Symmetry-preserving matchings

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Abstract

In the literature, the matchings between spacetimes have been most of the times implicitly assumed to preserve some of the symmetries of the problem involved. But no definition for this kind of matching was given until recently. Loosely speaking, the matching hypersurface is restricted to be tangent to the orbits of a desired local group of symmetries admitted at both sides of the matching and thus admitted by the whole matched spacetime. This general definition is shown to lead to conditions on the properties of the preserved groups. First, the algebraic type of the preserved group must be kept at both sides of the matching hypersurface. Secondly, the orthogonal transivity of two-dimensional conformal (in particular isometry) groups is shown to be preserved (in a way made precise below) on the matching hypersurface. This result has in particular direct implications on the studies of axially symmetric isolated bodies in equilibrium in General Relativity, by making up the first condition that determines the suitability of convective interiors to be matched to vacuum exteriors. The definition and most of the results presented in this paper do not depend on the dimension of the manifolds involved nor the signature of the metric, and their applicability to other situations and other higher dimensional theories is manifest.

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1 Introduction

Many practical problems of physical interest involve idealised models constructed by joining two regions, both admitting a certain symmetry\(^1\), in such a way that the whole model admits that symmetry globally. These matchings are then said to preserve the symmetry, and have been extensively used, for instance, in the description of astrophysical objects, the issue of the influence of cosmological (global) dynamics on local systems and the description of hypersurface layers, such as domain walls and 'brane-world' scenarios.

Although the concept of symmetry–preserving matching has an intuitive clear meaning, sometimes its application has not fully properly accounted for all the possibilities. For instance in some situations the matching hypersurface has been prescribed ‘by hand’, making ‘hidden’ implicit assumptions, such as a preassigned meaning of the coordinates or the preservation of geometrical properties, sometimes inferring physical constraints to the model. The problem here is that the lack of generality prevents conclusive results, and hence the need for a general clear definition. There have also been some previous works \([1, 2]\) where a general symmetry–preserving matching, taking into account the orbits of the preserved local groups at both sides of the matching hypersurface, has been performed. Nevertheless, no definition for symmetry–preserving matching had been given in general form until recently \([3]\).

The purpose of this paper is twofold: In first place, to motivate and present the definition of symmetry–preserving matching as stated in \([3]\). And secondly, to show two immediate consequences on the algebraic and geometric properties of the group preserved by the matching. These results can lead to restrictions on the physical properties of the model. In fact, it will be shown here that some of the ‘hidden’ assumptions mentioned above can be derived from the junction conditions as a consequence of the preservation of the symmetry.

Loosely speaking, the definition states that the matching hypersurface is restricted to be tangent to the orbits of a local group wanted to be admitted by the whole matched spacetime. Its strict use leads to more general parametrisations than usual, which eventually manifests in more general ways of performing the matching, some of them with clear physical differences. The definition of symmetry–preserving matching has already been proven very useful in obtaining some general and conclusive results \([2, 3, 4, 5, 6]\). In this sense, in the study of the uniqueness problem of the exterior solution given a known interior of an isolated axially symmetric body in equilibrium in General Relativity \([2]\), the implicit use of the definition introduces two essential parameters giving rise to inequivalent exteriors (if they exist). One of these parameters represents, for example, the rotation of the isolated body as seen by the stationary observer at spatial infinity \([2]\). Another example where the introduction of parameters by the matching procedure is crucial has arisen in the study of the existence of locally cylindrically symmetric static regions inside spatially homogeneous spacetimes in four dimensions \([6]\). As it will be explained below, the existence of any static region of that kind in a particular class of spatially homogeneous spacetimes is only possible for non-zero values of one of the new parameters. One could summarise the situation by saying that the new parameters introduced by the matching procedure are \textit{locally}\(^2\)...

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\(^1\)In what follows only symmetries defined by local groups of diffeomorphisms will be considered.
meaningless, because they can always be absorbed in the coordinates, but \textit{globally} essential \cite{2, 3, 7, 6}.

The definition has also geometric consequences; first, the matching hypersurface inherits the preserved symmetry and its algebraic type, and as a result, the algebraic type of the preserved group will be necessarily kept across the matching. Secondly, the point-wise property that, when satisfied in an open set, ensures the orthogonal transitivity of any two dimensional conformal $G_2$ group (including, of course, any $G_2$ group of isometries) will also be kept on the matching hypersurface if the matching preserves that symmetry. For simplicity, and abusing the terminology, I will refer to this by saying that “the orthogonal transitivity is ‘preserved’ on the matching hypersurface” in what follows. In General Relativity, a very well known result \cite{8, 9, 10, 11} can be used then to ensure the extension of the orthogonal transitivity on a connected open set intersecting the matching hypersurface when the $G_2$ is Abelian and certain types of matter content are present. These types include $\Lambda$-term type of matter, in particular vacuum, and perfect fluids whose flow is orthogonal to the orbits (if the $G_2$ acts on spacelike $S_2$ surfaces) or tangent to the orbits (if the $G_2$ acts on timelike $T_2$ surfaces).

The ‘preservation’ of the orthogonal transitivity on the matching hypersurface will have a clear implication, for instance, on the studies of global models describing rotating axially symmetric objects in equilibrium mentioned above. These models are based on stationary and axisymmetric spacetimes (admitting thus a $G_2$ on $T_2$) consisting of the matching of a vacuum exterior region to an interior region across a timelike matching hypersurface that preserves both the stationarity and the axial symmetry. This hypersurface represents the surface of the body at all times. As a result of the existence of the axis of symmetry, the $G_2$ on $T_2$ group on the exterior vacuum region acts orthogonally transitively because the Ricci tensor vanishes there \cite{8}. For convenience, it has been usually assumed an orthogonally transitive $G_2$ also in the interior. This so-called \textit{circularity condition} is equivalent to the absence of convective motions in the interior, and therefore restricts the physics of the astrophysical object. Dropping the circularity condition in the interior problem in order to deal with more general situations, the result shown in this paper will imply that the aforementioned property coupled with the orthogonal transitivity holds on the matching hypersurface for the $G_2$ on $T_2$ in the interior region. In practical terms, this will simplify the structure of the matching conditions for the more general problem.

Section 2 is devoted to a brief review of the matching procedure in order to motivate and present the definition of symmetry-preserving matching in Section 3, where the preservation of the algebraic type is also shown. The ‘preservation’ of the orthogonal transitivity for $G_2$ conformal groups on the matching hypersurface is addressed in Section 4.

Although the aim of this paper has been originally focused on General Relativity, the definition and results presented in this paper, except Corollary 4.2, neither depend on the dimension of the manifolds involved nor on the signature of the metric. Whenever the word “spacetime” is used in the definition and the results, it can perfectly be replaced by “manifold with metric”. The applicability to other situations and other higher dimensional theories is manifest.
2 Matching procedure

From the theoretical point of view, given two \( d+1 \)-dimensional \( C^3 \) spacetimes \((\mathcal{W}^+, g^+)\) and \((\mathcal{W}^-, g^-)\) each of them with oriented boundary \( \Sigma^+ \) and \( \Sigma^- \) [12], respectively, such that \( \Sigma^+ \) and \( \Sigma^- \) are diffeomorphic, the whole matched spacetime \( \mathcal{V} \) is the disjoint union of both \( \mathcal{W}^\pm \) with diffeomorphically related points in \( \Sigma^\pm \) identified and such that the junction conditions are satisfied [13, 14, 15, 16, 17]. In some cases, though, certain junction conditions will be relaxed in order to allow for distribution layers, see below. A brief review of the procedures involved in the practical setting of the matching problem together with these junction conditions will be given explicitly in what follows.

