Extended Operator Algebra and Reducibility in the WZW Permutation Orbifolds

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Recently the operator algebra, including the twisted affine primary fields, and a set of twisted KZ equations were given for the WZW permutation orbifolds. In the first part of this paper we extend this operator algebra to include the so-called orbifold Virasoro algebra of each WZW permutation orbifold. These algebras generalize the orbifold Virasoro algebras (twisted Virasoro operators) found some years ago in the cyclic permutation orbifolds. In the second part, we discuss the reducibility of the twisted affine primary fields of the WZW permutation orbifolds, obtaining a simpler set of single-cycle twisted KZ equations. Finally we combine the orbifold Virasoro algebra and the single-cycle twisted KZ equations to investigate the spectrum of each orbifold, identifying the analogues of the principal primary states and fields also seen earlier in cyclic permutation orbifolds. Some remarks about general WZW orbifolds are also included.

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1 Introduction

In the last few years there has been a quiet revolution in the local theory of current-algebraic orbifolds. Building on the discovery of orbifold affine algebras [1, 2] in the cyclic permutation orbifolds, Refs.[3-5] gave the twisted currents and stress tensor in each twisted sector of any current-algebraic conformal field theory [6-11] with a finite symmetry group $H$. The construction treats all current-algebraic orbifolds at the same time, using the method of eigenfields and the principle of local isomorphisms to map OPEs in the symmetric theory to OPEs in the orbifold. The orbifold results are expressed in terms of a set of twisted tensors or duality transformations,
which are discrete Fourier transforms constructed from the eigendata of the $H$-eigenvalue problem.

More recently, the special case of the $WZW$ orbifolds $A_g(H)/H$ was worked out in further detail [12, 13], while extending the operator algebra in this case to include the twisted affine primary fields, twisted vertex operator equations and twisted Knizhnik-Zamolodchikov equations for each sector of every $WZW$ orbifold:

- The WZW permutation orbifolds [12, 13]
- Inner-automorphic WZW orbifolds [12]
- The (outer-automorphic) charge conjugation orbifold on $\mathfrak{su}(n \geq 3)$ [13]
- Other outer-automorphic WZW orbifolds on simple $g$ [13]

Ref. [13] also solved the twisted vertex operator equations in an abelian limit to obtain the twisted vertex operators for each sector of a very large class of abelian orbifolds. For discussion of $WZW$ orbifolds at the level of characters, see Refs. [14, 1, 15, 16].

In addition to the operator formulation, the general $WZW$ orbifold action was also given in Ref. [12], with applications to special cases in Refs. [12, 13]. The general $WZW$ orbifold action provides the classical description of each sector of every $WZW$ orbifold in terms of appropriate group orbifold elements, which are the classical limit of the twisted affine primary fields. Moreover, Ref. [17] gauged the general $WZW$ orbifold action by general twisted gauge groups to obtain the general coset orbifold action.

In this paper, we consider the local theory of all $WZW$ permutation orbifolds in further detail, with special attention to the generalization of certain features [1]

- orbifold Virasoro algebra (twisted Virasoro operators)
- orbifold $\mathfrak{sl}(2)$ Ward identities
- principal primary states and fields

observed earlier by different methods in $\mathbb{Z}_\lambda$ cyclic permutation orbifolds with $\lambda =$ prime. The orbifold Virasoro algebras for $\mathbb{Z}_\lambda$, $\lambda =$ prime were discovered independently in Refs. [1, 18], and a recent application of these algebras is found in Ref. [19]. As we shall see, there is a disjoint cycle structure underlying all of these features, which is closely related to the reducibility of the twisted affine primary fields and the twisted KZ equations.

In further detail, Sec. 2 applies the principle of local isomorphisms to extend the operator algebra to include the general orbifold Virasoro algebra of the general $WZW$ permutation orbifold. As in Ref. [1], the general orbifold Virasoro algebra is associated to orbifold $\mathfrak{sl}(2)$ Ward identities for the correlators of the twisted affine primary fields, and we find
that these Ward identities are satisfied by any solution of the twisted KZ system.

After some discussion of the action formulation [12, 17] and an abelian limit [13] of the WZW permutation orbifolds, we argue in Sec. 3 that the twisted affine primary fields of the WZW permutation orbifolds are generically reducible. This leads us to a new, simpler set of single-cycle twisted KZ equations for the correlators of the blocks of the twisted affine primary fields (see Eq. (3.30)).

In Sec. 4, we combine the extended operator algebra and the new single-cycle twisted KZ system to study aspects of the spectrum of each WZW permutation orbifold. Following Ref. [1], we find in particular that the asymptotic formulae for in- and out-states are modified relative to the conventional asymptotic formulae of untwisted CFT. Moreover, using these orbifold-modified asymptotic formulae, we find that the twisted affine primary fields create twisted affine primary states, which are also primary under the orbifold Virasoro algebra. The twisted affine primary states also serve as base states for certain sets of so-called principal primary states [1] in the modules of the orbifold Virasoro algebra. We are also able to identify the matrix components of the twisted affine primary fields as the analogues of the old principal primary fields of Ref. [1].

Finally, Sec. 5 considers asymptotic formulae and the reducibility of the twisted affine primary fields and states of general WZW orbifolds.

Because we are studying features specific to the WZW permutation orbifolds, this paper is in large part self-contained – using primarily the notation appropriate [5, 12, 13] to the WZW permutation orbifolds. To understand this paper in the context of the general orbifold program, see Refs. [3, 5, 12, 13] and the general remarks in Sec. 5.

2 Extended Operator Algebra in WZW Permutation Orbifolds

2.1 Permutation-Invariant WZW Systems

In this subsection, we will extend the usual chiral algebra of permutation-invariant WZW systems to include what we will call the partial stress tensors of such systems.

To begin, we recall the notation of Ref. [12] for permutation-invariant WZW systems:

\[ g = \bigoplus_I g^I, \quad g^I \simeq g, \quad k^I = k \quad (2.1a) \]

\[ f_{aI,bJ}^c = f_{ab}^c (\tau_I)^J_L, \quad G_{aI,bJ} = k \eta_{ab} \delta_{IJ} \quad (2.1b) \]

\[ T = \bigoplus_I T^I, \quad T^a_I = T^a \tau_I, \quad [T_a, T_b] = i f_{ab}^c T^c, \quad (\tau_I)^J_L \equiv \delta_{IJ} \delta_{L^L}, \quad \tau_I \tau_J = \delta_{IJ} \tau_I \quad (2.1c) \]
Here the index $I$ labels the copies $g'$ of any simple $g$, whose structure constants and Killing metric are $f_{ab}^c$ and $\eta_{ab}$ respectively, and $\{T_a\}$ is any matrix irrep of $g$. Then Ref. [12] gives the left-mover OPEs of the system:

$$J_{aI}(z)J_{bJ}(w) = \frac{\delta_{IJK} \eta_{ab}}{(z-w)^2} + \frac{if_{abc} J_{cI}(w)}{z-w} + \mathcal{O}(z-w)^0$$  \hspace{1cm} (2.2a)

$$T(z)J_{aI}(w) = \left( \frac{1}{(z-w)^2} + \frac{\partial_w}{z-w} \right) J_{aI}(w) + \mathcal{O}(z-w)^0$$  \hspace{1cm} (2.2b)

$$T(z)T(w) = \frac{c_g/2}{(z-w)^2} + \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) T(w) + \mathcal{O}(z-w)^0$$  \hspace{1cm} (2.2c)

$$J_{aI}(z)g(T,\bar{w},w) = \frac{g(T,\bar{w},w)}{z-w} T_a \tau_I + \mathcal{O}(z-w)^0$$  \hspace{1cm} (2.2d)

$$T(z)g(T,\bar{w},w) = g(T,\bar{w},w) \Delta_g(T) + \frac{\partial_w}{z-w} g(T,\bar{w},w) + \mathcal{O}(z-w)^0$$  \hspace{1cm} (2.2e)

$$\frac{\eta_{ab}}{2k + Q_g} T_a T_b \equiv \Delta_g(T) \mathbb{1}.$$  \hspace{1cm} (2.2f)

Here $\{J_{aI}(z)\}$ are the currents [20, 21, 6] of affine $g$, $g(T,\bar{z},z)$ is the affine primary field corresponding to matrix irrep $T$ of $g$, and the constant $\Delta_g(T)$ is the conformal weight of irrep $T$ under the affine-Sugawara construction $T_g(z)$ on $g$. The affine primary fields are matrix-valued, with index structure

$$g(T,\bar{z},z)_{aI}^\beta J, \quad \alpha, \beta = 1, \ldots, \text{dim } T, \quad I, J = 0, \ldots, K-1.$$  \hspace{1cm} (2.3)

In fact, the affine primary fields are block-diagonal, with blocks labelled by the copies $I$, but we shall not use this fact explicitly until Sec. 3. The stress tensor $T(z)$ in (2.2) is the affine-Sugawara construction [6, 7, 22-24] on $g$

$$T(z) \equiv T_g(z) = \frac{\eta_{ab}}{2k + Q_g} \sum_I : J_{aI}(z)J_{bI}(z) :$$  \hspace{1cm} (2.4a)

$$c_g \equiv K c_{g'}, \quad c_g \equiv x \frac{\text{dim } g}{x + \hbar_g}, \quad x = \frac{2k}{\psi_0^2}, \quad \hbar_g = \frac{Q_g}{\psi_0^2}.$$  \hspace{1cm} (2.4b)
The right-mover analogue of the system (2.2), (2.4) is discussed in Ref. [12].

In this paper we will extend the system (2.2)-(2.4) to include the set of partial stress tensors \( T_I(z) \)

\[
T_I(z) \equiv \frac{\eta^{ab}}{2k + Q_g} : J_{aI}(z) J_{bI}(z) : , \quad T_I(z) \simeq T_g(z)
\] (2.5a)

\[
T(z) = \sum_I T_I(z)
\] (2.5b)

which sum to the full stress tensor \( T(z) \). The partial stress tensors are the individual affine-Sugawara constructions on the copies \( g^I \), and they satisfy the additional OPEs:

\[
T_I(z) J_{aJ}(w) = \delta_{IJ} \left( \frac{1}{(z-w)^2} + \frac{\partial_w}{z-w} \right) J_{aI}(w) + \mathcal{O}(z-w)^0
\] (2.6a)

\[
T_I(z) T_J(w) = \delta_{IJ} \left( \frac{c_g/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) T_I(w) \right) + \mathcal{O}(z-w)^0
\] (2.6b)

\[
T(z) T_I(w) = \frac{c_g/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) T_I(w) + \mathcal{O}(z-w)^0
\] (2.6c)

\[
T_I(z) g(T, \bar{w}, w) = \left( \frac{g(T, \bar{w}, w) \Delta_g(T)}{(z-w)^2} + \frac{\partial_w g(T, \bar{w}, w)}{(z-w)} \right) \tau_I + \mathcal{O}(z-w)^0.
\] (2.6d)

The relations (2.2b), (2.2c), (2.2e) for the stress tensor \( T(z) \) are obtainable from these OPE’s by the identity (2.5b).

