers, the decay rate is exponentially small, and the particle is essentially confined to the region outside the barrier. In this limit, the decay process is described by the stationary wave function $\psi(x)$ and the decay rate is $\lambda = \frac{\hbar^2}{2m} V(x)$. The stationary wave function is given by

\[ \psi(x) = \begin{cases} 0 & \text{for } x > 1, \\ \frac{\hbar}{\sqrt{2m}} e^{-\frac{\hbar}{\sqrt{2m} V(x)}} & \text{for } 0 < x < 1, \\ 1 + \frac{\hbar}{\sqrt{2m}} e^{-\frac{\hbar}{\sqrt{2m} V(x)}} & \text{for } -1 < x < 0. \end{cases} \]

The decay rate is $\lambda = \frac{\hbar^2}{2m} V(x)$.

The decay process is accelerated by tunneling through the potential barrier. The rate of decay is given by the exponential function $e^{-\lambda t}$, where $\lambda$ is the decay rate. The decay rate is determined by the height of the potential barrier and the wavelength of the particle. The decay rate is smaller for higher potential barriers and for shorter wavelengths of the particle. In the limit of high potential barriers, the decay rate is exponentially small, and the particle is essentially confined to the region outside the barrier. In this limit, the decay process is described by the stationary wave function $\psi(x)$ and the decay rate is $\lambda = \frac{\hbar^2}{2m} V(x)$. The stationary wave function is given by

\[ \psi(x) = \begin{cases} 0 & \text{for } x > 1, \\ \frac{\hbar}{\sqrt{2m}} e^{-\frac{\hbar}{\sqrt{2m} V(x)}} & \text{for } 0 < x < 1, \\ 1 + \frac{\hbar}{\sqrt{2m}} e^{-\frac{\hbar}{\sqrt{2m} V(x)}} & \text{for } -1 < x < 0. \end{cases} \]

The decay rate is $\lambda = \frac{\hbar^2}{2m} V(x)$.
If we set $a = 1 + w$, $P(1 + w, t)$ represents the probability that the particle is confined by the potential barrier. For $w$ we consider a few different values. In order to compare the results for different values of $w$, we set $a$ such that $a > 1 + w_{\text{max}}$. Throughout this paper, we take $a = 4$.

The $P(a, t)$ is a kind of the nonecase probability, which is the probability that the particle has not escaped from the potential by time $t$ [11, 12].

Next we introduce function $g(a, t)$ defined by

$$g(a, t) = \frac{dP(a, t)/dt}{P(a, t)} \quad (5)$$

If probability $P(a, t)$ decays exponentially, that is, if $P(a, t) \propto e^{-\gamma t}$, then $g(a, t)$ is independent of time,

$$g(a, t) = -\gamma.$$ If $P(a, t)$ obeys the Gaussian decay law, that is, if $P \propto e^{-t^2/\tau}$, $g(a, t)$ is proportional to $t$:

$$g(a, t) = -2t/\tau. \quad (6)$$

Thus, from the $t$ dependence of $g(a, t)$ we can see how well $P(a, t)$ obeys the exponential law or the Gaussian law.

Figure 1 shows the nonecase probability $P(a, t)$, where the potential height and width are taken as $h = 10$ and $w = 0.6$, respectively. In the initial period $0 \leq t \lesssim 0.3$, the decay is extremely slow. The period should correspond to the Gaussian decay process. In the period of $t \gtrsim 2$, the system is subject to the exponential decay law. Between the two stages, there is a period $(0.3 \lesssim t \lesssim 2)$ in which the decay process is neither Gaussian nor exponential. This is the transition period. In the period the decay of the nonecase probability seems to be much faster than that in the exponential period.

The function $g(4, t)$ calculated from $P(4, t)$ of Fig. 1 is shown by the solid line in Fig. 2. In the period of $t \gtrsim 2$, $g(4, t)$ is almost constant, which means that the decay process is exponential. In the period of $0 \leq t \lesssim 0.3$, $g(4, t)$ is not exactly proportional to $t$, i.e., the decay process in the initial stage slightly deviates from the Gaussian decay law. However, the decay speed in this period is still smaller than that in the exponential period. Corresponding to the rapid decay process seen in $P(t)$ of Fig. 1, the maximum of $\bar{g}(4, t)$ is obtained in the transition period ($t \sim 0.6$). The $g(4, t)$ starts from zero at $t = 0$. Thus, the quantum Zeno effect is possible when we repeat measurements with a sufficiently small time interval. On the other hand, the anti-Zeno effect is possible by repeated measurements only when the net decay rate of the fast decay process in the transition period is large compared to that of the slow decay process in the Gaussian period (see footnote [13]).
In order to see how the decay process in the transition period depends on the strength of the potential barrier, we examine $g(4,t)$ for various potential heights and widths. The dashed and dotted-dashed lines in Fig. 2 show the $g(4,t)$ for $h = 20$ and $h = 30$, respectively with a fixed width $w = 0.6$. The decay process in the Gaussian period $0 \leq t \leq 0.3$ does not depend strongly on the potential height. This is because in the initial stage the higher energy components of the initial wave function relative to the potential height contribute mainly to the decay of the system. On the other hand, the decay speed in the exponential period becomes much smaller as the confinement becomes stronger. The fast decay in the transition period depends strongly on the potential height. It tends to be suppressed as the confinement becomes stronger.