Let us consider the practical problem [18] of the matching of two given \( C^3 \) spacetimes \((\mathcal{V}^+, g^+)\) and \((\mathcal{V}^-, g^-)\). First, we need the \( d \)-dimensional hypersurfaces \( \Sigma^+ \subset \mathcal{V}^+ \) and \( \Sigma^- \subset \mathcal{V}^- \) which are going to be identified. As \( \Sigma^\pm \) must be diffeomorphic to each other, they can also be considered as diffeomorphic to an abstract \( d \)-dimensional oriented \( C^3 \) manifold \( \sigma \) which can be appropriately embedded both in \( \mathcal{V}^+ \) and \( \mathcal{V}^- \).

Letting \( \{\lambda^a\} \) (Latin indexes \( a, b, \ldots = 1, \ldots, d \)) be any local coordinate system on \( \sigma \) and \( \{x^{\pm a}\} \) (Greek indexes \( \alpha, \beta, \ldots = 0, \ldots, d \)) local coordinates on \( \mathcal{V}^\pm \), respectively, the two embeddings are given by the \( C^3 \) maps

\[
\Phi^\pm : \sigma \rightarrow \mathcal{V}^\pm \\
\lambda^a \mapsto x^{\pm \alpha} = \Phi^\pm_\alpha (\lambda^a),
\]

such that \( \Sigma^\pm = \Phi^\pm (\sigma) \). The diffeomorphism from \( \Sigma^+ \) to \( \Sigma^- \) is trivially \( \Phi^- \circ \Phi^+^{-1} \).

The hypersurfaces \( \Sigma^\pm \) split locally each corresponding spacetime \((\mathcal{V}^\pm, g^\pm)\) into two complementary parts. These parts are then the spacetimes with oriented boundary \((\mathcal{W}_1^\pm, g^\pm, \Sigma^\pm)\) and \((\mathcal{W}_2^\pm, g^\pm, \Sigma^\pm)\) which are to be matched, and thus, four possible different matchings are possible in principle, although only two of them will be actually inequivalent [18]. In order to simplify notations, whatever part \( \mathcal{W}_1^+ \) or \( \mathcal{W}_2^+ \) is chosen to be matched with \( \mathcal{W}_1^- \) or \( \mathcal{W}_2^- \), they will correspond to the theoretical \( \mathcal{W}_1^+ \) and \( \mathcal{W}_1^- \).

Given the natural basis \( \{\partial/\partial \lambda^a\} \) of the tangent bundle \( T\sigma \), the rank-\( d \) differential maps \( d\Phi^\pm \) apply \( \{\partial/\partial \lambda^a\} \) into \( d \) linearly independent vector fields at \( \Sigma^\pm \), denoted by \( \vec{e}^\pm_a |_{\Sigma^\pm} \), defined only on the corresponding hypersurfaces \( \Sigma^\pm \), as follows

\[
d\Phi^\pm \left( \frac{\partial}{\partial \lambda^a} \right) = \frac{\partial \Phi^\pm_{\mu}}{\partial \lambda^a} \left( \frac{\partial}{\partial x^{\pm \mu}} \right) \bigg|_{\Sigma^\pm} = \vec{e}^\pm_a |_{\Sigma^\pm}.
\]

Using the pull-backs \( \Phi^{\pm*} \), the metrics \( g^\pm \) can be mapped to \( \sigma \) providing two symmetric 2-covariant tensor fields defined by \( \bar{g}^\pm \equiv \Phi^{\pm*} (g^\pm |_{\Sigma^\pm}) \). These are the first fundamental forms of \( \Sigma^\pm \), and in components they read

\[
\bar{g}^\pm_{ab} = \vec{e}^\pm_a \vec{e}^\pm_b g^\pm_{\mu\nu} |_{\Sigma^\pm}.
\]

As shown in [16, 17], the necessary an sufficient condition such that there exists a continuous extension \( g \) of the metric to the whole manifold \( \mathcal{V} \) and such that \( g|_{\mathcal{W}^+} = g^+ \) and \( g|_{\mathcal{W}^-} = g^- \) is that

\[
\bar{g}^+ = \bar{g}^-.
\]
These relations were called the preliminary junction conditions\textsuperscript{2} in [17] and they have been traditionally used since the work of Darmois [13], Lichnerowicz [14] and Israel [15]. Once the above construction has been carried out, after identifying the points $\Phi^+(p) = \Phi^-(p) \equiv q$ for all $p \in \sigma$, the bases $\{\vec{e}^+_a|_q\}$ and $\{\vec{e}^-_a|_q\}$ can also be identified at every $q$. I will denote simply by $\Sigma (\equiv \Sigma^+ = \Sigma^-)$ the hypersurface in the final matched manifold $\mathcal{V}$.

The one-forms normal to the hypersurfaces, denoted by $n^\pm$, are defined on $\Sigma^\pm$ up to a multiplicative non-zero factor through the condition

$$n^\pm(\vec{e}_a^\pm) = 0.$$ \hspace{1cm} (3)

In addition, $n^+$ and $n^-$ must have the same norm in order to ensure their eventual proper identification on the final matched spacetime $\mathcal{V}$. The arbitrariness in the sign in both $n^+$ and $n^-$ will account for their possible relative orientations [21, 18], see below.

In order to deal with general hypersurfaces (changing its causal character from point to point), one also needs two transversal $C^2$ vector fields $\vec{\ell}^\pm$ defined on $\Sigma^\pm$, the so-called rigging vectors [22]. Obviously, for the case of non-null hypersurfaces the normal vector is itself a rigging. The riggings can be chosen by

$$n^\pm(\vec{\ell}^\pm) = 1$$ \hspace{1cm} (4)

Recalling that the preliminary conditions allowed us to identify $\{\vec{e}_a^+\}$ with $\{\vec{e}_a^-\}$, at this point there only remains to choose the riggings such that $\{\vec{\ell}^\pm, \vec{e}_a^\pm\}$ are both bases with the same orientation and such that

$$\ell^\mu \ell^{+\mu} \Sigma \equiv \ell^- \ell^{-\mu} \Sigma, \quad \ell^\mu e^+_a \Sigma \equiv \ell^- e^-_a \Sigma,$$ \hspace{1cm} (5)

where $\Sigma$ means that both sides of the equality must be evaluated on $\Sigma^\pm$ respectively. Then, we can identify the whole $d + 1$-dimensional tangent spaces of $\mathcal{V}^\pm$ at $\Sigma$, $\{\vec{\ell}^+, \vec{e}_a^+\} = \{\vec{\ell}^-, \vec{e}_a^-\} \equiv (\vec{\ell}, \vec{e}_a)$, as well as their respective dual co-bases $\{n^+, w^{+a}\} = \{n^-, w^{-a}\} \equiv (n, w^a)$. It must be taken into account that the first relation in (5) is implied by (3) provided that the preliminary junction conditions and the second in (5) hold.

The existence of a continuous $g$ allows for the treatment of Einstein’s equations in the distributional sense [15, 23, 24, 16, 17]. Now, the vanishing of the singular part of the Riemann tensor distribution is equivalent to the equality of the tensor fields defined by

$$\mathcal{H}^\pm_{ab} = e^A_a e^B_b \nabla_\mu \ell^\pm \nu.$$ \hspace{1cm} (6)

\textsuperscript{2}Actually, the sufficient conditions for the continuous extension of the metric at points where the matching hypersurface is null needs also the existence of the two rigging vector fields with proper orientations as are defined later. Their existence was erroneously stated in [16]. I refer to [20] for a detailed discussion and examples.
at both sides of $\Sigma$ [17]. Of course, these latter junction conditions are omitted in the studies focused on ‘surface’ layer distributions, such as topological defects, ‘brane-world’ scenarios, and others. For the case of non-null hypersurfaces, $H^\pm_{ab}$ coincide (up to a sign) with the second fundamental forms $K^\pm_{ab} = \epsilon_a^\pm \epsilon_b^\pm \nabla_a^\pm n_b^\pm$ inherited by $\Sigma^\pm$ from $V^\pm$ [13, 16, 17] by choosing $\vec{\ell} = \pm \vec{n}$ (plus sign when $\vec{n}$ is spacelike and minus when timelike). The junction conditions

$$H^+_a b \Sigma = H^-_{ab},$$

(7)
do not depend on the specific choice of the rigging [17].

