Master equations for effective Hamiltonians

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Abstract. We reexamine on a general method for obtaining effective Hamiltonians that describe different nonlinear optical processes. The method exploits the existence of a nonlinear deformation of the su(2) algebra that arises as the dynamical symmetry of the original model. When some physical parameter (usually related to the dispersive limit) becomes small, we immediately get a diagonal effective Hamiltonian that represents correctly the dynamics for arbitrary states and long times. We apply the same technique to obtain how the noise terms in the original model transform under this scheme, providing a systematic way of including damping effects in processes described in terms of effective Hamiltonians.

PACS numbers: 03.65.Fd, 42.50.Ct, 42.50.Hz

1. Introduction

Effective Hamiltonians are employed frequently in quantum optics. They appear most often in calculations involving multiphoton processes [1, 2] or nonlinear optical effects [3, 4]. Notwithstanding that the authors who have taken recourse to these Hamiltonians are truly innumerable, owing to the great advantages they present in practice, the procedure of their derivation has hardly been given the attention which it merits. Even worse, sometimes they are regarded as phenomenological in nature and are not properly derived from the underlying microscopic theory. For example, in nonlinear optics it is customary to justify effective Hamiltonians on the basis of a quantized macroscopic theory of electrodynamics in a nonlinear dielectric medium [5], which is not entirely satisfactory, especially when one attempts to describe dispersive media [6].

Perhaps, the most common way of obtaining effective Hamiltonians is via adiabatic elimination [7–9]. This approach was pioneered by Graham and Haken [10] for the parametric oscillator with a medium consisting of two-level atoms and the general strategy can be stated as follows: starting from the exact equations governing the interaction of a discrete set of field modes with an ensemble of atoms, one invokes the assumption that the atomic polarization is a fast variable controlled by the slow motion of the field amplitudes and follows its evolution adiabatically (damping is negligible in this time scale). Under this hypothesis the atomic degrees of freedom can be eliminated and the resulting equations appear to be suffering a nonlinear interaction
that can be reinterpreted as the Heisenberg equations of motion for the field operator under the dynamics of an effective nonlinear Hamiltonian.

This is a physically appealing picture, but presents some serious drawbacks: first, it does not provide a general prescription for finding effective Hamiltonians since the particular details strongly depend on the model considered. Second, it could become very cumbersome, and explicit but enormously complicated expressions for the different orders of approximation can be found in many original publications [11, 12]. Third, the procedure is not uniquely defined: depending on the term eliminated, the outcome of the final Hamiltonian could be different [13–15].

For these compelling reasons, other methods of deriving effective Hamiltonians exist. For quantum optics, the ones devised in [16] and [17] are especially germane. A thorough review of alternative approaches may be found in [18]. Roughly speaking, all the methods have in common that at some point in their implementation one applies a unitary (or canonical) transformation to the total Hamiltonian and keeps only terms up to some fixed order. Although sometimes the results can be rather complicated, important examples of application may be given [19].

Recently [20, 21], we have provided a setting to support the theory of effective Hamiltonians by resorting to some elementary notions of group theory. The key point for that is the simple observation that most of the Hamiltonians in nonlinear quantum optics contain cubic or higher terms in creation and annihilation operators. Among others, typical examples are \( k \)th harmonic generation, \( k \)-wave mixing, and generalized Dicke models [9]. The common mathematical structure underlying all these cases is a nonlinear or polynomial deformation of \( su(2) \), which arises as the dynamical symmetry algebra of the corresponding Hamiltonian. This nonlinear algebra has recently found an important place in quantum optics [22–26] because it allows us to handle problems in very close analogy with the usual treatment of an angular momentum. In particular, we get a decomposition of the Hilbert space into direct sums of invariant subspaces and the dynamical problem generated by the corresponding Hamiltonian can be reduced to the diagonalization of a finite-dimensional matrix.