We will also need the associated left-mover vertex operator equation (see e.g. Ref. [25])

\[
\partial g(T, \bar{z}, z) = \frac{2\eta^{ab}}{2k + Q_g} \sum_I : J_{aI}(z) g(T, \bar{z}, z) : T_b \tau_I
\] (2.7)

which follows from the OPE’s above.

We turn next to the action of the symmetry group

\[
H(\text{permutation}) \subset \text{Aut}(g), \quad H(\text{permutation}) \subset S_N, \quad K \leq N
\] (2.8)

of the WZW model defined above. Picking one representative \( h_\sigma \in H \) of each conjugacy class, the action is [12]

\[
J_{aI}(z)' = \omega(h_\sigma) J_{aI}(z), \quad \sigma = 0, \ldots, N_c - 1
\] (2.9a)
\[
g(T, \bar{z}, z)' = W(T, h_\sigma)g(T, \bar{z}, z)W^\dagger(T, h_\sigma), \quad W(T, h_\sigma)_{\alpha I}^\beta J = \delta_\alpha^\beta \omega(h_\sigma)_{I}^J \tag{2.9b}
\]

\[
T_I(z)' = \omega(h_\sigma)_I^J T_J(z), \quad T(z)' = T(z) \tag{2.9c}
\]

where \( \omega(h_\sigma) \) is the action in the adjoint representation of \( h_\sigma \in H \), \( W(T, \sigma) \) is the action of \( h_\sigma \) in irrep \( T \) and \( N_c \) is the number of conjugacy classes of \( H \). The action (2.9) is an automorphism of the extended OPE’s (2.2), (2.6) if and only if three conditions hold

\[
\sum_I \omega(h_\sigma)_I^J = \sum_J \omega(h_\sigma)_J^I = 1, \quad \omega(h_\sigma)_I^L \omega(h_\sigma)_J^M \delta_{LM} = \delta_{IJ} \tag{2.10a}
\]

\[
\omega(h_\sigma)_I^L \omega(h_\sigma)_J^M (\tau_L)_M^N \omega^\dagger(h_\sigma)_N^K = (\tau_I)_J^K. \tag{2.10b}
\]

These conditions are satisfied by all permutation matrices, and only by permutation matrices. Together, (2.10b) and the second part of (2.9b) provide the solution to the linkage relation [12] for permutation groups.

### 2.2 Eigenfields and Duality Transformations

In current-algebraic orbifold theory, we consider next the \( H \)-eigenvalue problem [3, 5], which diagonalizes the action of \( \omega(h_\sigma) \) in each sector. For permutation groups, the \( H \)-eigenvalue problem has the reduced form [5]

\[
\omega(h_\sigma)_I^J U^\dagger(\sigma)_J^{n(r)j} = U^\dagger(\sigma)_I^{n(r)j} E_{n(r)}(\sigma), \quad \sigma = 0, \ldots, N_c - 1 \tag{2.11a}
\]

\[
E_{n(r)}(\sigma) = e^{-2\pi i \frac{\rho(\sigma)}{\rho(\sigma)}} \equiv n(r(\sigma)) \in \mathbb{Z} \tag{2.11b}
\]

where \( \rho(\sigma) \) is the order of \( h_\sigma \in H \). All quantities are periodic \( n(r) \rightarrow n(r) \pm \rho(\sigma) \) in the spectral indices \( n(r) \), and, correspondingly, the generic eigenvector matrix \( U^\dagger(\sigma) \) is a discrete Fourier element.

The solution to the \( H \)-eigenvalue problem allows us to define the standard eigencurrents [3, 5] and affine eigenprimary fields [12] of sector \( \sigma \)

\[
J_{n(r)aj}(z, \sigma) \equiv \chi(\sigma)_{n(r)j} U(\sigma)_{n(r)j}^I J_{al}(z), \quad \sigma = 0, \ldots, N_c - 1 \tag{2.12a}
\]

\[
g(T, \bar{z}, z, \sigma) \equiv U(T, \sigma) g(T, \bar{z}, z) U^\dagger(T, \sigma), \quad U(T, \sigma)_{n(r)aj}^\beta I \equiv \delta_\alpha^\beta U(\sigma)_{n(r)j}^\alpha I \tag{2.12b}
\]

\[
g(T, \bar{z}, z, \sigma)_{n(r)aj}^{n(s)\beta I} = U(\sigma)_{n(r)j}^{\alpha I} g(T, \bar{z}, z)_{\alpha I}^{\beta J} U^\dagger(\sigma)_{J}^{n(s)j}, \quad \alpha, \beta = 1, \ldots, \dim T \tag{2.12c}
\]
where following Ref. [13] we have chosen an \( a \)-independent normalization \( \chi \). To these, we may add the set of \textit{partial eigenstress tensors} \( \Theta_{n(r)j} \).

\[
\Theta_{n(r)j}(z, \sigma) = \chi(\sigma)_{n(r)j} U(\sigma)_{n(r)j}^T T_I(z) \tag{2.13a}
\]

\[
T(z) = A^{n(r)j}(\sigma) \Theta_{n(r)j}(z, \sigma) = \sum_j A^{0j}(\sigma) \Theta_{0j}(z, \sigma) \tag{2.13b}
\]

\[
A^{n(r)j}(\sigma) \equiv \chi(\sigma)^{-1}_{n(r)j} \sum_I U^\dagger(\sigma)_{I}^{n(r)j} = \delta_{n(r),0 \mod \rho(\sigma)} A^{0j}(\sigma) \tag{2.13c}
\]

where the \( A \)-selection rule in (2.13c) is implied by Eqs. (2.10) and (2.11). The eigenfields \( J, \mathcal{G} \), and \( \Theta \) are constructed to have the following diagonal responses

\[
J_{n(r)aj}(z, \sigma)' = E_{n(r)}(\sigma) J_{n(r)aj}(z, \sigma) = e^{-2\pi i^{\bar{n}(s)}_{\rho(\sigma)}} J_{n(r)aj}(z, \sigma) \tag{2.14a}
\]

\[
\mathcal{G}(T, \bar{T}, z, \sigma)' = E(T, \sigma) \mathcal{G}(T, \bar{T}, z, \sigma) E(T, \sigma)^* \tag{2.14b}
\]

\[
E(T, \sigma)_{n(r)aj}^{n(s)\beta l} \equiv \delta_{\alpha}^{\beta} \delta_{j}^{l} \delta_{n(r)-n(s),0 \mod \rho(\sigma)} E_{n(r)}(\sigma) \tag{2.14c}
\]

\[
\Theta_{n(r)j}(z, \sigma)' = E_{n(r)}(\sigma) \Theta_{n(r)j}(z, \sigma) = e^{-2\pi i^{\bar{n}(s)}_{\rho(\sigma)}} \Theta_{n(r)j}(z, \sigma) \tag{2.14d}
\]

\[
T(z)' = T(z) \tag{2.14e}
\]

to the automorphism group \( H \).

The next step is to reexpress everything in terms of the eigenfields. For example, we may express the stress tensor and partial eigenstress tensors in terms of the eigencurrents as follows:

\[
T(z) = \mathcal{L}^{n(s)al:n(t)bm}(\sigma) : J_{n(s)al}(z, \sigma) J_{n(t)bm}(z, \sigma) : \tag{2.15a}
\]

\[
\mathcal{L}^{n(s)al:n(t)bm}(\sigma) \equiv \chi(\sigma)^{-1}_{n(s)al} \chi(\sigma)^{-1}_{n(t)bm} \frac{\eta^{ab}}{2k + Q_g} \sum_I U^\dagger(\sigma)_{I}^{n(s)j} U^\dagger(\sigma)_{I}^{n(t)m} \tag{2.15b}
\]

\[
\Theta_{n(r)j}(z, \sigma) = \mathcal{L}^{n(s)al:n(t)bm}(\sigma) : J_{n(s)al}(z, \sigma) J_{n(t)bm}(z, \sigma) : \tag{2.15c}
\]
\[ \mathcal{L}_{n(r)j}^{n(s)al;n(t)bm}(\sigma) \equiv \frac{\chi(\sigma)_{n(r)j}}{\lambda_n(\sigma)_{al} \lambda(\sigma)_{n(t)bm}} \frac{\eta^{ab}}{2k + Q_g} \sum_I U(\sigma)_{n(r)j} I U^\dagger(\sigma)_I n(s)I U^\dagger(\sigma)_I n(t)m \]

\[ = \delta_{n(r)-n(s)-n(t),0} \rho(\sigma) \mathcal{L}_{n(r)j}^{n(s)al;n(r)-n(s),bm}(\sigma) \quad (2.15d) \]

\[ \mathcal{L}_{\tilde{g}(\sigma)}^{n(s)al;n(t)bm}(\sigma) = \mathcal{A}^{n(r)j}(\sigma) \mathcal{L}_{n(r)j}^{n(s)al;n(t)bm}(\sigma). \quad (2.15e) \]

Here the forms in (2.15a,b) are well-known [3, 5] and \( \mathcal{L}_{\tilde{g}(\sigma)}(\sigma) \) is called the twisted inverse inertia tensor. The twisted tensors \( \mathcal{L}_{\tilde{g}(\sigma)}(\sigma) \) and \( \mathcal{L}(\sigma) \) in (2.15d) are examples of dualities, which are discrete Fourier transformations of tensors from the untwisted theory.