Notice that the choice of the same orientation for both bases $\{\vec{\ell}^\pm, \vec{e}^\pm_a\}$ is the same as choosing the riggings in such a way that if $\vec{\ell}^-$ points from $W^-$ outwards, then $\vec{\ell}^+$ points inwards onto $W^+$, or vice versa [18]. The relative orientation of the rigging vectors clearly translate to the relative orientations of the normals through (3).

### 3 Definition of symmetry–preserving matching

In many practical problems of physical interest one looks for a final whole spacetime having some symmetries and matched across a hypersurface “naturally” defined by them. This situation requires not only that both spacetimes to be matched must contain these particular symmetries, but also that the matching hypersurface inherits the symmetry. This type of matching as well as the matching hypersurface are usually said to preserve the given symmetry. Illustrative simple examples of matchings that preserve the symmetry are given by the traditional spherically symmetric ones performed across any time-dependent invariantly defined 2-sphere. Of course, the idea behind all this is that we demand that the matching hypersurface be tangent to the orbits of the local symmetry group to be preserved.

Suppose that we are given two initial spacetimes $(V^\pm, g^\pm)$ admitting the local groups of symmetries $G^+_n$ and $G^-_n$ respectively. Suppose also that we want the final matched spacetime $(V, g)$ to have a local group of symmetries $G_m$ which is a subgroup of both $G^\pm_n$, so that $m \leq \min\{n^+, n^\pm\}$. Here by local symmetries we mean not only isometries but more general ones such as homothetic and conformal motions, etc (see e.g. [22, 19]). The motivation for the definition as presented here comes from the consideration of symmetries that involve the metric (including the conformal structure). Despite this fact, the definition does not depend on the kind of symmetry, and thus symmetries that concern other objects, as for instance collineations, projective symmetries, etc (see e.g. [25]) could also be considered.

Take any generator $\vec{\xi}$ of the subgroup $G_m$, such that $\mathcal{L}_{\vec{\xi}} g$ has the corresponding form (for instance, zero if $\vec{\xi}$ is a Killing vector field), and assume that the restriction of $\vec{\xi}$ to the hypersurface $\Sigma$ is tangent to $\Sigma$. This means that

$$\vec{\xi}|_\Sigma = \xi^a \vec{e}_a|_\Sigma,$$

(8)

where the $\xi^a$ are three functions defined on $\Sigma \subset V$. Therefore, $\vec{\xi}$ provides in a natural way a unique vector field on $\sigma$ [22] denoted by $\vec{\gamma}$ such that

$$d\Phi(\vec{\gamma}) = \vec{\xi}|_\Sigma.$$
A straightforward calculation shows then that [22]

\[ \mathcal{L}_{\vec{\gamma}} \bar{g} = \Phi^* \left( \mathcal{L}_{\vec{\xi}} g \big|_{\Sigma} \right). \]  

(10)

This result means that \( \vec{\gamma} \) is a symmetry in \((\sigma, \bar{g})\) of the same (or more specialised) type as \( \vec{\xi} \) is in \((\mathcal{V}, g)\). Notice that the fundamental property in all this construction is the tangency of \( \vec{\xi} \) to \( \Sigma \). In short, the symmetry defined by \( \vec{\xi} \) in \((\mathcal{V}, g)\) is inherited by \( \sigma \) whenever \( \vec{\xi} \) is tangent to its image \( \Sigma \subset \mathcal{V} \). In fact, equation (10) allows us to show the following:

**Lemma 3.1** Let \( \vec{\xi}^+ \) and \( \vec{\xi}^- \) be two conformal Killing vector fields acting on \((\mathcal{V}^+, g^+)\) and \((\mathcal{V}^-, g^-)\) respectively: \( \mathcal{L}_{\vec{\xi}^\pm} g^\pm = \alpha^\pm g^\pm \) for given, possibly zero, functions \( \alpha^\pm \). If \((\mathcal{V}^+, g^+)\) and \((\mathcal{V}^-, g^-)\) are (preliminary) matched across a matching hypersurface \( \Sigma \equiv \Sigma^+ = \Sigma^- \) diffeomorphic to \( \sigma \) by (1) such that there is a vector field \( \vec{\gamma} \) satisfying (9) for both \( \vec{\xi}^+ \) and \( \vec{\xi}^- \), then \( \alpha^+ \Sigma = \alpha^- \).

**Proof.** Equation (10) for the (+) part reads

\[ \mathcal{L}_{\vec{\gamma}} \bar{g}^+ = \Phi^{+*} \left( \mathcal{L}_{\vec{\xi}^+} g^+ \big|_{\Sigma^+} \right) = \alpha^+|_{\Sigma^+} \Phi^{+*} (g^+|_{\Sigma^+}) = \alpha^+|_{\Sigma^+} \bar{g}^+, \]

and analogously for the (−) part. The preliminary junction conditions (2) clearly imply \( \alpha^+ \Sigma = \alpha^- \) after the identification \( \Sigma \equiv \Sigma^+ = \Sigma^- \). □

After the above discussion, it seems natural to give the following definition of symmetry–preserving matching:

**Definition 1** Let \((\mathcal{V}, g)\) be a spacetime arising from the matching of two oriented \( C^3 \) spacetimes \((\mathcal{V}^\pm, g^\pm)\) admitting a \( G_{n^+} \) and \( G_{n^-} \) local group of symmetries, respectively, and with respective boundaries \( \Sigma^\pm \) given by the embeddings (1). Then, \((\mathcal{V}, g)\) preserves the symmetry defined by the subgroup \( G_m \) with \( m \leq \min\{n^+, n^-\} \) when first, this group is admitted by both \((\mathcal{V}^\pm, g^\pm)\), and second, the differential maps \( d\Phi^\pm \) send \( m \) vector fields \( \vec{\gamma}_A \) (\( A = 1 \ldots m \)) on \( \sigma \) to the restrictions of the generators \( \vec{\xi}^\pm_A \) of \( G_m \) to \( \Sigma^\pm \).

**Remark:** It must be taken into account that if there is an intrinsically distinguished generator of \( G_m \) on \( \mathcal{V}^+ \) and \( \mathcal{V}^- \), then the symmetry–preserving matching must ensure its identification at \( \Sigma \).

From the definition, and depending on the kind of symmetry involved, it may follow that \( m \) has a maximum. In the case of isometries, for instance, this corresponds to the case when the matching hypersurface \( \Sigma \) is maximally symmetric, and hence \( m \leq d(d+1)/2 \) in this case, being \( d \) the dimension of \( \Sigma \).