Unfortunately it is impossible to obtain, in general, analytic expressions for the eigenvalues and eigenstates of those matrices. Therefore, from the dynamical viewpoint this algebraic structure does not solve the problem. However, to some extent it can help to remedy it: if one applies a “nonlinear rotation” from this \( su(2) \) deformed algebra to the original Hamiltonian, the transformed one is rather involved, but when some physical parameter (dictated by the model under consideration) becomes small, one obtain an effective Hamiltonian that is diagonal and represents correctly the dynamics for arbitrary states and even for long times.

In this paper we wish to go one step further showing a unique advantage of our method. We stress that all the aforementioned techniques implicitly assume that the fields vary much more slowly than atomic damping times. This means that the resulting effective Hamiltonian can be used to describe only short-time evolution; otherwise losses begin to play an important role and cannot be avoided [27, 28].

On the contrary, our method allows one to take into account damping mechanisms in a natural way. Indeed, starting from the original microscopic model, we couple it to a reservoir and then follow a standard procedure [29]: from the Liouville equation for the system coupled to the reservoir, we trace over the reservoir variables and, after a Markovian approximation, we end up with a master equation. This equation can be transformed by the same nonlinear rotation as before, obtaining in this way what we call an effective master equation. In this paper we investigate this method and
present some relevant examples to illustrate the procedure.

2. Nonlinear rotations, effective Hamiltonians and effective master equations

To keep the discussion as self-contained as possible and to introduce the physical ideas underlying the method, let us start with the simplest case of some physical system whose Hamiltonian can be written as

\[ H = H_0 + H_{\text{int}} \] (2.1)

where \( H_0 \) describes the free dynamics and

\[ H_{\text{int}} = \Delta X_3 + g(X_+ + X_-), \] (2.2)

where \( g \) is a coupling constant and \( \Delta \) is a parameter usually representing the detuning between frequencies of different subsystems, although it is not necessary. The operators \( X_\pm \) and \( X_3 \) satisfy

\[ [X_3, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = P(X_3), \] (2.3)

where \( P(X_3) \) refers to an arbitrary polynomial function of the diagonal operator \( X_3 \) with coefficients perhaps depending on some integrals of motion \( N_j \). These commutation relations correspond to the so-called polynomial deformation of \( su(2) \). This kind of nonlinear algebras were discovered by Higgs \[30\] and Sklyanin \[31\] and have already played an important role in several aspects of quantum mechanics \[32–35\].

Now suppose that for some physical reasons (depending on the particular model under consideration) the condition

\[ \epsilon = \frac{g}{\Delta} \ll 1 \] (2.4)

is fulfilled. Then, it is clear that (2.2) is almost diagonal in the basis that diagonalizes \( X_3 \). In fact, a standard perturbation analysis immediately shows that the first-order corrections introduced by the nondiagonal part \( g(X_+ + X_-) \) to the eigenvalues of \( X_3 \) vanish and those of second order are proportional to \( \epsilon \ll 1 \). According to the technique developed in \[20\], we apply the following unitary transformation to equation (2.2)

\[ U = \exp[\epsilon(X_+ - X_-)], \] (2.5)

which, in fact, is a small nonlinear rotation, in such a way that

\[ H_{\text{eff}} = U H_{\text{int}} U^\dagger. \] (2.6)

Using the standard expansion \( e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots \), and after keeping terms up to order \( \epsilon^2 \), we get

\[ H_{\text{eff}} = \Delta X_3 + \frac{g^2}{\Delta} P(X_3). \] (2.7)