In Fig. 3, we show $g(4,t)$ for various potential widths and a fixed height $h = 15$. The decay process in the Gaussian period $0 \leq t \leq 0.3$ does not seem to depend strongly on the potential width. On the other hand, the decay rate of the exponential period becomes much smaller as the potential width becomes broader. This is due to the increase of the confinement strength. The decay speed in the transition period becomes larger as the potential width becomes narrower.

As we have shown, the speed of the fast decay process becomes smaller as the potential barrier becomes stronger. Thus, we might guess that the fastest decay will be obtained in the decay process with no potential barrier. However, as we will show in the following, potential barriers with appropriate widths and heights can accelerate the decay process.

In Fig. 4, the solid line shows $g(4,t)$ calculated with no potential barrier. The dashed, dot-dashed and dotted lines exhibit $g(4,t)$ calculated with the potential barriers with $h = 10, 20$ and $30$ and fixed width $w = 0.2$, respectively. It should be noted that $w = 0.2$ is much thinner than those used in Figs. 2 and 3, and therefore, in this case, the confinement is very weak compared with that in Figs. 2 and 3. In the Gaussian period, the potential dependence of the decay speed is not very appreciable. However, in the transition region, the decay speed becomes faster as the potential height becomes lower. For $h = 10$, the maximum decay speed at $t \sim 1.8$ exceeds that for no potential barrier at $t \sim 0.7$.

As shown in Fig. 5, such an acceleration of the decay speed by tunneling gives rise to an appreciable difference in the time evolution of $P(4,t)$. The solid and dashed lines are the $P(4,t)$ for no potential barrier and for the potential barrier with $w = 0.2$ and $h = 10$, respectively. The nonescape probability for $h = 10$ becomes smaller than that for no potential barrier at $t \sim 1.5$. At this time the residual nonescape probability is still about ten percent. In this sense, the effect of this acceleration cannot be ignored. On the other hand, for $h = 20$, at the time region in which the nonescape probability becomes smaller than that for no potential barrier, the residual nonescape probability is negligibly small.

The fluctuations in the behaviors of $g(4,t)$ indicate that the decay processes are still in the transition period from the Gaussian to exponential period. However, as shown in Fig. 5, the decay process has been almost completed before $t = 4$. Therefore, even if the decay process proceeds to the exponential period eventually, the exponential decay has no importance in this case. The stability of quantum system is usually characterized by the magnitude of the imaginary part of the pole that gives the inverse of the lifetime of the exponential period [4, 5, 6]. The result that we have shown implies that such a pole analysis may not be effective for highly unstable systems [14].
FIG. 6: The time-evolution of $g(4, t)$ with a fixed height $h = 10$. The dashed, dot-dashed and dotted lines are the $g(4, t)$ for the heights $w = 0.2, 0.4$ and 0.6, respectively. The solid line shows the $g(4, t)$ for no potential barrier. The units are such that $h = 1$ and $2m = 1$.

Next, we investigate the acceleration with a fixed height. We examine the time evolution of $g(4, t)$ for $h = 10$ [15]. The dashed, dot-dashed and dotted lines in Fig. 6 exhibit $g(4, t)$ calculated for the potential barriers with $w = 0.2, 0.4$ and 0.6 and fixed height $h = 10$, respectively. One can see that the decay rate becomes larger for thinner potential widths. For $w = 0.2$, the maximum decay speed exceeds that for no potential barrier. Thus, one sees that the acceleration of the decay speed by tunneling can be obtained when the strength of the confinement by the potential barrier is sufficiently weak. In our illustrations, for $h \lesssim 10$ and $w \lesssim 0.2$, the accelerations are remarkable.

Finally, we mention that the $g(4, t)$ represented by dotted line in Fig. 4 ($w = 0.2$ and $h = 10$) takes positive values around $t \sim 3.5$. Recall that the probability current $j(a, t)$ is related to the non-escape probability $P(a, t) = \frac{1}{2} \left[ 1 + \frac{g(a, t)}{2} \right]$ with $P(a, t) = -dP(a, t)/dt = -g(a, t) P(a, t)$. This means that, if $g(a, t)$ is positive, $j(a, t)$ is negative. However, in [5, 16] the negative currents were obtained at very late time region after the exponential decay period. Our result implies that in a highly unstable quantum system the negative current can occur even at the initial stage after the Gaussian period.

We have investigated the fast decay process in the transition period between the Gaussian and exponential decay processes. In most cases of the tunneling process, the decay speed becomes smaller as the potential barrier becomes stronger. As a special case, we have found that the fast decay process can be remarkably accelerated by tunneling through potential barriers with appropriately small widths and heights. A detailed analysis of the acceleration of the fast decay process by tunneling is a future project.

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[9] The transition region between the exponential and power periods has been examined; see W. van Dijk and Y. Nogami, Phys. Rev. C65, 024008 (2002).
[13] In order to observe the anti-Zeno effect, we have to repeat the measurements with a time interval $\tau$ that satisfies the condition $F(\tau) = \int_0^\tau dt g(4, t + \tau_{exp}) < 0$, where $\gamma_{exp}$ is a decay rate in the exponential region.
[15] Notice that we consider the tunneling phenomena, so we do not investigate the lower height than $H = 10$.