Of course, this definition is nothing but the typical procedure used more or less explicitly in the works on matchings that preserve symmetries. In fact, in [1] Shaver and Lake already performed a matching preserving the cylindrical symmetry taking into account the orbits of the groups acting at both sides of the junction (see also the less explicit procedures in [26]). Nevertheless, sometimes the conditions arising from a strict use of the above definition leads to parametrisations more general than the usual ones.
in which the hypersurface is assumed to preserve some further particular features of the symmetries, such as orthogonal transitivity, the meaning of the ignorable coordinates, and others (see, for instance, [2, 3, 7, 6]). This definition has already been proven determinant in recent works focused on General Relativity. Thus, for instance, in [2], where the uniqueness of the exterior solution given a known interior in a matching of stationary axisymmetric spacetimes is treated, it is shown how the matching following Definition 1 introduces two essential new parameters giving raise to different exteriors, as it will be seen below. Another example arises in the study of the matching of $G_4$ on $S_3$ locally rotationally symmetric spacetimes (LRS) with static cylindrically symmetric ones preserving the cylindrical symmetry [6]. In this case, if it were not by an essential parameter introduced by the matching, no non-static LRS model containing a $G_3$ on $S_3$ subgroup of Bianchi types $V$ and $VII_h$ could be matched to any cylindrically symmetric static spacetime. In fact, in the same reference, the definition is also used in order to show the existence of the axis of symmetry at both sides of the matching hypersurface if one of the halves is spatially homogeneous and either that part or the other represents a spatially compact and simply-connected region [6].

The generation of the parameterisations by the matching procedure in these examples can be seen schematically as follows. Let us have two 4-dimensional space-times with Lorentzian metric $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ admitting a $G_2$ and a $G_3$ respectively, both Abelian and including an axial symmetry, to be matched preserving an Abelian $G_2$ containing the axial symmetry. We know there exist coordinate systems on $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ where the Killing vector fields take the form $\{\partial/\partial \phi, \partial/\partial x\}$ and $\{\partial/\partial \tilde{\phi}, \partial/\partial \tilde{x}, \partial/\partial \tilde{y}\}$ respectively, where both $\partial/\partial \phi$ and $\partial/\partial \tilde{\phi}$ generate an axial symmetry. One starts by defining $\Sigma^+$. Without loss of generality one can choose a coordinate $\phi$ on $\sigma$ such that $d\Phi^+(\partial/\partial \phi) = \partial/\partial \phi$. Since the group is Abelian, and after a suitable coordinate change on $\sigma$ leaving $\partial/\partial \phi$ ‘unchanged’, one can also choose a coordinate $\zeta$ on $\sigma$ such that $d\Phi^+(\partial/\partial \zeta) = \partial/\partial x$. Note that here we have used Lemma 3.2 below, which ensures an Abelian $G_2$ on $\sigma$. The remaining coordinate $\lambda$ can finally be chosen, without loss of generality, such that $\partial/\partial \lambda$ is sent to any vector field orthogonal to both $\partial/\partial \phi$ and $\partial/\partial x$. All this determines $\Sigma^+$ as $\{\phi = \phi, x = \zeta\}$ after a suitable choice of origin of coordinates for $\phi$ and $\zeta$, plus two arbitrary functions of $\lambda$ for the other two coordinates on $\mathcal{V}^+$. Now, since the axial symmetry is uniquely defined [27, 11, 28], the vector $\partial/\partial \phi$ must be sent to the axial generator in $\mathcal{V}^-$, i.e. $d\Phi^-(\partial/\partial \phi) = \partial/\partial \tilde{\phi}$. On the other hand, $\partial/\partial \zeta$ has to be mapped to any Killing vector field that generates an Abelian $G_2$ together with $\partial/\partial \phi$. This implies that

$$d\Phi^-(\partial/\partial \zeta) = a \frac{\partial}{\partial \tilde{\phi}} + b \frac{\partial}{\partial \tilde{x}} + c \frac{\partial}{\partial \tilde{y}}_{\Sigma^-},$$

(11)

where $a, b, c$ are constants. The image of $\partial/\partial \lambda$ through $d\Phi^-$ will be a vector orthogonal to the images of the other two. As before, $\Sigma^-$ is finally determined up to two arbitrary functions, although its explicit expression in more involved than that of $\Sigma^+$.

Three parameters have been introduced by the matching in (11), and although they can eventually be restricted by the junction conditions, they are free in principle. The final remaining freedom introduced by these parameters could be, in some cases where the metric $g^-$ is unknown, absorbed by the metric functions after a coordinate change.
of the kind\textsuperscript{3}
\begin{equation}
\phi' = \phi - \frac{a}{b} \bar{x}, \quad \bar{x}' = \frac{1}{b} \bar{x}, \quad \bar{y}' = \bar{y} - \frac{c}{b} \bar{x}.
\end{equation}

This transforms the vector at the right hand side of (11) onto $\partial/\partial \bar{x}'$ and leaves $\partial/\partial \bar{\phi}' = \partial/\partial \bar{\phi}$. Nevertheless, as happens in [2, 6], the intrinsic meaning of some of the coordinates $\{\bar{\phi}, \bar{x}, \bar{y}\}$ or the geometrical properties of the generators invalidates the absorption of the parameters after the change (12) has been performed in most cases.

More explicitly, in [2] both $(V^\pm, g^\pm)$ are stationary and axisymmetric spacetimes. Although in this case we do not necessarily have a $G_3$ in $(V^-, g^-)$, the procedure followed there can be described with the above construction just by not considering $\partial/\partial \bar{y}$ a Killing vector. We have to put then $c = 0$ in (11). The stationarity is accounted for by taking $x$ and $\bar{x}$ to be timelike coordinates. In the asymptotically flat exterior $(V^-, g^-)$, there is a timelike coordinate, say $\bar{x}$, with an intrinsic meaning: it measures proper time at infinity. Then, quoting from [2], if the interior describes a fluid with velocity vector $\bar{u} = N(\partial/\partial x + w\partial/\partial \phi)$, where $N$ and $w$ are two functions that do not depend on $\phi$, $\bar{u}$ on $\Sigma^-$ becomes $\bar{u}_{|\Sigma} = N/b [\partial/\partial \bar{x} + (wb - a)\partial/\partial \bar{\phi}]_{|\Sigma}$ by (11).

The proper angular velocity of the fluid on $\Sigma$ is then $(wb - a)$, which depends on the parameters introduced by the matching $a, b$. The change (12) (with $c = 0$) could be used to absorb the parameters locally, by redefining the metric functions, but the global meaning of the coordinates clearly implies a global meaning for the parameters $a, b$.

In [6], $(V^+, g^+)$ is taken to be a $G_4$ on $S_3$ LRS spacetime. There is only a $G_2$ subgroup containing the axial symmetry, hence Abelian [29], which will be called $C_2$ in what follows. On the other hand, $(V^-, g^-)$ corresponds to a static cylindrically symmetric spacetime. The preserved symmetry is the Abelian $G_2$ (cylindrical symmetry), and thus the above schematic construction makes sense by taking $x$ and $\bar{x}$ to be space-like coordinates and $\bar{y}$ a timelike coordinate. In this case, and since there is no intrinsic meaning for the coordinates a priori (and the metrics are unknown at both sides), the parameters $a, b$ can indeed be absorbed after the change (12) into redefined metric functions so that we can put $a = 0, b = 1$ in (11) without loss of generality. Nevertheless, the parameter $c$ has an important role to play: In [6] it is also assumed that $(V^-, g^-)$ admits an orthogonally transitive $G_3$ (see below), generated by, say, $\partial/\partial \bar{\phi}$ and $\partial/\partial \bar{x}$. The key point here is that when the $G_4$ in $(V^+, g^+)$ contains a $G_3$ subgroup of Bianchi types $V$ and $VII_h$ ($h \neq 0$), the subgroup $C_2$ does not act orthogonally transitively. The preservation of the orthogonal transitivity on $\Sigma$, which is going to be shown in Section 4, implies in this case that if $c = 0$ then, for non-static LRS $(V^+, g^+)$, the subgroup $C_2$ necessarily acts orthogonally transitively. In short, not including $c$ in the matching procedure would prevent non-static LRS spacetimes $(V^+, g^+)$ with Bianchi types $V$ and $VII_h$ to be matched to the static $(V^-, g^-)$.