The essential point is that we have got a Hamiltonian that is diagonal in the basis of eigenstates of \( X_3 \). In consequence, the evolution (as well as the spectral) problem is completely solved in this approximation. Note that because (2.7) has the form of an expansion in the small parameter \( \epsilon \), its eigenvalues will coincide with those obtained using the standard perturbation theory in the same order of approximation. However, we stress that the method is fully operatorial and avoids the tedious work of computing the successive corrections as sums over all the accessible states.
To take into account damping mechanisms in the original model, we couple it to a reservoir and then follow a standard procedure: from the Liouville equation for the total system we trace over the reservoir variables and, after a Markovian approximation, we end up with a master equation of the form \[36,37\]

\[
\dot{\rho} = -i[H_{\text{int}}, \rho] + \sum_m \gamma_m \mathcal{L}[C_m] \rho, \tag{2.8}
\]

where \(\gamma_m\) are real parameters determined by the reservoir and \(\mathcal{L}[C_m]\) is known as the Lindblad superoperator

\[
\mathcal{L}[C_m] \rho = 2C_m \rho C_m^\dagger - \{C_m^\dagger C_m, \rho\}. \tag{2.9}
\]

Here \(C_m\) are eigenoperators of the system satisfying (in units \(\hbar = 1\), which we shall use throughout all this paper)

\[
[H_0, C_m] = \omega_m C_m, \tag{2.10}
\]

and their explicit form depends on the model under consideration.

It seems natural to ask how this equation is transformed by the same small rotation leading to the effective Hamiltonian. Instead of discussing an abstract formalism, we shall illustrate the main ideas by resorting to some selected models that are representative enough in quantum optics.

3. Two-mode coupled oscillators

The model of two coupled time-dependent harmonic oscillators has been studied by many authors, who have applied it to various problems of quantum mechanics and quantum optics. For instance, it was used to describe quantum amplifiers and converters in [38–40]. The explicit exact solutions and propagators of the Schrödinger equation, as well as solutions of the Heisenberg equations of motion, were considered in [41–45]. Squeezing, photon statistics, and entanglement in the system of two coupled oscillators were studied in [46–51]. The model has been used recently to illustrate the exchange of nonclassical properties [52] and the exchange of quantum states without energy transfer between two modes of the electromagnetic field [53].

The Hamiltonian for the model we are discussing is

\[
H = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a^\dagger b + b^\dagger a), \tag{3.1}
\]

where \(a(b)\) and \(a^\dagger(b^\dagger)\) are annihilation and creation operators for each of one of the interacting modes of frequency \(\omega_a\) and \(\omega_b\) and \(g\) is the coupling constant.

Because the excitation number \(N = a^\dagger a + b^\dagger b\) is an integral of motion we can recast (3.1) in the form (2.1) with

\[
H_0 = \frac{1}{2}(\omega_a + \omega_b)N, \tag{3.2}
\]

\[
H_{\text{int}} = \frac{\Delta}{2}(b^\dagger b - a^\dagger a) + g(a^\dagger b + b^\dagger a),
\]

where the detuning is \(\Delta = \omega_b - \omega_a\). But this is exactly of the form (2.2) when we introduce the operators

\[
X_+ = b^\dagger a, \quad X_- = b a^\dagger, \quad X_3 = \frac{1}{2}(b^\dagger b - a^\dagger a), \tag{3.3}
\]
which coincide with the standard generators of the su(2) algebra (without deformation).

Now suppose that one of the modes, say mode $b$, is lossy, while mode $a$ is lossless. Coupling the quantum oscillator $b$ to a standard reservoir leads to the following master equation for the density matrix in the interaction picture

$$\dot{\rho} = -i[H_{\text{int}}, \rho] + \frac{\gamma}{2} \mathcal{L}[b] \rho,$$

which has been thoroughly studied in the literature [7, 8].