We also need to compute the OPE’s of the eigenfields,

\[ \mathcal{J}_{n(r)aj}(z, \sigma) \mathcal{J}_{n(s)bl}(w, \sigma) = \frac{\mathcal{G}_{n(r)aj;n(s)bl}(\sigma)}{(z-w)^2} + \]

\[ + \frac{i \mathcal{F}_{n(r)aj;n(s)bl} n(t)cm(\sigma) \mathcal{J}_{n(t)cm}(w, \sigma)}{z-w} + \mathcal{O}(z-w)^0 \quad (2.16a) \]

\[ \Theta_{n(r)j}(z, \sigma) \Theta_{n(s)l}(w, \sigma) = \tilde{\mathcal{G}}_{n(r)j;n(s)l}(\sigma) \frac{c_g/2}{(z-w)^4} + \]

\[ + \tilde{\mathcal{F}}_{n(r)j;n(s)l} n(t)lm(\sigma) \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) \Theta_{n(t)lm}(w, \sigma) + \mathcal{O}(z-w)^0 \quad (2.16b) \]

\[ T(z) \Theta_{n(r)j}(w, \sigma) = \mathcal{B}_{n(r)j}(\sigma) \frac{c_g/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{z-w} \right) \Theta_{n(r)j}(w, \sigma) + \mathcal{O}(z-w)^0 \quad (2.16c) \]

\[ T(z) \mathcal{G}(T, \tilde{w}, w, \sigma) = \mathcal{G}(T, \tilde{w}, w, \sigma) \left( \frac{\mathcal{D}_{\tilde{g}(\sigma)}(T(T, \sigma))}{(z-w)^2} + \frac{\partial_{\tilde{w}}}{z-w} \right) + \mathcal{O}(z-w)^0 \quad (2.16d) \]

where \( \mathcal{G}, \mathcal{F}, \) and \( \mathcal{D}_{\tilde{g}(\sigma)} \) are called respectively the twisted metric [3, 5], the twisted structure constants [3, 5], and the twisted conformal weight matrix [12]. The twisted tensors \( \tilde{\mathcal{G}}, \tilde{\mathcal{F}}, \) and \( \mathcal{B} \) appear for the first time in this paper. The explicit forms of all these duality transformations are:

\[ \mathcal{G}_{n(r)aj;n(s)bl}(\sigma) = k \eta_{ab} \tilde{\mathcal{G}}_{n(r)j;n(s)l}(\sigma) \quad (2.17a) \]

\[ \tilde{\mathcal{G}}_{n(r)j;n(s)l}(\sigma) \equiv \chi(\sigma)_{n(r)j} \chi(\sigma)_{n(s)l} \sum_I U(\sigma)_{n(r)j} U(\sigma)_{n(s)l} \quad (2.17b) \]
\[ F_{n(r)aj;n(s)bl}^{n(t)cm} (\sigma) = f_{ab}^c \tilde{F}_{n(r)j;n(s)l}^{n(t)m} (\sigma) \]  
\[ \tilde{F}_{n(r)j;n(s)l}^{n(t)m} (\sigma) \equiv \frac{\chi(\sigma)_{n(r)j} \chi(\sigma)_{n(s)l}}{\chi(\sigma)_{n(t)m}} U(\sigma)_{n(r)j}^I U(\sigma)_{n(s)l}^J (\tau_I) J L U^\dagger(\sigma)_L^{n(t)m} \]  
\[ B_{n(r)j} (\sigma) \equiv A^{n(s)l} (\sigma) \tilde{G}_{n(s)l;n(r)j} (\sigma) = \chi(\sigma)_{n(r)j} \sum_I U(\sigma)_{n(r)j}^I, \quad A^{n(r)j} (\sigma) B_{n(r)j} (\sigma) = K \]  
\[ D_{g(\sigma)} (T(\sigma)) \equiv U(T, \sigma) \Delta_g (T) U^\dagger (\sigma) = L_{g(\sigma)}^{n(r)aj;n(s)bl} T_{n(r)aj} (T, \sigma) T_{n(s)bl} (T, \sigma). \]  
In (2.17f) we also have the duality transformations called the twisted representation matrices \( T \), which in this case have the form 
\[ T_{n(r)aj} (T, \sigma) \equiv \chi(\sigma)_{n(r)j} U(\sigma)_{n(r)j}^I U(\sigma)_{n(s)l}^J (\tau_I) J K U^\dagger(\sigma)_K^{n(t)m} \]  
\[ t_{n(r)jl} (\sigma)_{n(s)l}^{n(t)m} = \chi(\sigma)_{n(r)j} U(\sigma)_{n(r)j}^I U(\sigma)_{n(s)l}^J (\tau_I) J K U^\dagger(\sigma)_K^{n(t)m} \]  
where the matrices \( \{\tau_I\} \) were defined in (2.1c).

To solve the \( H \)-eigenvalue problem and evaluate the duality transformations, it is convenient to use the (untwisted but \( \sigma \)-dependent) cycle basis defined in Ref. [13]. In this basis, each element \( h_\sigma \) is expressed in terms of disjoint cycles of size \( f_j(\sigma) \), where \( j \) indexes the cycles and \( \hat{j} \) indexes the position within the \( j \)th cycle. Then we have [13]
\[ I \rightarrow \hat{j}, \quad J_{al}(z) \rightarrow \hat{J}_{aj}(z, \sigma), \quad T_I(z) \rightarrow \hat{T}_{j}(z, \sigma) \]  
\[ n(r)j \rightarrow \hat{j}, \quad J_{n(r)aj}(z, \sigma) \rightarrow J_{jaj}(z, \sigma), \quad \Theta_{n(r)j}(z, \sigma) \rightarrow \Theta_{jaj}(z, \sigma) \]  
\[ \frac{n(r)}{\rho(\sigma)} = \frac{N(r)}{R(\sigma)} = \frac{\hat{j}}{f_j(\sigma)}, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1, \quad \sum_j f_j(\sigma) = K \]  
\[ U(\sigma)_{n(r)j}^I \rightarrow U(\sigma)_{n(r)\hat{j}}^\dagger \rightarrow U(\sigma)^{\hat{j}} \rightarrow \frac{\delta_{\hat{j}\hat{i}}}{\sqrt{f_j(\sigma)}} e^{\frac{2\pi i j}{f_j(\sigma)}}, \quad E_j(\sigma) = e^{-\frac{2\pi i j}{f_j(\sigma)}} \]  
\[ \chi(\sigma)_{jaj} = \chi(\sigma)_{\hat{j}\hat{j}} = \sqrt{f_j(\sigma)} \]
where \( \tilde{j} \) is the pullback of the spectral index \( j \) to the fundamental domain. As a computational aide for the reader, we list some well-known examples

\[
\mathbb{Z}_\lambda: K = \lambda, \quad f_j(\sigma) = \rho(\sigma), \quad \tilde{j} = 0, \ldots, \rho(\sigma) - 1, \quad j = 0, \ldots, \frac{\lambda}{\rho(\sigma)} - 1, \quad \sigma = 0, \ldots, \rho(\sigma) - 1
\]  
\hspace{10cm} (2.20a)

\[
\mathbb{Z}_\lambda, \quad \lambda = \text{prime}: \quad \rho(\sigma) = \lambda, \quad \tilde{j} = 0, \ldots, \lambda - 1, \quad j = 0, \ldots, \lambda - 1
\]  
\hspace{10cm} (2.20b)

\[
S_N: \quad K = N, \quad f_j(\sigma) = \sigma_j, \quad \sigma_{j+1} \leq \sigma_j, \quad j = 0, \ldots, n(\bar{\sigma}) - 1, \quad \sum_{j=0}^{n(\bar{\sigma})-1} \sigma_j = N
\]  
\hspace{10cm} (2.20c)

so that e.g. the sectors of the \( S_N \) permutation orbifolds are labelled by the ordered partitions of \( N \).

All the eigenfields are independent of the untwisted basis, e.g.

\[
\mathcal{J}_{n(\tau)aj}(z, \sigma) \longrightarrow \mathcal{J}_{ja,j}(z, \sigma) = \chi(\sigma) \delta_{aj} U(\sigma) \hat{J}_{a;j}(z, \sigma)
\]

\[
= \chi(\sigma) \delta_{aj} U(\sigma) \hat{J}_{a;j}(z, \sigma) = \sum_{j'=0}^{f_j(\sigma)-1} e^{2\pi i j' j} \hat{J}_{a;j'}(z, \sigma). \quad (2.21)
\]

Similarly, each duality transformation is independent of the choice of untwisted basis, and hence all of the OPEs of the eigenfields are likewise basis-independent.

Using the explicit forms of \( U(\sigma) \) and \( \chi(\sigma) \) in (2.19), we may evaluate all the duality transformations that appear in our development:

\[
\mathcal{A}_{\hat{j}j}(\sigma) = \delta_{\hat{j},0 \mod f_j(\sigma)}, \quad \mathcal{B}_{\hat{j}j}(\sigma) = f_j(\sigma) \delta_{\hat{j},0 \mod f_j(\sigma)} \]  
\hspace{10cm} (2.22a)

\[
\mathcal{G}_{\hat{j}a;j;bl}(\sigma) = k \eta_{ab} f_j(\sigma) \delta_{\hat{j}l} \delta_{j+1,0 \mod f_j(\sigma)}, \quad \mathcal{\tilde{G}}_{\hat{j}j;il}(\sigma) = f_j(\sigma) \delta_{ij} \delta_{\hat{j}l} \delta_{j+1,0 \mod f_j(\sigma)} \]  
\hspace{10cm} (2.22b)

\[
\mathcal{F}_{\hat{j}a;j;bl}^{nmc}(\sigma) = f_{ab} \eta_{\hat{j}l} \delta_{\hat{j}n} \delta_{\hat{j}l+m,0 \mod f_j(\sigma)}, \quad \mathcal{\tilde{F}}_{\hat{j}j;il}^{nm}(\sigma) = \delta_{ij} \delta_{\hat{j}l} \delta_{\hat{j}j+1,0 \mod f_j(\sigma)} \]  
\hspace{10cm} (2.22c)

\[
\mathcal{D}_{\hat{\sigma}}(\tau, \sigma)_{\alpha;j} = \delta_{\alpha}^{\beta} \delta_{\hat{j}l} \delta_{\hat{j}j+1,0 \mod f_j(\sigma)} \Delta_{\hat{\sigma}}(\tau)
\]  
\hspace{10cm} (2.22d)

\[
\mathcal{L}_{\hat{\sigma}}^{a;\hat{\sigma}}(\sigma) = \frac{\eta_{ab}}{2k + Q \hat{\sigma} f_j(\sigma)} \delta_{\hat{j}l} \delta_{\hat{j}j+1,0 \mod f_j(\sigma)}
\]  
\hspace{10cm} (2.22e)

\[
\mathcal{L}_{\hat{j}j}^{a;\hat{\sigma}bm}(\sigma) = \frac{\eta_{ab}}{2k + Q \hat{\sigma}} f_j(\sigma) \delta_{\hat{j}l} \delta_{\hat{j}j+1,0 \mod f_j(\sigma)}
\]  
\hspace{10cm} (2.22f)
\[ T_{jaj}(T, \sigma) = T_a t_{jj}(\sigma), \quad t_{jj}(\sigma)|_{l m} \equiv \delta_{jl} \delta_{m,l-\hat{m},0 \mod f_j(\sigma)}, \quad t_{j\pm f_j(\sigma),j}(\sigma) = t_{jj}(\sigma) \quad (2.22g) \]

\[ [T_a, T_b] = if_{abc}, \quad t_{jj}(\sigma)t_{li}(\sigma) = \delta_{jl} t_{j\pm f_j(\sigma),j}(\sigma), \quad [t_{jj}(\sigma), t_{ll}(\sigma)] = 0, \quad \forall j, l. \quad (2.22h) \]

Here \( T \) is an irrep of \( g \) and \( \Delta g(T) \) is the conformal weight of representation \( T \) under the affine-Sugawara construction on \( g \). All these quantities are periodic \( \hat{j} \rightarrow \hat{j} \pm f_j(\sigma) \) in any spectral index, as seen for example in (2.22g). In what follows we often abbreviate \( T \equiv T(T, \sigma) \).