The effect of the parameter $c$ is that the $G_2$ generated by $\partial/\partial \bar{\phi}$ and $\partial/\partial \bar{x} + c\partial/\partial \bar{y}$ for $c \neq 0$ is not orthogonally transitive. As shown in [6], having a general non-vanishing $c$ allows LRS spacetimes $(V^+, g^+)$ with non-orthogonal transitive $C_2$ to be matched to the static $(V^-, g^-)$ spacetimes (which indeed contain an orthogonally transitive $G_2$).

\textsuperscript{3}Clearly we do not want $b = c = 0$ and hence we can take $b \neq 0$ by interchanging $\bar{x}$ and $\bar{y}$ if necessary.
3.1 Preservation of the algebraic type

The final matched manifold \((\mathcal{V}, g)\) is at least \(C^1\) (with \(C^0\) metric), but not necessarily \(C^2\) across \(\Sigma\). Therefore the vector fields generating the \(G_m\) group at \(\mathcal{V}\) are not differentiable across \(\Sigma\) in general. As a consequence the continuity of the algebraic type of the \(G_m\) group at both sides of \(\Sigma\) cannot be ensured yet. However, if \(\Sigma\) preserves the symmetry the algebraic type of \(G_m\) must be also preserved across \(\Sigma\). This is obvious from the fact that the commutators for \(\vec{\gamma}_A\) are mapped by \(d\Phi^\pm\) to the commutators of their respective \(\vec{\xi}_A\) (9). More explicitly, taking \([\vec{\xi}_A, \vec{\xi}_B]\) = \(C^{\pm} C_{AB} \vec{\xi}_C\), for any \(C^1\) function defined on \(\Sigma\), we have [22]

\[
[[\vec{\gamma}_A, \vec{\gamma}_B](\Phi^* f) = d\Phi ([[\vec{\gamma}_A, \vec{\gamma}_B]](f) = \left[\vec{\xi}_A, \vec{\xi}_B\right]|_{\Sigma} (f) =
\]

\[
C^{\pm} C_{AB} \vec{\xi}_C|_{\Sigma}(f) = C^{\pm} C_{AB} \vec{\gamma}_C(\Phi^* f),
\]

for both embeddings \(\Phi^+\) and \(\Phi^-\). Hence \(\sigma\) inherits the algebraic type from both sides, i.e. \(C^{\pm} C_{AB} \vec{\gamma}_C = [\vec{\gamma}_A, \vec{\gamma}_B] = C^{\pm} C_{AB} \vec{\gamma}_C\), and thus the structure constants must coincide at both sides for both sets of generators \(\vec{\xi}_A\) and \(\vec{\gamma}_A\) by construction. Therefore, the definition above readily implies that the group \(G_m\) will have indeed the same algebraic type at both sides of \(\Sigma\). This is summarised in the following lemma:

**Lemma 3.2** In a symmetry-preserving matching the algebraic type of the preserved group \(G_m\) is the same at both sides of the matching hypersurface. If the \(G_m\) corresponds to a conformal –not necessarily proper– symmetry, the matching hypersurface also inherits that symmetry (or a more specialised one) and its algebraic type.

This result may seem obvious, and it arises naturally in the process of imposing the preliminary junction conditions in a practical problem. Nevertheless, it is essential in order to choose the right coordinates on \(\sigma\) adapted to the symmetry given by \(G_m\) and its algebraic type. This has been more or less implicitly assumed in the literature, but the explicit procedure would follow the lines of the example concerning the preservation of an Abelian \(G_2\) above.

Furthermore, the setting of the matching hypersurface can be simplified from the very beginning using Lemma 3.2, see for instance [6]. As an example, let us consider again the above situation, but changing the algebraic type of the \(G_3\) group in \((\mathcal{V}^-, g^-)\) from being Abelian to a more general one, for instance a Bianchi \(V\). The \(G_3\) group contains then an Abelian \(G_2\) subgroup, and again, this is the group we will preserve. The coordinates \(\{\tilde{x}^\alpha\}\) can be taken such that the generators are written as \(\{\partial/\partial \tilde{\varphi}, \partial/\partial \tilde{x}, \tilde{v}\}\) and \([\tilde{v}, \partial/\partial \tilde{\varphi}] = \partial/\partial \tilde{\varphi}, [\tilde{v}, \partial/\partial \tilde{x}] = \partial/\partial \tilde{x}\). The same procedure follows until we get the analogous to equation (11). We now have

\[
d\Phi^- \left( \frac{\partial}{\partial \zeta} \right) = a \frac{\partial}{\partial \tilde{\varphi}} + b \frac{\partial}{\partial \tilde{x}} + c \tilde{v} \right|_{\Sigma^-}.
\]

(13)

We could carry on with all the matching procedure, but Lemma 3.2 readily implies that we must have \(c = 0\), because the vector at the right hand side in (13) will have to commute with the image of \(\partial/\partial \tilde{\varphi}\), this is \(\partial/\partial \tilde{\varphi}\).
4 Orthogonal transitivity of preserved conformal $G_2$ groups

Given a $d + 1$-dimensional spacetime admitting a two-dimensional $G_2$ local group of symmetries acting on non-null orbits, the $G_2$ is said to be acting orthogonally transitively (say, in an open set $U$) if there exists a family of $d - 1$-surfaces which are orthogonal to the orbits of the group. Denoting by $\xi, \eta$ two independent vector fields generating the group, this happens iff the two 4-forms defined by $\xi \wedge \eta \wedge d\eta$ and $\xi \wedge \eta \wedge d\xi$ vanish in $U$ (see e.g. [22]). In components, this is expressed as

$$\xi_{[\alpha} \eta_{\beta]} \nabla_\mu \eta_{\nu] = \xi_{[\alpha} \eta_{\beta]} \nabla_\mu \xi_{\nu]} = 0,$$

where the square brackets stand for the usual antisymmetrisation.

This geometric property has important physical implications: Global models for astrophysical self-gravitating rotating bodies in equilibrium in General Relativity consist of stationary and axisymmetric spacetimes, thus admitting a $G_2$ local group of isometries acting on timelike surfaces $T_2$, composed of two main regions: an interior region $(W^I, g^I)$, that is to describe the spatially compact and simply connected object, and an exterior region $(W^E, g^E)$. The two regions are matched across a timelike hypersurface $\Sigma$, which describes the limiting surface of the body at all times. The metric $g^I$ is taken to be a solution of the Einstein field equations with matter, whereas $g^E$ satisfies the equations for vacuum $R_{\alpha\beta} = 0$. Incidentally, if the model is to describe an isolated body, then the exterior region is also taken to be asymptotically flat.