From now on we shall be mainly interested in the dispersive regimen, when

$$g\sqrt{(\bar{n}_a + 1)(\bar{n}_b + 1)} \ll |\Delta|,$$

where $\bar{n}_a$ and $\bar{n}_b$ are the average number of excitations in each mode. Then, we can apply the small rotation (2.5) up to second-order terms obtaining

$$H_{\text{eff}} = \Delta b^\dagger b + \frac{g^2}{\Delta} (b^\dagger b - a^\dagger a).$$

As stated before, our goal is to examine how the master equation transforms under the same small rotation. To this end, let us denote

$$\rho_{\text{eff}} = U \rho U^\dagger.$$ (3.7)

Then, after some calculations, one gets that the new master equation becomes

$$\dot{\rho}_{\text{eff}} = -i[H_{\text{eff}}, \rho_{\text{eff}}] + \frac{\gamma}{2} \left(1 - \frac{g^2}{2\Delta^2}\right) \mathcal{L}[b] \rho_{\text{eff}} + \frac{\gamma g^2}{2 \Delta^2} \mathcal{L}[a] \rho_{\text{eff}}.$$ (3.8)

g5gg We can see here an effective transfer of decoherence for the lossy mode $b$ to mode $a$, which is of order $g^2/\Delta^2$. Note that second-order corrections to $H_{\text{eff}}$ are of the same order of magnitude, but they play no significant role because are diagonal.

If we further assume that mode $b$ is initially in vacuum we have no effective contributions from the normally-ordered terms like $b^\dagger b$ and the effective Hamiltonian reduces to

$$H_{\text{eff}} = -\frac{g^2}{\Delta} a^\dagger a,$$ (3.9)

while the master equation reduces to

$$\dot{\rho}_{\text{eff}} = -i[H_{\text{eff}}, \rho_{\text{eff}}] + \frac{\gamma g^2}{2 \Delta^2} \mathcal{L}[a] \rho_{\text{eff}}.$$ (3.10)

The transfer of decoherence between modes is here even more striking. Some efforts, especially with the method of stochastic unravellings [54], have been made to elucidate the physical contents of this kind of equations. We stress, however, that is not the scope of this paper to work out their solutions, but rather to show how they appear in the context of effective Hamiltonians.

4. Second-harmonic generation

Second-harmonic generation is the simplest nonlinear optical process. It exhibits a rich spectrum of nonclassical features such as photon antibunching, squeezing, or collapses and revivals [55–59]. Nikitin and Masalov [60] have shown that, at resonance, the quantum state of the fundamental mode evolves into a superposition of two macroscopically distinguishable states; i.e., a Schrödinger cat state. This point may
be considered unrealistic, because the stringent experimental constraints of having perfect phase matching and low decoherence could be rather difficult to attain.

In [61] we have considered the dispersive limit of second-harmonic generation, which seems to be almost ignored in the literature. Without assuming perfect resonance, this process is described by the following model Hamiltonian

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a^2 b^\dagger + a^{\dagger 2} b).$$

(4.1)

The excitation number

$$N = a^\dagger a + 2 b^\dagger b$$

is also a constant of motion. If the detuning is taken now as $\Delta = \omega_b - 2 \omega_a$, the Hamiltonian (4.1) can be rewritten in the general form (2.1) with

$$H_0 = \frac{1}{3}(\omega_b + \omega_a)N,$$

(4.2)

$$H_{\text{int}} = \frac{\Delta}{3} (b^\dagger b - a^\dagger a) + g(a^2 b^\dagger + a^{\dagger 2} b).$$

The polynomial deformation of su(2) naturally emerges when we rewrite

$$X^+_3 = \frac{1}{3}(b^\dagger b - a^\dagger a),$$

(4.3)

$$X^+_3 = \frac{1}{3} (b^\dagger b - a^\dagger a).$$

Let us next assume that we are in the dispersive limit

$$g(\bar{n}_a + 1) \sqrt{\bar{n}_b + 1} \ll |\Delta|,$$

(4.4)

and apply the small nonlinear rotation (2.5). The final result is [61]

$$H_{\text{eff}} = \frac{\Delta}{3} (b^\dagger b - a^\dagger a) + \frac{g^2}{\Delta} [4b^\dagger b a^\dagger a - (a^\dagger a)^2].$$

(4.5)

This Hamiltonian is diagonal, which implies that there is no population transfer between the modes, as it would be expected in the far-off resonant limit. The first term does not affect the dynamics and just leads to a rapid oscillation of the wave function.