### 2.3 Local Isomorphisms and Twisted Operator Product Expansions

The next step is an application of the principle of local isomorphisms [3, 5, 12] which is a map from the untwisted theory in the eigenfield basis to twisted sector \( \sigma \) of the orbifold. In the present case, the local isomorphisms are written as:

\[ J_{jaj}(z, \sigma) \rightarrow \hat{J}_{jaj}(z, \sigma), \quad G(T, \bar{z}, z, \sigma)_{a, j}^{\beta, l} \rightarrow \hat{g}(T, \bar{z}, z, \sigma)_{a, j}^{\beta, l} \quad (2.23a) \]

\[ \Theta_{jj}(z, \sigma) \rightarrow \hat{\Theta}_{jj}(z, \sigma), \quad T(z) \rightarrow \hat{T}_\sigma(z) \quad (2.23b) \]

\[ a = 1, \ldots, \dim g, \quad \alpha, \beta = 1, \ldots, \dim T, \quad j, l = 0, \ldots, f_j(\sigma) - 1, \quad \sigma = 0, \ldots, N_c - 1. \quad (2.23c) \]

Here the hatted fields \( \hat{J}, \hat{g}, \) and \( \hat{\Theta} \) are called respectively the twisted currents, the twisted affine primary fields, and the twisted partial stress tensors. The principle of local isomorphisms tells us first that the monodromies of the twisted fields

\[ \hat{J}_{jaj}(ze^{2\pi i}, \sigma) = E_j(\sigma)\hat{J}_{jaj}(z, \sigma) = e^{-2\pi i \tilde{j}/f_j(\sigma)} \hat{J}_{jaj}(z, \sigma) \quad (2.24a) \]

\[ \hat{\Theta}_{jj}(ze^{2\pi i}, \sigma) = E_j(\sigma)\hat{\Theta}_{jj}(z, \sigma) = e^{-2\pi i \tilde{j}/f_j(\sigma)} \hat{\Theta}_{jj}(z, \sigma) \quad (2.24b) \]

\[ \hat{T}_\sigma(ze^{2\pi i}) = \hat{T}_\sigma(z) \quad (2.24c) \]

have the same form as the automorphic responses (2.14) of the eigenfields. The monodromies of the twisted affine primary fields \( \hat{g} \) are determined by the twisted KZ equations. We remind the reader that all quantities are periodic \( \hat{j} \rightarrow \hat{j} \pm f_j(\sigma) \) in the spectral indices.
The principle of local isomorphisms also tells us that the OPEs of the twisted fields are the same as the OPEs of the eigenfields. For example, the OPEs of the standard twisted fields are given in Refs. [12, 13].

\[
\hat{J}_{j_1 j_2} (z, \sigma) \hat{J}_{\bar{l} \bar{d}} (w, \sigma) = \delta_{j_1 \bar{l}} \left( \frac{k f_j (\sigma) \eta_{ab} \delta_{j+1,0} f_j (\Sigma)}{(z - w)^2} + \frac{i f_{\sigma} \hat{J}_{j_1 j_2} (w, \sigma)}{z - w} \right) + O(z - w)^0 \tag{2.25a}
\]

\[
\hat{T}_\sigma (z) \hat{J}_{j_1 j_2} (w, \sigma) = \left( \frac{1}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \hat{J}_{j_1 j_2} (w, \sigma) + O(z - w)^0 \tag{2.25b}
\]

\[
\hat{T}_\sigma (z) \hat{T}_\sigma (w) = \frac{c_g / 2}{(z - w)^4} + \left( \frac{2}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \hat{T}_\sigma (w) + O(z - w)^0 \tag{2.25c}
\]

\[
\hat{J}_{j_1 j_2} (z, \sigma) \hat{g} (T, \bar{w}, w, \sigma) = \frac{\hat{g} (T, \bar{w}, w, \sigma) T_a t_{j_1 j_2}}{z - w} + O(z - w)^0 \tag{2.25d}
\]

\[
\hat{T}_\sigma (z) \hat{g} (T, \bar{w}, w, \sigma) = \frac{\hat{g} (T, \bar{w}, w, \sigma) \Delta_g (T)}{(z - w)^2} + \frac{\partial_w \hat{g} (T, \bar{w}, w, \sigma)}{z - w} + O(z - w)^0 \tag{2.25e}
\]

are given in Refs. [12, 13].

In our extension, we also obtain the OPEs involving the twisted partial stress tensors

\[
\hat{\Theta}_{j_1 j_2} (z, \sigma) \hat{J}_{\bar{l} \bar{d}} (w, \sigma) = \delta_{j_1 \bar{l}} \left( \frac{1}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \hat{J}_{j_1 j_2} (w, \sigma) + O(z - w)^0 \tag{2.26a}
\]

\[
\hat{\Theta}_{j_1 j_2} (z, \sigma) \hat{\Theta}_{\bar{u} \bar{v}} (w, \sigma) = \delta_{j_1 \bar{u}} \left[ \frac{\delta_{j_1 + 1,0} f_j (\Sigma) c_g f_j (\Sigma)}{(z - w)^4} \right] + \left( \frac{2}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \hat{\Theta}_{j_1 j_2} (w, \sigma) + O(z - w)^0 \tag{2.26b}
\]

\[
\hat{T}_\sigma (z) \hat{\Theta}_{j_1 j_2} (w, \sigma) = \frac{\delta_{j_1 + 1,0} f_j (\Sigma) c_g f_j (\Sigma)}{(z - w)^4} + \left( \frac{2}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \hat{\Theta}_{j_1 j_2} (w, \sigma) + O(z - w)^0 \tag{2.26c}
\]

\[
\hat{\Theta}_{j_1 j_2} (z, \sigma) \hat{g} (T, \bar{w}, w, \sigma) = \left[ \frac{\hat{g} (T, \bar{w}, w, \sigma) \Delta_g (T)}{(z - w)^2} + \frac{\partial_w \hat{g} (T, \bar{w}, w, \sigma)}{z - w} \right] t_{j_1 j_2} (\sigma) + O(z - w)^0. \tag{2.26d}
\]
In terms of the twisted currents, the explicit forms of the stress tensor and the twisted partial stress tensors are

\[ \hat{T}_\sigma(z) = \frac{\eta^{ab}}{2k + Q_\{g\}} \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} : \hat{J}_{j+a}(z,\sigma)\hat{J}_{-j,b}(z,\sigma) : \]  

(2.27a)

\[ \hat{\Theta}_{\hat{j}j}(z,\sigma) = \frac{\eta^{ab}}{2k + Q_\{g\}} \sum_{j=0}^{f_j(\sigma)-1} : \hat{J}_{i+a}(z,\sigma)\hat{J}_{-i,b}(z,\sigma) : \]  

(2.27b)

\[ \hat{T}_\sigma(z) = \sum_j \hat{\Theta}_{0j}(z,\sigma) \]  

(2.27c)

where (2.27a) was first given in Ref. [13]. The normal ordering here is operator product normal ordering [2, 3, 5] of the twisted currents.

Note in particular the \( \delta_{jl} \) factor in (2.25a), (2.26a), and (2.26b). Taken together, these relations tell us that although each sector \( \sigma \) may contain many cycles \( j \), the dynamics of each cycle is independent.

As a final local result in the permutation orbifolds, we give the twisted left-mover vertex operator equation [12, 13]

\[ \partial \hat{g}(T, \bar{z}, z, \sigma) = \frac{2\eta^{ab}}{2k + Q_\{g\}} \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} : \hat{J}_{j+a}(z)\hat{g}(T, \bar{z}, z, \sigma) : M T_b t_{-j,j}(\sigma) \]

\[ - \frac{\Delta_b(T)}{z} \sum_j (1 - \frac{1}{f_j(\sigma)})\hat{g}(T, \bar{z}, z, \sigma) t_{0j}(\sigma) \]  

(2.28)

where this mode normal ordering is defined in Ref. [12].

2.4 Extended Operator Algebra and the Orbifold Virasoro Subalgebra

Using the monodromies (2.24), we may now expand the twisted fields in terms of modes

\[ \hat{J}_{j+a}(z,\sigma) = \sum_{m \in \mathbb{Z}} \hat{J}_{j+a}(m + \frac{j}{f_j(\sigma)}) z^{-(m+\frac{j}{f_j(\sigma)})-1} \]  

(2.29a)

\[ \hat{J}_{j+\pm f_j(\sigma),a}(m + \frac{j \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{J}_{j+a}(m + 1 + \frac{j}{f_j(\sigma)}) \]  