The existence of the axis of symmetry in the vacuum exterior $(W^E, g^E)$ implies that the $G_2$ on $T_2$ must act orthogonally transitively there [8]. This very well known result is based on the following identities [9, 10, 11, 19], which are in fact valid for arbitrary dimension and signature, for two Killing vector fields $\xi$ and $\eta$:

$$\nabla^\mu (\eta_{[\alpha} \xi_{\beta]} \xi_{\lambda} \eta_{\rho]} = -\frac{1}{2} \xi^\rho R_{\rho[\lambda} \eta_{\alpha} \xi_{\beta]} + \frac{1}{4} \left( [\tilde{\eta}, \tilde{\xi}]_{[\alpha} \nabla_\lambda \xi_{\beta]} + \xi_{[\alpha} \nabla_\lambda [\tilde{\eta}, \tilde{\xi}]_{\beta]} \right),$$

(14)

plus the $\xi \leftrightarrow \eta$ analogous. The first term in the right hand side vanishes if and only if the Ricci tensor has an invariant 2-plane which is spanned by the Killing vector fields at each point, i.e.

$$R^\alpha_{\rho\beta} \xi^\rho = a_1(x^\beta) \xi^\alpha + b_1(x^\beta) \eta^\alpha,$n
$$R^\alpha_{\rho\beta} \eta^\rho = a_2(x^\beta) \xi^\alpha + b_2(x^\beta) \eta^\alpha,$$

(15)

where the functions $a$'s and $b$'s need to satisfy some relations to account for the symmetric character of $R_{\alpha\beta}$. In the situation here, the $G_2$ on $T_2$ groups acting in both $(W^I, g^I)$ and $(W^E, g^E)$ must be Abelian because of the cyclic (axial) symmetry [10, 30, 29], which leaves only the first term on the right hand side in (14). Now, equations (15) hold, for instance, when the Ricci tensor (and hence the energy-momentum tensor) is either proportional to the metric (so-called $\Lambda$-term type, which includes vacuum) or of perfect-fluid type and such that the fluid flow is orthogonal to the orbits if the $G_2$ group acts on spacelike $S_2$ surfaces, or lies on the orbits if the $G_2$ acts on timelike $T_2$ surfaces. This latter property corresponds to the absence of convective motions [10].

In the vacuum exterior region $(W^E, g^E)$ equations (15) hold trivially. Denoting by $\{\tilde{\xi}^E, \tilde{\eta}^E\}$ the generators of the axial symmetry and stationarity in $(W^E, g^E)$ respec-
tively, $\xi^E \wedge \eta^E \wedge d\eta^E$ and $\xi^E \wedge \eta^E \wedge d\xi^E$ are thus constant\(^4\) all over $(W^E, g^E)$, and equal to zero, because $\xi^E$ vanish at the axis of symmetry.

But regarding the interior region $(W^I, g^I)$ and denoting by $\{\xi_I^+, \eta_I^+\}$ the generators of the $G_2$ group there which are eventually identified with $\{\xi^E, \eta^E\}$ on $\Sigma$, although $\xi^I \wedge \eta^I \wedge d\eta^I$ and $\xi^I \wedge \eta^I \wedge d\xi^I$ also vanish at the axis, they do not necessarily vanish everywhere in $(W^I, g^I)$, because equations (15) do not hold in principle. Hence, from equation (14) it does not necessarily follow that the $G_2$ on $(W^I, g^I)$ acts orthogonally transitively. Nevertheless, in all the studies on global models describing rotating objects in equilibrium it has been usually assumed that the $G_2$ on $T_2$ group acts orthogonally also on the interior region [31, 2, 32, 33]. This is the so-called circularity condition, and, as mentioned above, implies the absence of convective motions in fluids [10].

The matchings of spacetimes involved in the constructions of such models have been also always implicitly assumed to preserve both the stationarity and the axial symmetry across $\Sigma$. It is natural to ask then whether or not the symmetry–preserving matching implies the preservation of the orthogonal transitivity at the exterior across $\Sigma$ and hence leads to restrictions of the $G_2$ on $T_2$ in the interior. Any result in this direction would turn the circularity condition into a consequence of the symmetry–preserving matching.

The following more general result ensures that the two 4-forms $\xi^I \wedge \eta^I \wedge d\eta^I$ and $\xi^I \wedge \eta^I \wedge d\xi^I$ defined above, or equivalently the corresponding Hodge-dual (*) $d-3$-forms (i.e. functions in four dimensions), for the $G_2$ group at the interior must also vanish everywhere on $\Sigma$.

**Theorem 4.1** Given a matching preserving the symmetry of a $G_2$ local conformal group –not necessarily proper– as defined above, and choosing $\{\xi^+, \eta^+\}$ and $\{\xi^-, \eta^-\}$ as the sets of generators of the $G_2$ groups at $(V^+, g^+)$ and $(V^-, g^-)$ respectively such that $d\Phi^+(\gamma_1) = \xi^+_I|_{\Sigma^+}$ and $d\Phi^+(\gamma_2) = \eta^+_I|_{\Sigma^+}$, then

$$
\begin{align*}
*\left(\eta^+ \wedge \xi^+ \wedge d\xi^+\right) & \equiv \left(\eta^- \wedge \xi^- \wedge d\xi^+\right), \\
*\left(\xi^+ \wedge \eta^+ \wedge d\eta^+\right) & \equiv \left(\xi^- \wedge \eta^- \wedge d\eta^+\right).
\end{align*}
$$

**Proof.** Let us consider first $(V^+, g^+)$. The restriction of any $1$-form field $\xi^+$ to $\Sigma^+$ can be expressed in a co-basis $\{n^+, w^a\}$ defined on $\Sigma^+$ and dual to $\{\ell^+, e^+_a\}$ as

$$
\xi^+|_{\Sigma^+} = \xi^+_a \ w^a + \xi^+ n^+,
$$

where $\xi^+_a = \xi^+(e^+_a)$ and $\xi^+_\ell = \xi^+(\ell^+)$. Its exterior differential $2$-form can be cast then as follows

$$
d\xi^+|_{\Sigma^+} = A^+_a \ w^a + B^+_a \ n^+ \wedge w^a
$$

with

$$
\begin{align*}
A^+_a &= -A^-_{ba} = e^+_a (\xi^+_b), \\
B^+_a &= 2\ell^+ e^+_a \nabla^+_a \xi^+_\ell - \ell^+ (\xi^+_a) + \xi^+_a \ n^+ \nabla^+_a \ell^+ + \xi^+_a \ e^+_a \nabla^+_a \ell^+.
\end{align*}
$$

\(^4\)In four dimensions, the codifferential of 4-forms vanish iff the 4-forms are constant.
Another convenient expression for the latter is

\[ B_a^+ = \ell^+ a_e^+ [\nabla a_\beta^+ - 2a_e^+ (\xi_\ell^+ + 2\xi_a^+ \nabla \ell^+ a)]. \tag{21} \]

Taking the analogous expressions for another arbitrary 1-form \( \eta^+ \), the 4-form we look for reads then

\[ \eta^+ \wedge \xi^+ \wedge d\xi^+ |_{\Sigma^+} = \left\{ \eta_{\ell}^+ \xi_c^+ A_{ab} - \eta_c^+ \left( \xi^+ A_{ab} - \xi_a^+ B_b^+ \right) \right\} n^+ \wedge w^+ \wedge w^+ \wedge w^+, \tag{22} \]

on \( \Sigma^+ \). The analogous expressions follow when considering the \((-)\) counterpart.

We assume now that the preliminary junction conditions hold, so that, following the construction as explained in Section 2, we have a whole matched spacetime \((\mathcal{V}, g)\) with \(\Sigma = \Sigma^+ = \Sigma^-\) splitting it into \(\mathcal{W}^+(\subset \mathcal{V}^+)\) and \(\mathcal{W}^-(\subset \mathcal{V}^-)\), and such that the metric \(g\) is continuous on the whole \(\mathcal{V}\) and at least of class \(C^2\) in both \(\mathcal{W}^+\) and \(\mathcal{W}^-\). The tangent bases \(\{\tilde{\xi}^+, \tilde{\xi}^\pm\}\) and \(\{\tilde{\xi}^-, \tilde{\xi}^\pm\}\) have been identified then to give \(\{\tilde{\xi}, \tilde{\xi}^\pm\}\) at every \(q \equiv \Phi^-(p) = \Phi^+(p),\) as well as their respective cotangent bases have been identified as \(\{n, w^\pm\}\). In other words, this means that \(d\Phi^+ (\partial/\partial \lambda) = d\Phi^- (\partial/\partial \lambda).