When the harmonic mode $b$ is initially in the vacuum, the two terms containing $b^\dagger b$ do not contribute. In addition, the linear term $a^\dagger a$ leads just to a $c$-number phase shift and can be also omitted. In consequence, (4.5) reduces to

$$H_{\text{eff}} = - \frac{g^2}{\Delta} (a^\dagger a)^2,$$

(4.6)

which is nothing but the interaction Hamiltonian that governs the state evolution of the single-mode field $a$ in a Kerr medium. We emphasize that Kerr effect provides a nonlinearity of particular interest for generating field cat states [62–66]. Apart from their intrinsic simplicity, Kerr-based schemes have the specific advantage of not relying on conditional measurements. However, the realistic values of Kerr coefficients are quite small, thus requiring a large interaction length. Then, losses become significant and may destroy the very delicate quantum superpositions. In short, although very appealing from the physical viewpoint, Kerr schemes are not generally considered to be realistic.

For this reason, the identification proposed in (4.6) could be more than an academic curiosity: second-harmonic generation is, for a variety of reasons [67], more robust than the Kerr effect as for noise-limiting factors. In consequence, this scheme
could be an experimentally feasible proposal to generate optical cat states. However, it is clear that losses should be taken into account. While this is standard in a Kerr medium, for the case of the dispersive limit of second-harmonic generation is far from being known. In our approach, by transforming the standard master equation (under the same condition of harmonic mode $b$ initially in the vacuum) one easily gets that

$$\dot{\rho}_{\text{eff}} = -i[H_{\text{eff}}, \rho_{\text{eff}}] + \frac{\gamma g^2}{2 \Delta^2} \mathcal{L}[a^2] \rho_{\text{eff}},$$

(4.7)

which, in fact, agrees with the standard form of dealing with decoherence effects in a Kerr medium, in which all the dissipation (up to order $g^2/\Delta^2$) takes place via two-photon process.

### 5. Dicke model

Finally, let us consider the example of the well-known Dicke model describing the interaction of a single-mode field of frequency $\omega_f$ with a collection of $A$ identical two-level atoms with transition frequency $\omega_0$. Making the standard dipole and rotating-wave approximations, the model Hamiltonian reads as

$$H = \omega_f a^\dagger a + \omega_0 S_3 + g(a S_+ + a^\dagger S_-),$$

(5.1)

where $(S_+, S_-)$ are collective atomic operators forming an $(A + 1)$-dimensional representation of $\text{su}(2)$.

Now the excitation number reads as $N = a^\dagger a + S_3$, and once again we can recast (5.1) in the form (2.1) with

$$H_0 = \omega_f N,$$

(5.2)

$$H_{\text{int}} = \Delta S_3 + g(a S_+ + a^\dagger S_-),$$

the detuning being $\Delta = \omega_0 - \omega_f$. The operators defining the $\text{su}(2)$ deformation are given by

$$X_+ = a S_+,$$

$$X_- = a^\dagger S_-,$$

(5.3)

$$X_3 = S_3,$$

and then the interaction part of the Dicke Hamiltonian looks exactly as in (2.2).

To keep our discussion as realistic as possible, let us consider the case in which the atomic system and the field interact in a lossy cavity, in such a way that photons leakage occurs say through a partially transmitting mirror with a decay rate given by $\gamma$. In the case of optical frequencies, thermal excitation from the environment of the continuum of modes outside the cavity is negligible and the dynamics is well described by the master equation [68]

$$\dot{\rho} = -i[H_{\text{int}}, \rho] + \frac{\gamma}{2} \mathcal{L}[a] \rho.$$  

(5.4)

This decay is also responsible for the rapid decay of any eventual quantum coherence generated within the cavity.