(2.29b)
\[
\hat{\Theta}_{jj}(z, \sigma) = \sum_{m \in \mathbb{Z}} \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)})z^{-(m + \frac{\hat{j}}{f_j(\sigma)})^{-2}}
\] (2.29c)

\[
\hat{L}_{j \pm f_j(\sigma)j}(m + \frac{\hat{j} \pm f_j(\sigma)}{f_j(\sigma)}) = \hat{L}_{jj}(m \pm 1 + \frac{\hat{j}}{f_j(\sigma)})
\] (2.29d)

\[
\hat{T}_\sigma(z) = \sum_{m \in \mathbb{Z}} L_\sigma(m)z^{-m-2}, \quad L_\sigma(m) = \sum_j \hat{L}_{0j}(m)
\] (2.29e)

and then the algebra of these twisted modes follows from the twisted OPE system (2.25), (2.26). The algebra of the twisted current modes and the Virasoro generators \(L_\sigma(m)\)

\[
\left[ \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{lb}(n + \frac{\hat{i}}{f_l(\sigma)}) \right] = \\
= \delta_{jl} \left( if_{ab} c_\sigma \hat{J}_{j+1, ej}(m + n + \frac{\hat{j} + \hat{i}}{f_j(\sigma)}) + \eta_{abk} f_j(\sigma)(m + \frac{\hat{j}}{f_j(\sigma)})\delta_{m+n, \frac{\hat{j}}{f_j(\sigma)}-2}, \delta_{m+n, 0} \right) (2.30a)
\]

\[
\left[ L_\sigma(m), \hat{J}_{ja}(n + \frac{\hat{j}}{f_j(\sigma)}) \right] = -(n + \frac{\hat{j}}{f_j(\sigma)})\hat{J}_{ja}(m + n + \frac{\hat{j}}{f_j(\sigma)})
\] (2.30b)

\[
[L_\sigma(m), L_\sigma(n)] = (m - n)L_\sigma(m + n) + \frac{c_2}{12} m(m^2 - 1)\delta_{m+n, 0} (2.30c)
\]

\[
\left[ \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}(T(T, \sigma), \bar{z}, z, \sigma) \right] = \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)T_a t_{jj}(\sigma)z^{m + \frac{\hat{j}}{f_j(\sigma)}}
\] (2.30d)

\[
[L_\sigma(m), \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)] = \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)(\bar{\partial} z + (m + 1)\Delta_0(T))
\] (2.30e)

was given in Refs. [12, 13]. Here (2.30a) is the general orbifold affine algebra [1-3, 5, 12, 13]. Note that the simple components \(\hat{J}_{ja}\) of the orbifold affine algebra act entirely within disjoint cycle \(j\). The adjoint of these operators

\[
\hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)})^\dagger = \hat{J}_{-j, a}(-m - \frac{\hat{j}}{f_j(\sigma)}), \quad L_\sigma(m)^\dagger = L_\sigma(-m)
\] (2.31)

was given in Refs. [1, 5, 12].

In our extension, we also obtain the algebra of the twisted Virasoro modes \(\hat{L}_{jj},\)

\[
\left[ \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{J}_{ld}(n + \frac{\hat{i}}{f_l(\sigma)}) \right] = -\delta_{jl}(n + \frac{\hat{i}}{f_l(\sigma)})\hat{J}_{j+1, dj}(m + n + \frac{\hat{j} + \hat{i}}{f_j(\sigma)})
\] (2.32a)
\[
\left[ \hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{L}_{il}(n + \frac{i}{f_i(\sigma)}) \right] = \delta_{jl} \{(m - n + \frac{j}{f_j(\sigma)} - \frac{i}{f_i(\sigma)})\hat{L}_{ij+l,j}(m + n + \frac{j + i}{f_j(\sigma)}) + \\
+ \frac{c_j f_j(\sigma)}{12} (m + \frac{j}{f_j(\sigma)})(m + \frac{j}{f_j(\sigma)})^2 - 1)\delta_{m+n,\frac{j+i}{f_j(\sigma)},0} \} \quad (2.32b)
\]

\[
\left[ L_\sigma(m), \hat{L}_{jj}(n + \frac{j}{f_j(\sigma)}) \right] = (m - n - \frac{j}{f_j(\sigma)})\hat{L}_{jj}(m + n + \frac{j}{f_j(\sigma)}) + \\
+ f_j(\sigma)\frac{c_j}{12} m(m^2 - 1)\delta_{m+n,\frac{j+i}{f_j(\sigma)},0} \quad (2.32c)
\]

\[
\left[ \hat{L}_{jj}(m + \frac{j}{f_j(\sigma)}), \hat{g}(T(T,\sigma), \bar{z}, z, \sigma) \right] = \\
\hat{g}(T(T,\sigma), \bar{z}, z, \sigma) \left( \bar{\partial}_z z + ((m + \frac{j}{f_j(\sigma)}) + 1)\Delta_g(T) \right) t_{jj}(\sigma) z^{m+n+\frac{j}{f_j(\sigma)}} \quad (2.32d)
\]

where (2.32b) is the general orbifold Virasoro algebra. The orbifold Virasoro algebra for the special case \( H = \mathbb{Z}_\lambda, \lambda = \text{prime} \)

\[ \hat{L}^{(r)}(m + \frac{r}{\lambda}) \equiv \hat{L}_{j,j=0}(m + \frac{j}{\lambda})|_{j=r}, \quad \hat{r} = 0, \ldots, \lambda - 1, \quad \sigma = 1, \ldots, \lambda - 1 \quad (2.33a) \]

\[
[\hat{L}^{(r)}(m + \frac{r}{\lambda}), \hat{L}^{(s)}(n + \frac{s}{\lambda})] = (m - n + \frac{r - s}{\lambda})\hat{L}^{(r+s)}(m + n + \frac{r + s}{\lambda}) + \\
+ \frac{\lambda c_j}{12} (m + \frac{r}{\lambda})(m + \frac{r}{\lambda})^2 - 1)\delta_{m+n,\frac{j}{f_j(\sigma)},0} \quad (2.33b)
\]

was given earlier in Ref. [1] and independently in Ref. [18]. Of course, we obtain in this paper only the special case of (2.33) with the affine-Sugawara central charge \( c = c_g \). By the same token, the orbifold Virasoro algebra (2.32b) with \( c_g \rightarrow c \) will also occur in more general copy permutation orbifolds, where \( c \) is the central charge of each copy of a general affine-Virasoro construction [9-11] on affine \( g \).

We comment briefly on subalgebras of the orbifold Virasoro algebra. We note first the semisimple integral Virasoro subalgebra

\[ \hat{j} = 0: \quad [\hat{L}_{0j}(m), \hat{L}_{0\ell}(n)] = \delta_{jl} \{(m - n)\hat{L}_{0j}(m + n) + \frac{f_j(\sigma) c_g}{12} m(m^2 - 1)\delta_{m+n,0} \} \quad (2.34) \]
whose generators are in fact the Virasoro operators of each cycle separately (see (2.29e)). The orbifold Virasoro algebra also contains a closed subalgebra whose generators are

\[ \hat{L}_{0,j}(0), \hat{L}_{1,j}(\frac{1}{f_j(\sigma)}), \hat{L}_{f_j(\sigma)-1,j}(-1 + \frac{f_j(\sigma) - 1}{f_j(\sigma)}) = \hat{L}_{-1,j}(0 + \frac{-1}{f_j(\sigma)}), \forall j \] (2.35)

and one finds that this subalgebra is a set of mutually-commuting centrally-extended \( \mathfrak{sl}(2) \) algebras:

\[ \left[ \hat{L}_{1,j}(\frac{1}{f_j(\sigma)}), \hat{L}_{f_i(\sigma)-1,l}(-1 + \frac{f_i(\sigma) - 1}{f_i(\sigma)}) \right] = \delta_{jl} \left( \frac{2}{f_j(\sigma)} \hat{L}_{0,j}(0) + \frac{c_a}{12} \left( \frac{1}{f_j(\sigma)^2} - 1 \right) \right) \] (2.36a)

\[ \left[ \hat{L}_{0,j}(0), \hat{L}_{1,l}(\frac{1}{f_l(\sigma)}) \right] = -\delta_{jl} \frac{1}{f_j(\sigma)} \hat{L}_{1,j}(\frac{1}{f_j(\sigma)}) \] (2.36b)

\[ \left[ \hat{L}_{0,j}(0), \hat{L}_{f_i(\sigma)-1,l}(-1 + \frac{f_i(\sigma) - 1}{f_i(\sigma)}) \right] = \delta_{jl} \frac{1}{f_j(\sigma)} \hat{L}_{f_i(\sigma)-1,j}(-1 + \frac{f_j(\sigma) - 1}{f_j(\sigma)}). \] (2.36c)

Relative to Ref. [1], we see that each of these \( \mathfrak{sl}(2) \)'s is associated to a fixed cycle \( j \). As we shall see in the following subsections, there are \( \mathfrak{sl}(2) \) Ward identities [1] associated to this subalgebra – but not to the standard \( \mathfrak{sl}(2) \) subalgebra generated by \( \{ L_\sigma(|m| \leq 1) \} \).

### 2.5 The Scalar Twist-Field States

For the orbifold affine algebras, it is known [1-3], that the scalar twist-field state \( \tau_\sigma(0)|0\rangle = |0\rangle_\sigma \) of sector \( \sigma \) satisfies

\[ \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0)|0\rangle_\sigma = \sigma(0)|\hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)} \leq 0) = 0 \] (2.37a)

\[ a = 1, \ldots, \text{dim} \mathfrak{g}, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1, \quad \sigma = 0, \ldots, N_c - 1 \] (2.37b)

and the scalar twist-field state is in fact the ground state of sector \( \sigma \) under the Virasoro generator \( L_\sigma(0) \). Then a more useful form of the Virasoro generators is the mode-ordered form \([5, 12, 13]\)

\[ : \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{lb}(n + \frac{\hat{i}}{f_l(\sigma)}) : \equiv \theta(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0) \hat{J}_{lb}(n + \frac{\hat{i}}{f_l(\sigma)}) \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}) + \]

\[ + \theta(m + \frac{\hat{j}}{f_j(\sigma)} < 0) \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{lb}(n + \frac{\hat{i}}{f_l(\sigma)}) \] (2.38a)
\[ L_\sigma(m) = \left( \frac{\eta^{ab}}{2k + Q_\sigma} \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} \hat{J}_{j\alpha}(p + \frac{\hat{j}}{f_j(\sigma)}) \hat{J}_{-j\beta}(m - p - \frac{\hat{j}}{f_j(\sigma)}):M \right) + \hat{\Delta}_0(\sigma) \delta_{m,0} \]  

(2.38b)

\[ L_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0} \hat{\Delta}_0(\sigma)|0\rangle_\sigma, \quad \langle 0|L_\sigma(m \leq 0) = \langle 0|\hat{\Delta}_0(\sigma)\delta_{m,0} \]  

(2.38c)

\[ \hat{\Delta}_0(\sigma) = \frac{c_\sigma}{24} \sum_j (f_j(\sigma) - \frac{1}{f_j(\sigma)}) = \frac{c_\sigma}{24} (K - \sum_j \frac{1}{f_j(\sigma)}) \]  

(2.38d)

where \( \hat{\Delta}_0(\sigma) \) is the conformal weight of the scalar twist-field state of sector \( \sigma \). Similarly, we find for the generators of the orbifold Virasoro algebra

\[ \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) = \left( \frac{\eta^{ab}}{2k + Q_\sigma} \frac{1}{f_j(\sigma)} \sum_{l=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \hat{J}_{l\alpha}(p + \frac{\hat{l}}{f_j(\sigma)}) \hat{J}_{-l\beta}(m - p + \frac{\hat{l}}{f_j(\sigma)}):M \right) + \hat{\Delta}_{0j}(\sigma) \delta_{m+\hat{j}(\sigma),0} \]  