Taking now \(\tilde{\xi}^+\) and \(\tilde{\xi}^-\) to be two vector fields defined on \(\mathcal{V}^+\) and \(\mathcal{V}^-\) respectively such that their restrictions to \(\Sigma^+\) and \(\Sigma^-\) resp. are the images through \(d\Phi^+\) and \(d\Phi^-\) of the same vector \(\tilde{\gamma}_\ell\) on \(\sigma\), as follows from Definition 1, the pair of vector fields \(\tilde{\xi}^+\) and \(\tilde{\xi}^-\) necessarily coincide on \(\Sigma\). And the same for \(\tilde{\eta}^+\) and \(\tilde{\eta}^-\), which are images of the same \(\tilde{\gamma}_\ell\). Since \(g\) is continuous, then it obviously follows that \(\tilde{\xi}_\ell^+ = \tilde{\eta}_\ell = \tilde{\xi}_\ell\), i.e. \(\xi^+ = \xi^\pm\), and analogously for \(\eta_a\) and \(\eta_c\), at every point \(q \in \Sigma\). Defining \([f] \equiv f^+ - f^-\) as the difference of the values of the function \(f\) on \(\Sigma\) as taken on \(\Sigma^+\) and \(\Sigma^-\), all this can be expressed as

\[ [\xi_a] = 0, \quad [\xi_a] = 0, \quad [\eta_a] = 0 \quad [\eta_c] = 0. \]

Any derivative along \(\Sigma\) of a function \(f\) satisfying \([f] = 0\) also coincide as coming from either side, i.e. \([\tilde{\xi}^\pm_a(f)] = 0\), and thus, in particular

\[ [\tilde{\xi}^\pm_a(\xi_b)] = 0, \quad [\tilde{\xi}^\pm_a(\xi_b)] = 0. \tag{23} \]

As a result, one gets

\[ [A_{ab}] = 0. \]

Therefore, on \(\Sigma\) one has

\[ \eta^+ \wedge \xi^+ \wedge d\xi^+ |_{\Sigma} = \eta^- \wedge \xi^- \wedge d\xi^- |_{\Sigma} = \left\{ \eta_{\ell} \xi_c A_{ab} - \eta_c (\xi^+ A_{ab} - \xi_a^+ B_b) \right\} n^+ \wedge w^+ \wedge w^+ \wedge w^+ = \eta_a \xi_a B_b n^+ \wedge w^+ \wedge w^+ \wedge w^+. \tag{24} \]

This expression could have been obviously written as \(\eta \wedge \xi \wedge (d\xi^+ - d\xi^-)\), but I have preferred to keep the \((\pm)\) signs on \(\xi\) and \(\eta\) throughout the proof for the sake of clarity. The coincidence of the two 4-forms across \(\Sigma\) is then equivalent to

\[ \eta_a \xi_b [B_c] = 0. \tag{25} \]

From (20) and (21) and their \((-)\) counterpart, we have

\[ [B_a] = \ell^+ a_e [\nabla a_\beta^+ - 2a_e^+ (\xi_\ell^+ + 2\xi_a^+ \nabla \ell^+ a)] = \ell^+ a_e [\nabla a_\beta^+ - 2\xi_a^+ \nabla \ell^+ a] = \ell^+ a_e [\nabla a_\beta^+ - 2\xi_a^+ \nabla \ell^+ a], \tag{26} \]
where (23) has been used. The first term in both expressions in (26) is not zero in general, because it contains derivatives of the components of $\vec{\xi}$ along $\vec{\ell}$, this is, off $\Sigma$. The second term in both expressions can be rewritten using (8) as $\xi^b e_a [e_{a}^\beta \nabla_\beta \ell_\alpha]$, and thus, by the definition of the generalised second fundamental form (6), (26) can be re-expressed as

$$[B_a] = \ell^a e_a^\beta [\nabla_\alpha \xi_\beta] + \xi^b [\mathcal{H}_{ab}] = \ell^a e_a^\beta [\mathcal{L}_{\xi} g_{a\beta}] + 2 \xi^b [\mathcal{H}_{ab}]. \quad (27)$$

Equation (24) reads then

$$\eta^+ \wedge \xi^+ \wedge d\xi^+ \mid_\Sigma - \eta^- \wedge \xi^- \wedge d\xi^- \mid_\Sigma = \eta_c \xi_a \left( \ell^a e_a^\beta [\mathcal{L}_{\xi} g_{a\beta}] + 2 \xi^d [\mathcal{H}_{bd}] \right)\ n \wedge w^a \wedge w^b \wedge w^c. \quad (28)$$

So far we have only used the preliminary junction conditions and the fact that $\vec{\xi}$ and $\vec{\eta}$ at both sides are images of the same $\vec{\gamma}_1$ and $\vec{\gamma}_2$ respectively. The second term at the right hand side in (28) clearly vanishes when imposing the rest of the junction conditions (7), $[\mathcal{H}_{ab}] = 0$. On the other hand, the first term at the right hand side in (28) vanishes if and only if $\eta_a [\xi_b e_c]^\beta [\ell^a [\mathcal{L}_{\xi} g_{a\beta}]] = 0$, i.e.

$$\ell^a e_a^\beta [\mathcal{L}_{\xi} g_{a\beta}] = f_1 \xi_a + f_2 \eta_a, \quad (29)$$

where the $f$'s are arbitrary functions defined on $\Sigma$. In particular, if $\vec{\xi}^\pm$ are conformal Killing vector fields, Lemma 3.1, together with (5), implies that the left hand side of (29) vanishes, and thus (29) is trivially satisfied for $f_1 = f_2 = 0$. In this case we have then

$$\eta^+ \wedge \xi^+ \wedge d\xi^+ \mid_\Sigma - \eta^- \wedge \xi^- \wedge d\xi^- \mid_\Sigma = 0. \quad (30)$$

Since the metric $g$ is continuous and the orientation has been preserved across $\Sigma$ we can take now the Hodge-dual of this last expression. This allows us to write it as the equality of the corresponding Hodge-dual $d-3$-forms, being $d+1$ the dimension of the manifolds. Of course, the usefulness of this last step is apparent in four dimensions.

The analogous equations and arguments follow for the difference of the other $4$-form $\xi \wedge \eta \wedge d\eta$ at both sides of $\Sigma$ by interchanging $\xi$ and $\eta$ in all the expressions above. ■

In particular, then, we have obtained the following:

**Corollary 4.1** Given a matching preserving a $G_2$ local conformal group --not necessarily proper-- as defined above and such that the $G_2$ acts orthogonally transitively at one side of $\Sigma$, say at $(\mathcal{W}^+, g^+)$, then

$$\ast (\xi^I \wedge \eta^I \wedge d\eta^I) \cong 0,$$

$$\ast (\xi^I \wedge \eta^I \wedge d\xi^I) \cong 0, \quad (31)$$

where $\{\vec{\xi}^-, \vec{\eta}^-\}$ are any two independent generators of the $G_2$ group on $(\mathcal{W}^-, g^-)$. ■
As a first remark, let me stress again the fact that in order to obtain (16)-(17) the assumptions on $\vec{\xi}$ and $\vec{\eta}$ being conformal Killing vector fields as well as the junction conditions can be relaxed. One only needs (25), with (27) (and their $\xi \leftrightarrow \eta$ analogous), to be satisfied. For this, as has been seen above, once all the junction conditions are satisfied, then only (30) (or, equivalently (29)) is needed.