The dispersive limit may be stated now in the form [69]

$$Ag\sqrt{n + 1} \ll |\Delta|,$$

(5.5)
where $\bar{n}$ is the average number of photons in the field. Then, the nonlinear rotation (2.5) transforms the Hamiltonian (5.2) into [20]

$$H_{\text{eff}} = \Delta S_3 + \frac{g^2}{\Delta}[S_3^2 - 2(a^\dagger a + 1)S_3 - C_2],$$

(5.6)

where $C_2 = A/2(A/2 + 1)$ is the value of the Casimir operator for $\text{su}(2)$. The Hamiltonian (5.6) was previously obtained in [70] by quite a different method (see also [71]) and, due to the presence of the nonlinear term $S_3^2$, has been considered as a candidate for the generation of atomic Schrödinger cats and squeezed states [72].

Transforming the master equation (5.4) by the operator (2.5) we obtain

$$\dot{\rho}_{\text{eff}} = -i[H_{\text{eff}}, \rho_{\text{eff}}] + \mathcal{L}_1[\rho_{\text{eff}}] + \varepsilon \mathcal{L}_2[\rho_{\text{eff}}] + \varepsilon^2 \mathcal{L}_2[\rho_{\text{eff}}],$$

(5.7)

where

$$\mathcal{L}_1[\rho_{\text{eff}}] = 2(S_- \rho_{\text{eff}} a^\dagger + a \rho_{\text{eff}} S_+) - a^\dagger S_- \rho_{\text{eff}} - S_+ a \rho_{\text{eff}} - \rho_{\text{eff}} a^\dagger S_- \rho_{\text{eff}} S_+, \quad \mathcal{L}_2[\rho_{\text{eff}}] = 2S_- \rho_{\text{eff}} S_+ - S_+ S_- \rho_{\text{eff}} - \rho_{\text{eff}} S_+ S_- + 2a S_3 \rho_{\text{eff}} a^\dagger - 2 \rho_{\text{eff}} S_3 a^\dagger a^\dagger \rho_{\text{eff}} a.$$  

(5.8)

In the rotating frame, the term $\mathcal{L}_1$ oscillates very rapidly and consequently can be eliminated by the corresponding transformation to a rotating frame of $\rho_{\text{eff}}$. Surprisingly, the term $\mathcal{L}_2$ contains a resonant part that is time independent in the rotating frame. This term generates effective atomic dissipation, leading to a complete decoherence for times of the order of $1/(\varepsilon^2 \gamma)$.

To further proceed, let us assume that initially the field is in vacuum, while the atom is in an arbitrary atomic state $\rho_{\text{at}}(0)$. One could expect that in this dispersive limit, because there is no net energy transfer between the atoms and the field, the atomic system would not feel the field. One can check that

$$\dot{\rho}_{\text{eff}} = -i[H_{\text{eff}}, \rho_{\text{eff}}] + \frac{\varepsilon^2}{\gamma}(S_- \rho_{\text{eff}} S_+ - S_+ S_- \rho_{\text{eff}} - \rho_{\text{eff}} S_+ S_-),$$

(5.9)

which shows the appearance of an effective atomic dissipation due to the low-amplitude atom-field transitions generated by corrections to the initial state due to the transformation by the nonlinear rotation $U$.

6. Concluding remarks

In this paper we have studied a systematic way of including damping in processes described by effective Hamiltonians. Against the widespread opinion that noise cannot be included in this kind of theories (and, therefore, they describe only short-time evolution) we have shown how to obtain an effective master equation of the Lindblad type that allows one to take into account the dissipation from first principles and not merely adding phenomenological terms.

Here we have considered three specific models of interest in quantum optics, although it is easy to convince oneself that our results can be generalized to any system that can be described in terms of a polynomial deformation of the algebra $\text{su}(2)$. We finally stress that we have not discussed in detail the effects of this dissipation for each model under consideration, since our main goal has been only to provide the adequate framework to deal with such effects.
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[41] Orszag M 2000 Quantum Optics (Berlin: Springer)