(2.39a)

\[ \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) \geq -\frac{1}{f_j(\sigma)}|0\rangle_\sigma = \delta_{m,0} \hat{\Delta}_{0j}(\sigma)|0\rangle_\sigma \]  

(2.39b)

\[ \langle 0|\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) \leq \frac{1}{f_j(\sigma)} = \langle 0|\hat{\Delta}_{0j}(\sigma)\delta_{m,0} \]  

(2.39c)

\[ \hat{\Delta}_{0j}(\sigma) = \frac{c_\sigma}{24} (f_j(\sigma) - \frac{1}{f_j(\sigma)}), \quad \hat{\Delta}_0(\sigma) = \sum_j \hat{\Delta}_{0j}(\sigma) \]  

(2.39d)

where \( \hat{\Delta}_{0j}(\sigma) \) are the partial conformal weights of sector \( \sigma \). In further detail, (2.37a) and (2.39a) were used to establish (2.39b,c), which tell us that the scalar twist-field state is primary under the orbifold Virasoro algebra. The adjoint [1, 5] of the twisted Virasoro operators

\[ \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)})^\dagger = \hat{L}_{-j,j}(-m - \frac{\hat{j}}{f_j(\sigma)}) \]  

(2.40)

follows from Eqs.(2.31) and (2.39a).
2.6 The Orbifold $\mathfrak{sl}(2)$ Ward Identities and the Twisted KZ System

For the generators (2.35) of the centrally-extended $\mathfrak{sl}(2)$, the relations (2.39b) reduce to

$$\hat{L}_{1,j}(\frac{1}{f_j(\sigma)})|0\rangle_\sigma = \sigma|0\rangle_\sigma \hat{L}_{1,j}(\frac{1}{f_j(\sigma)}) = 0 \quad (2.41a)$$

$$\hat{L}_{f_j(\sigma)-1,j}(-1 + \frac{f_j(\sigma) - 1}{f_j(\sigma)})|0\rangle_\sigma = \sigma|0\rangle_\sigma \hat{L}_{f_j(\sigma)-1,j}(-1 + \frac{f_j(\sigma) - 1}{f_j(\sigma)}) = 0 \quad (2.41b)$$

$$\hat{L}_{0,j}(0)|0\rangle_\sigma = \hat{\Delta}_{0j}(\sigma)|0\rangle_\sigma, \quad \sigma|0\rangle_\sigma \hat{L}_{0,j}(0) = \sigma|0\rangle_\sigma \hat{\Delta}_{0j}(\sigma) \quad (2.41c)$$

where the partial conformal weight $\hat{\Delta}_{0j}(\sigma)$ is given in (2.39d). Although the scalar twist-field state is not invariant under this $\mathfrak{sl}(2)$, nevertheless [1] we find the following identities

$$\hat{A}(T, \bar{z}, z, \sigma) \equiv \sigma|0\rangle_\sigma \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma) \ldots \hat{g}(T^{(N)}, \bar{z}_N, z_N, \sigma)|0\rangle_\sigma$$

$$\equiv \langle \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma) \ldots \hat{g}(T^{(N)}, \bar{z}_N, z_N, \sigma) \rangle_\sigma \quad (2.42)$$

$$\langle [\hat{L}_{0,j}(0), \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma) \ldots \hat{g}(T^{(N)}, \bar{z}_N, z_N, \sigma)] \rangle_\sigma = 0 \quad (2.43a)$$

$$\langle [\hat{L}_{1,j}(\frac{1}{f_j(\sigma)}), \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma) \ldots \hat{g}(T^{(N)}, \bar{z}_N, z_N, \sigma)] \rangle_\sigma = 0 \quad (2.43b)$$

$$\langle [\hat{L}_{f_j(\sigma)-1,j}(-1 + \frac{f_j(\sigma) - 1}{f_j(\sigma)}), \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma) \ldots \hat{g}(T^{(N)}, \bar{z}_N, z_N, \sigma)] \rangle_\sigma = 0 \quad (2.43c)$$

for the correlators $\hat{A}$ in the scalar twist-field state of sector $\sigma$. These give the orbifold $\mathfrak{sl}(2)$ Ward identities

$$\hat{A}(T, \bar{z}, z, \sigma) \sum_{\mu=1}^N \left( \hat{\partial}_\mu z_\mu + \hat{\Delta}_{0}(T^{(\mu)}) \right) t^{(\mu)}_{0j}(\sigma) = 0, \quad \forall j \quad (2.44a)$$

$$\hat{A}(T, \bar{z}, z, \sigma) \sum_{\mu=1}^N \left[ \left( \hat{\partial}_\mu z_\mu + (1 + \frac{1}{f_j(\sigma)})\hat{\Delta}_{0}(T^{(\mu)}) \right) t^{(\mu)}_{1j}(\sigma) z^{\frac{1}{f_j(\sigma)}} \right] = 0, \quad \forall j \quad (2.44b)$$

$$\hat{A}(T, \bar{z}, z, \sigma) \sum_{\mu=1}^N \left[ \left( \hat{\partial}_\mu z_\mu + (1 - \frac{1}{f_j(\sigma)})\hat{\Delta}_{0}(T^{(\mu)}) \right) t^{(\mu)}_{f_j(\sigma)-1,j}(\sigma) z^{\frac{1}{f_j(\sigma)}} \right] = 0, \quad \forall j \quad (2.44c)$$
which are the analogues of the orbifold $\mathfrak{sl}(2)$ Ward identities given for $H = \mathbb{Z}_\lambda, \lambda = \text{prime}$ in Ref. [1]. The Ward identity in (2.44a) implies the known $L_\sigma(0)$ Ward identity [12]

$$\langle [L_\sigma(0), \tilde{g}(T^{(1)}, z_1, \sigma) \ldots \tilde{g}(T^{(N)}, z_N, \sigma)] \rangle_\sigma = 0 \quad (2.45a)$$

$$\hat{A}(T, \bar{z}, z, \sigma) \sum_{\mu=1}^{N} \left( \frac{\delta^\mu_{\bar{z}} z_\mu + \Delta_\sigma(T^{(\mu)})}{z_\mu} \right) = 0 \quad (2.45b)$$

as a consequence of Eq. (2.29e) and the relation $\sum_j t_{0j}(\sigma) = 1$.

It is known [12] that the $L_\sigma(0)$ Ward identity (2.45) follows from the twisted left-mover KZ system [12, 13]

$$\partial_\mu \hat{A}(T, \bar{z}, z, \sigma) = \hat{A}(T, \bar{z}, z, \sigma) \hat{W}_\mu(T, z, \sigma), \quad \hat{W}_\mu(T, z, \sigma) = \sum_j \hat{W}_\mu^j(T, z, \sigma) \quad (2.46a)$$

$$\hat{W}_\mu^j(T, z, \sigma) \equiv \frac{2}{2k + Q_\mathfrak{g}} \sum_{\nu \neq \mu} \eta^{ab} T_a^{(\nu)} T_b^{(\mu)} f_j(\sigma) z_{\nu\mu} \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{z_\nu}{z_\mu} \right)^{\frac{j}{f_j(\sigma)}} t_{0j}^{(\mu)}(\sigma) t_j^{(\nu)}(\sigma)$$

$$- \frac{\Delta_\sigma(T^{(\mu)})}{z_\mu} (1 - \frac{1}{f_j(\sigma)}) t_{0j}^{(\mu)}(\sigma) \quad (2.46b)$$

$$\hat{A}(T, \bar{z}, z, \sigma) \left( \sum_{\mu=1}^{N} T_\mu^{(\mu)} t_{0j}^{(\mu)}(\sigma) \right) = 0, \quad \forall j, a = 1, \ldots, \dim \mathfrak{g}, \quad \sigma = 0, \ldots, N_\mathfrak{g} - 1. \quad (2.46c)$$

In this notation, all matrix products $M^{(\nu)} N^{(\mu)} \equiv M^{(\nu)} \otimes N^{(\mu)}, \nu \neq \mu$ are tensor products. The form of the twisted KZ connection in (2.46a,b) is easily obtained from the original form given in Refs. [12, 13] by the relation $\sum_j t_{0j}(\sigma) = 1$. The twisted partial connections $\hat{W}_\mu^j$ defined above are abelian flat:

$$\partial_\mu \hat{W}_\nu^j - \partial_\nu \hat{W}_\mu^j = 0 \quad (2.47a)$$

$$\left[ \hat{W}_\mu^j(\sigma), \hat{W}_{\nu}^{j'}(\sigma) \right] = 0, \quad \text{when } j \neq j' \quad (2.47b)$$

$$\left[ \hat{W}_\mu^j(\sigma), \hat{W}_{\nu}^{j}(\sigma) \right] = 0, \quad \forall j, \mu, \nu. \quad (2.47c)$$

Here Eqs. (2.47a,b) are relatively easy to check, and the more difficult proof of (2.47c) is essentially identical to the proof given for $H = \mathbb{Z}_\lambda$ in Ref. [12]. We remind the reader that the $z_{\mu}^{-1}$ term in (2.46b) can be traced back to the fact that the twist-field states $|0\rangle_\sigma = \tau_\sigma(0)|0\rangle, L_\sigma(-1)|0\rangle_\sigma \neq 0$ are not translation invariant.

Moreover, one finds after some algebra (see App. A) that the entire system (2.44) of orbifold $\mathfrak{sl}(2)$ Ward identities also follows from the twisted KZ system (2.46). This phenomenon is familiar from the untwisted KZ system [23], but we emphasize that it is an unsolved problem to find the general solution of the orbifold $\mathfrak{sl}(2)$ Ward identities.
2.7 Right-Movers and Rectification

As current-algebraic orbifold theory \[1-5, 12, 13, 17\] is presently formulated, the action of the automorphism group on the right-mover fields

\[
\tilde{J}_{aI}(\tilde{z})' = \omega(h_\sigma) I^J \tilde{J}_{aJ}(\tilde{z}), \quad \tilde{T}_I(\tilde{z})' = \omega(h_\sigma) I^J \tilde{T}_J(\tilde{z})
\]

is taken to be the same as the action on the left-movers. This defines what can be called the class of current-algebraic \textit{symmetric} orbifolds, as contrasted with the class of current-algebraic asymmetric orbifolds which will not be studied here.