From the converse point of view, if (30) is satisfied the necessary and sufficient condition to obtain (16)-(17) is that $[\mathcal{H}_{bd}]$ has an analogous geometric property as that of the Ricci tensor in (15), i.e.

\[ \xi^d [\mathcal{H}_{bd}] = A_1 \xi_b + A_2 \eta_b, \]
\[ \eta^d [\mathcal{H}_{bd}] = B_1 \xi_b + B_2 \eta_b, \]

(32)

where, as above, the functions $A$'s and $B$'s defined on $\Sigma$ satisfy certain relations to account for the symmetric character of $[\mathcal{H}_{ab}]$.\(^5\) This would be useful when relaxing Theorem 4.1 from matchings of spacetimes to “matchings” allowing for distributional parts on the Riemann tensor, as for instance, in the description of topological defects and ‘brane-world’ models.

As a second remark, note that the result in Corollary 4.1 only ensures the vanishing of the forms on a hypersurface ($\Sigma$). The identities (14) and its $\xi \leftrightarrow \eta$ analogous can be used then to extend this result off $\Sigma$ in four dimensions. If the Ricci tensor (and thus the energy momentum tensor in General Relativity) satisfies (15) in $\mathcal{W}^-$ then (14) (and the $\xi \leftrightarrow \eta$ analogous) imply that $*(\eta^- \wedge \xi^- \wedge d\xi^-) = *(\xi^- \wedge \eta^- \wedge d\eta^-) = 0$ hold all over $\mathcal{W}^-$ and thus the $G_2$ conformal local group acts orthogonally transitively there. This can be summarised as follows:

**Corollary 4.2** Given a matching in four dimensions preserving a $G_2$ local conformal group –not necessarily proper– as defined above and such that the $G_2$ acts orthogonally transitively at one side of $\Sigma$, say at $(\mathcal{W}^+, g^+)$, if the Ricci tensor at the other side $(\mathcal{W}^-, g^-)$ has an invariant 2-plane spanned by two Killing vector fields generating the $G_2$ at each point, then the $G_2$ acts orthogonally transitively in $(\mathcal{W}^-, g^-)$. \(\blacksquare\)

Corollary 4.1 gives, in particular, two necessary conditions for the existence of the matching hypersurface. If a spacetime admits a non-orthogonally transitive $G_2$ such that (31) cannot be satisfied anywhere, then it cannot be matched to a spacetime admitting an orthogonally transitive $G_2$ whenever the matching preserves those two $G_2$. As an example, one can consider the specialised van Stockum class of stationary cylindrically symmetric dust solutions [34, 19], where the cylindrical symmetry is defined by a non-orthogonally transitive $G_2$ on $S_2$ [30]. In this case it can be checked that (31) is impossible, and thus this solution cannot be matched to an orthogonally transitive cylindrically symmetric spacetime while preserving the cylindrical symmetry. Nevertheless, it can be matched to non-orthogonally transitive ones, as was shown by van Stockum in [34], see also [35] and references therein.

Regarding the study of stationary and axisymmetric models, the applicability of Corollary 4.1 is therefore immediate since it is the first condition that determines the suitability of convective interiors to be immersed in vacuum exteriors. On the other hand, $[\mathcal{H}_{ab}]$ is indeed symmetric, whereas $\mathcal{H}_{ab}$ is not in general [17].

\(^5\)[\mathcal{H}_{ab}] is indeed symmetric, whereas $\mathcal{H}_{ab}$ is not in general [17].
hand, it also provides a first step in the generalisation of the whole set of matching conditions in the stationary and axisymmetric problem in General Relativity. The whole set of matching conditions for the usual stationary and axisymmetric matchings with a vacuum exterior and a given orthogonally transitive interior can be reorganised in \[31, 2\] (a) conditions on the interior hypersurface, (b) exterior matching hypersurface and (c) boundary conditions for the exterior problem. The conditions in (a) \[31, 2\] correspond to an over-determined system of ordinary differential equations given in this case by the usual direct consequence of the junction conditions (7) on the discontinuities of the Einstein tensor \(G_{\alpha\beta},\)

\[n^\alpha [G_{\alpha\beta}] = 0.\]  \hspace{1cm} (33)

In the case of non-null matching hypersurfaces, these are the so-called Israel conditions [15]. In fact, and because of the symmetries and the orthogonal transitivity involved in this case, two of the four equations in (33) are satisfied identically [2]. Hence, two equations arise at most from (33), whose compatibility is then necessary for the existence of \(\Sigma^I\). In this case, and for perfect fluid interiors, the Israel conditions translate into the intuitive vanishing of the pressure at \(\Sigma^I\). The equation \(p \Sigma^I = 0\) defines then the matching hypersurface as seen from the interior \(\Sigma^I\) in an implicit manner. In some occasions the equations (33) are satisfied identically, as for example the case of dust. In these cases \(\Sigma^I\) is not determined and one could match, in principle, across any timelike hypersurface preserving the symmetry.

When \(\Sigma^I\) is uniquely defined, conditions (b) (see \[31, 2\]) determine \(\Sigma\) as seen from the exterior, i.e. \(\Sigma^E\). The rest of the matching conditions (c) provide the over-determined boundary conditions for the elliptic vacuum exterior problem by giving the values of the Ernst potential up to an additive constant and its normal derivatives on \(\Sigma^E\) [2].

If the circularity condition is dropped, and because neither \(\xi^I\) nor \(\eta^I\) vanish all over the matching hypersurface, the two necessary conditions (31) constitute two more equations defining \(\Sigma^I\). Furthermore, and because now there are more non-zero components of the Einstein tensor, equations (33) will result on more relations. But it still remains to be checked whether or not the Israel conditions (33) plus that in (31) account for all the matching conditions determining the existence and eventual definition of \(\Sigma^I\). The existence of dust solutions admitting non-orthogonally transitive \(G_2\) groups of isometries (see e.g. \[34, 35, 36, 19\]) ensures that the Israel conditions (33) do not imply the new conditions (31) in general.

5 Conclusions

After motivating and presenting the definition of symmetry-preserving matching as introduced in \[3\], this paper has dealt with the first consequences such definition has on the preserved group. A usual property of the differential maps has been implemented here to see how the algebraic type of the preserved group must be kept at both sides of the matching hypersurface.

It has also been shown the ‘preservation’ of the orthogonal transitivity of conformal \(G_2\) groups on the matching hypersurface. The implications of this result on the generalisation of the studies of stationary and axisymmetric models of isolated bodies
in General Relativity to allow for convective interiors have been discussed. The next step at this point should be to address the problem of the whole set of matching conditions when the interior admits a general Abelian $G_2$. Since the only new property, the non-orthogonal transitivity in the interior, is driven by two functions that have to vanish on $\Sigma^I$ by Corollary 4.1, it may seem that the only new terms to be added in the matching conditions would come from the normal derivatives of the two functions $\ast(\xi^I \wedge \eta^I \wedge d\eta^I)$ and $\ast(\xi^I \wedge \eta^I \wedge d\xi^I)$. These new terms would appear by modifying the expressions for the values of the normal derivatives of the exterior Ernst potential on $\Sigma$ as obtained in the usual non-convective case [2]. The study of the whole general set of matching conditions for a general Abelian $G_2$ in the interior is currently under investigation. Obtaining an analogous structure of equations for the matching in the general case as in the non-convective case would be very useful, because all the results and studies on uniqueness [2] and existence [32, 33] of the exterior vacuum problem could be implemented in a straightforward manner to the general case without the need of the circularity condition.

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