Because the orbifold is symmetric, the right-mover eigenfields have the same forms as the left-movers, e.g.

\[
\tilde{\Theta}_{n(r)j}(z, \sigma) = \chi(\sigma)_{n(r)j} U(\sigma)_{n(r)j} \tilde{T}_I(z) \quad (2.49)
\]

and the principle of local isomorphisms tells us that the twisted right-mover OPE system is a copy of the twisted left-mover system above. Moreover, the principle of local isomorphisms gives the monodromies of the twisted fields:

\[
\hat{\tilde{J}}_{jaj}(\tilde{z} e^{2\pi i}, \sigma) = E_j(\sigma) \hat{\tilde{J}}_{jaj}(\tilde{z}, \sigma) = e^{-\frac{2\pi i}{f_j(\sigma)}} \hat{\tilde{J}}_{jaj}(\tilde{z}, \sigma) \quad (2.50a)
\]

\[
\hat{\tilde{\Theta}}_{jj}(\tilde{z} e^{2\pi i}, \sigma) = e^{-\frac{2\pi i}{f_j(\sigma)}} \hat{\tilde{\Theta}}_{jj}(\tilde{z}, \sigma) \quad (2.50b)
\]

\[
\hat{\tilde{T}}(\tilde{z} e^{2\pi i}, \sigma) = \hat{\tilde{T}}(\tilde{z}, \sigma) \quad (2.50c)
\]

The rule here \[12\] is that the monodromies of the right-movers are the same as those of the left-movers when the same path is followed. The monodromies (2.50) give the mode expansions

\[
\hat{\tilde{J}}_{jaj}(\tilde{z}, \sigma) = \sum_{m \in \mathbb{Z}} \hat{\tilde{J}}_{jaj}(m + \frac{\hat{j}}{f_j(\sigma)}) \tilde{z}^{(m + \frac{\hat{j}}{f_j(\sigma)}) - 1}, \quad \hat{\tilde{T}}_\sigma(\tilde{z}) = \sum_{m \in \mathbb{Z}} \hat{\tilde{T}}_\sigma(m) \tilde{z}^{-m - 2} \quad (2.51a)
\]

\[
\hat{\tilde{\Theta}}_{jj}(\tilde{z}, \sigma) = \sum_{m \in \mathbb{Z}} \hat{\tilde{\Theta}}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}) \tilde{z}^{(m + \frac{\hat{j}}{f_j(\sigma)}) - 2} \quad (2.51b)
\]

and, together with the twisted right-mover OPE system, one obtains the right-mover chiral algebra of each sector of the orbifold.
As discussed on the sphere and the torus in Refs. [12, 13], the twisted right-mover current algebra shows a sign reversal of the central terms, and in fact we find the same phenomenon in the right-mover orbifold Virasoro algebra.

In the rectification problem [12, 13] one asks whether the twisted right-mover current algebra of sector \( \sigma \) is equivalent to a copy of the twisted left-mover current algebra of that sector. For the WZW permutation orbifolds in particular, this rectification has been demonstrated [12] and we have checked that a similar rectification is possible for the right-mover orbifold Virasoro algebra. The rectified right-mover operators \( \hat{J}_j^\sigma \), \( \hat{L}_j^\sigma \) are

\[
\hat{J}_{ja}(m + \frac{j}{f_j(\sigma)}) \equiv \hat{J}_{-ja}(-m - \frac{j}{f_j(\sigma)}), \quad \hat{L}_{j}(m + \frac{j}{f_j(\sigma)}) \equiv \hat{L}_{-j}(-m - \frac{j}{f_j(\sigma)}) \tag{2.52a}
\]

\[
\tilde{L}_\sigma(m) = \sum_j \tilde{L}_{0,j}(-m) = \sum_j \tilde{L}_{0,j}^\sigma(m) \tag{2.52b}
\]

\[
\begin{align*}
\left[ \hat{J}_{ja}^\sigma(m + \frac{j}{f_j(\sigma)}), \hat{J}_{ia}^\sigma(n + \frac{i}{f_i(\sigma)}) \right] &= \delta_{ji} \left( if_{ab}^c \hat{J}_{j+i,c}^\sigma(m + n + \frac{j + i}{f_j(\sigma)}) + \eta_{ab}^c f_j(\sigma)(m + \frac{j}{f_j(\sigma)}) \delta_{m+n,\frac{j+i}{f_j(\sigma)}} \right) \tag{2.53a} \\
\left[ \hat{L}_{ja}^\sigma(m + \frac{j}{f_j(\sigma)}), \hat{J}_{ia}^\sigma(n + \frac{i}{f_i(\sigma)}) \right] &= -\delta_{ji} (n + \frac{i}{f_i(\sigma)}) \hat{J}_{j+i,a}^\sigma(m + n + \frac{j + i}{f_j(\sigma)}) \tag{2.53b} \\
\left[ \hat{L}_{ja}^\sigma(m + \frac{j}{f_j(\sigma)}), \hat{L}_{ia}^\sigma(n + \frac{i}{f_i(\sigma)}) \right] &= \delta_{ji} \left( (m - n + \frac{j - i}{f_j(\sigma)}) \hat{L}_{j+i,a}^\sigma(m + n + \frac{j + i}{f_j(\sigma)}) + \right. \\
&\quad \left. + \frac{c_{ab}^c f_j(\sigma)}{12} (m + \frac{j}{f_j(\sigma)})(m + \frac{j}{f_j(\sigma)})^2 - 1) \delta_{m+n,\frac{j+i}{f_j(\sigma)}} \right) \tag{2.53c}
\end{align*}
\]

and the right-mover algebra commutes with the left-mover algebra. For completeness, we also give the form of the rectified orbifold Virasoro generators in terms of the rectified right-mover modes \( \hat{J}_j^\sigma \)

\[
\tilde{L}_\sigma(m) = \left( \frac{\eta_{ab}}{2k + Q_\sigma} \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} : \hat{J}_{ja}^\sigma(p + \frac{j}{f_j(\sigma)}) \hat{J}_{-j,bj}^\sigma(m - p - \frac{j}{f_j(\sigma)}) :_M \right) + \hat{A}_\sigma(m) \delta_{m,0} \tag{2.54a}
\]
\[ \hat{L}_{jj}^\sharp(m + \frac{\hat{j}}{f_j(\sigma)}) = \left( \frac{\eta^{ab} Q_{f_j(\sigma)} - 1}{2k + Q_{f_j(\sigma)}} \sum_{l=0}^{f_j(\sigma)-1} \sum_{p \in \mathbb{Z}} \hat{J}_{loj}^\sharp(p + \frac{l}{f_j(\sigma)}), \hat{J}_{j-ib_j}^\sharp(m - p + \frac{\hat{j} - l}{f_j(\sigma)}) : M \right) + \hat{\Delta}_{0j}(\sigma) \delta_{m+\frac{\hat{j}}{f_j(\sigma)}0} \] (2.54b)

where the \( M \) normal ordering used here for the modes of \( \hat{J}^\sharp \) is the same as that given in (2.38a) for the modes of \( \hat{J} \). These forms are nothing but right-mover copies of the corresponding left-mover results in Eqs. (2.38b) and (2.39a).

On the other hand, the algebra of the rectified right-mover modes is not exactly the same as that of the left-mover modes. For example, the relations

\[ [\hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)] = -\bar{z}^{(m + \frac{\hat{j}}{f_j(\sigma)})} t_{-\bar{j}, j}(\sigma) \hat{g}(T(T, \sigma), \bar{z}, z, \sigma) \] (2.55a)

\[ [\hat{L}_{jj}^\sharp(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)] = \bar{z}^{(m + \frac{\hat{j}}{f_j(\sigma)})} (\bar{z} \partial + ((m + \frac{\hat{j}}{f_j(\sigma)}) + 1) \Delta_g(T)) t_{-\bar{j}, j}(\sigma) \hat{g}(T(T, \sigma), \bar{z}, z, \sigma) \] (2.55b)

\[ [\hat{L}_a(m), \hat{g}(T(T, \sigma), \bar{z}, z, \sigma)] = \bar{z}^m (\bar{z} \partial + (m + 1) \Delta_g(T)) \hat{g}(T(T, \sigma), \bar{z}, z, \sigma) \] (2.55c)

are found for the commutators with the twisted affine primary fields.

In terms of the rectified right-mover current modes, the right-mover ground state conditions are copies of those for the left-movers:

\[ \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0 |0\rangle = \sigma |0\rangle \] (2.56a)

\[ \hat{L}_{jj}^\sharp(m + \frac{\hat{j}}{f_j(\sigma)}) \geq -\frac{1}{f_j(\sigma)} |0\rangle = \delta_{m+\frac{\hat{j}}{f_j(\sigma)}0} \hat{\Delta}_{0j}(\sigma) |0\rangle \] (2.56b)

\[ \sigma |0\rangle \hat{L}_{jj}^\sharp(m + \frac{\hat{j}}{f_j(\sigma)}) \leq \frac{1}{f_j(\sigma)} = \sigma |0\rangle \hat{\Delta}_{0j}(\sigma) \delta_{m+\frac{\hat{j}}{f_j(\sigma)}0} \] (2.56c)

This leads immediately to the right-mover orbifold \( \mathfrak{sl}(2) \) Ward identities

\[ \sum_{\mu=1}^N (\bar{z}_\mu \bar{\partial}_\mu + \Delta_g(T^{(\mu)})^t_{0j}(\sigma) \hat{A}(T, \bar{z}, z, \sigma) = 0, \forall j \] (2.57a)
\[
\sum_{\mu=1}^{N} \left[ \bar{z}_{\mu} f_j(\sigma) \left( \bar{z}_{\mu} \bar{\partial}_{\mu} + (1 + \frac{1}{f_j(\sigma)}) \Delta_{\theta}(T(\mu)) \right) t_{-1j}(\sigma) \right] \hat{A}(T, \bar{z}, z, \sigma) = 0, \quad \forall j \tag{2.57b}
\]

\[
\sum_{\mu=1}^{N} \left[ \bar{z}_{\mu} f_j(\sigma) \left( \bar{z}_{\mu} \bar{\partial}_{\mu} + (1 - \frac{1}{f_j(\sigma)}) \Delta_{\theta}(T(\mu)) \right) t_{_1j}(\sigma) \right] \hat{A}(T, \bar{z}, z, \sigma) = 0, \quad \forall j \tag{2.57c}
\]

\[
\sum_{\mu=1}^{N} \left( \bar{z}_{\mu} \bar{\partial}_{\mu} + \Delta_{\theta}(T(\mu)) \right) \hat{A}(T, \bar{z}, z, \sigma) = 0 \tag{2.57d}
\]

where Eq. (2.57d), which follows from (2.57a), was given earlier in Ref. [12].

The twisted right-mover vertex operator equation is [12, 13]

\[
\bar{\partial}_{\mu} \hat{g}(T, \bar{z}, z, \sigma) = -\frac{2\eta^{ab}}{2k + Q_g} \sum_{\nu \neq \mu} \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} T_b t_{jj} : \hat{j}_{jaj}(\bar{z}) \hat{g}(T, \bar{z}, z, \sigma) :_{\bar{M}}
\]

\[
- \frac{\Delta_{\theta}(T)}{\bar{z}} \sum_{j} (1 - \frac{1}{f_j(\sigma)}) t_{0j}(\sigma) \hat{g}(T, \bar{z}, z, \sigma) \tag{2.58}
\]

where \(\bar{M}\) normal ordering is defined in Ref. [12], and the twisted right-mover KZ system is [12, 13]:

\[
\bar{\partial}_{\mu} \hat{A}(T, \bar{z}, z, \sigma) = \hat{W}_{\mu}(T, \bar{z}, \sigma) \hat{A}(T, \bar{z}, z, \sigma), \quad \hat{W}_{\mu}(T, \bar{z}, \sigma) = \sum_{j} \hat{W}_{\mu}^{j}(T, \bar{z}, \sigma) \tag{2.59a}
\]

\[
\hat{W}_{\mu}^{j}(T, \bar{z}, \sigma) = \frac{2}{2k + Q_g} \sum_{\nu \neq \mu} \eta^{ab} T_{a}^{(\nu)} T_{b}^{(\mu)} T_{c}^{(\sigma)} f_j(\sigma) \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{\bar{z}_{\nu}}{\bar{z}_{\mu}} \right) t_{\bar{j}j}^{(\mu)}(\sigma) t_{-j\bar{c}}^{(\nu)}(\sigma)
\]

\[
- \frac{\Delta_{\theta}(T)}{\bar{z}_{\mu}} (1 - \frac{1}{f_j(\sigma)}) t_{0j}^{(\mu)}(\sigma) \tag{2.59b}
\]

\[
\left( \sum_{\mu=1}^{N} T_{a}^{(\mu)} t_{0j}^{(\mu)}(\sigma) \right) \hat{A}(T, \bar{z}, z, \sigma) = 0, \quad \forall j, \quad a = 1, \ldots, \dim g, \quad \sigma = 0, \ldots, N_c - 1. \tag{2.59c}
\]

Finally, we have checked that the right-mover orbifold \(\mathfrak{sl}(2)\) Ward identities (2.57) are satisfied as a consequence of this system.

This completes our discussion of the orbifold \(\mathfrak{sl}(2)\) Ward identities associated to the orbifold Virasoro algebra of each WZW permutation orbifold. As discussed in Sec. 4, the orbifold Virasoro algebras are also important tools in the study of the spectra of the WZW permutation orbifolds.
3 Reducibility in WZW Permutation Orbifolds

3.1 Reducibility of the Twisted Affine Primary Fields

We will argue in this subsection that the WZW permutation orbifolds admit a ‘natural’ solution in which the twisted affine primary fields are reducible according to the disjoint cycles \( j \) of \( h_\sigma \in H \), with the following block-diagonal structure

\[
\hat{g}(T(T,\sigma),\bar{z},z,\sigma)_{\alpha j}^{\beta j} = \delta_j^i \hat{g}_j(T(T,\sigma),\bar{z},z,\sigma)_{\alpha j}^{\beta i}, \quad \forall T \text{ for each } \sigma = 0, \ldots, N_c - 1
\]

(3.1a)

\[
\alpha, \beta = 1, \ldots, \dim T, \quad \bar{j}, \bar{l} = 0, \ldots, f_j(\sigma) - 1
\]

(3.1b)

where \( \hat{g}_j \) are the blocks. We will also argue (as in the untwisted case [25]) that the twisted affine primary fields can be factorized \( \hat{g}_j = \hat{g}_j^- \cdot \hat{g}_j^+ \) into twisted left- and right-mover fields. Finally we will see that all these fields are not functions of \( \bar{j}, \bar{l} \) independently, but only of the difference variable \( \bar{l} - \bar{j} \).

In fact, evidence has been accumulating for the reducibility of the twisted affine primary fields of the permutation orbifolds - in particular at the classical [12, 17] and at the abelian [13] levels. We review this evidence first by way of motivation.

• Motivation: Functional Integral Formulation

For all WZW orbifolds, the WZW orbifold action [12, 17] is a functional of the classical group orbifold elements which are the classical (high-level) limit of the twisted affine primary fields. In the case of the WZW permutation orbifolds it has been known for some time that the classical group orbifold elements are reducible according to the disjoint cycles, exactly as shown in (3.1a). For this discussion, it is conventional to work on the cylinder \((\xi, t)\), where one has the following relations [12, 17]:

\[
T_{j\alpha j}(T, \sigma) = T_{\alpha t j}(\sigma)
\]

(3.2a)

\[
\hat{g}(T(T,\sigma),\xi, t, \sigma) = e^{i \sum_j \hat{g}^{(j)}(\xi, t, \sigma) T_{j\alpha j}(T, \sigma)}, \quad \forall T \text{ for each } \sigma = 0, \ldots, N_c - 1
\]

(3.2b)

\[
\hat{g}(T, \xi + 2\pi, t, \sigma)_{\alpha j}^{\beta j} = e^{-2\pi i \hat{g}^{(j)}(\xi, t, \sigma)_{\alpha j}^{\beta j}}
\]

(3.2c)

\[
\hat{g}(T, \xi, t, \sigma)_{\alpha j}^{\beta j} = \delta_j^l \hat{g}_j(T, \xi, t, \sigma)_{\alpha j}^{\beta l}, \quad \hat{j}, \hat{l} = 0, \ldots, f_j(\sigma) - 1
\]

(3.2d)
the permutation orbifolds is separable

The twisted representation matrices $T$, and we have suppressed the Lie indices $\alpha, \beta$

Then one has factorization of the correlators according to distinct values of $L$

Moreover, separability of the action tells us that, at least in the classical limit, the dynamics of the block $\hat{g}_j$ is independent of the dynamics of the block $\hat{g}_l$ when $j \neq l$.

The quantum theory admits a ‘natural’ solution of the same type because the Virasoro operators $L_\sigma(m), \tilde{L}_\sigma(m)$ in (2.29e) and (2.52b) are sums of commuting terms $L_{0j}(m)$. In a functional integral formulation with the action (3.3), this can be realized by choosing the naive measure which is factorized in $j$,

Then one has factorization of the correlators according to distinct values of $j$, e.g. when $j \neq l$:

$$\langle \hat{g}_j(T(1), \xi_1, t_1, \sigma) \hat{g}_l(T(2), \xi_2, t_2, \sigma) \hat{g}_l(T(3), \xi_3, t_3, \sigma) \hat{g}_l(T(4), \xi_4, t_4, \sigma) \rangle_{\sigma} = \langle \hat{g}_j(T(1), \xi_1, t_1, \sigma) \hat{g}_j(T(3), \xi_3, t_3, \sigma) \rangle_{\sigma} \times \langle \hat{g}_l(T(2), \xi_2, t_2, \sigma) \hat{g}_l(T(4), \xi_4, t_4, \sigma) \rangle_{\sigma}. \quad (3.5)$$
Similarly, all correlators in the functional integral formulation can be expressed in terms of the independent “single-cycle” correlators

\[ \langle \hat{g}_j(T^{(1)}, \xi_1, t_1, \sigma) \ldots \hat{g}_j(T^{(N)}, \xi_N, t_N, \sigma) \rangle_\sigma \]  

for each fixed value of \( j \).

Finally we notice that the blocks of the classical group orbifold elements are not functions of \( \hat{j}, \hat{l} \) independently, but rather only of the difference variable \( \hat{l} - \hat{j} \):

\[ \hat{g}_j(T, \xi, t, \sigma)_{\hat{j}+\hat{m}}^{\hat{j}+\hat{m}} = \hat{g}_j(T, \xi, t, \sigma)_j^i, \quad \forall \hat{j}, \hat{l}, \hat{m}. \]  

This shift-invariance condition follows from the difference variable structure of the matrices \( \tau_j(j, \sigma) \) in (3.2f). As we shall see, the shift invariance will also generalize to the twisted affine primary fields in the operator formulation.

**Motivation: Abelian Permutation Orbifolds**

Ref. [12] gave the twisted vertex operator equations for all WZW orbifolds, and Ref. [13] solved those equations in an abelian limit to obtain the twisted vertex operators of a large class of abelian orbifolds

\[ \frac{A_{\text{Cartan}_g}(H)}{H}, \quad H \subset \text{Aut(Cartan}_g), \quad \text{Cartan}_g \subset g \]  

where the ambient algebra \( g \) supplies the representation space for the twisted sectors of each orbifold. We are interested here in the case of the abelian permutation orbifolds, where Cartan \( g \) has the form

\[ \text{Cartan}_g = \bigoplus_I \text{Cartan}_{g^I}, \quad \text{Cartan}_{g^I} \simeq \text{Cartan}_g \]  

and \( H(\text{permutation}) \) acts among the copies Cartan \( g^I \) of Cartan \( g \).

Our second reducibility argument is based on the explicit form of the twisted left-mover vertex operators given in Ref. [13] for the case of the abelian permutation orbifolds:

\[ \hat{g}_+(T, z, \sigma) = \hat{\Gamma}(T, \hat{J}_0(0), \sigma) \prod \hat{V}_j(T, z, \sigma), \quad T(T, \sigma) = Tt(\sigma) \]  

(3.10a)
\[ \dot{V}_j(T, z, \sigma) \equiv z^{-\Delta(T)}(1 - \frac{1}{n} \delta_{j,\alpha})(T_{\alpha j}) e^{it_j(T_{\alpha j} \dot{\phi}^{\alpha j}(z) z^{\eta^{ab} T_{\alpha j} T_{\beta j}(T_{\alpha j}(0)) T_{\beta j}(0)/k_j})} \times \exp \left( -\frac{\eta^{ab} T_{ab}}{k_j} \frac{1}{f_j} \frac{f_j}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma) - 1} \sum_{m \leq -1} \dot{J}_{j j}(m + \frac{j}{f_j(\sigma)}) z^{-(m + j_{f_j(\sigma)})} m + \frac{j}{f_j(\sigma)} t_{j j}(\sigma) \right) \times \exp \left( -\frac{\eta^{ab} T_{ab}}{k_j} \frac{1}{f_j} \frac{1}{f_j(\sigma)} \sum_{m \geq 1} \dot{J}_{o a j}(m) z^{m} m t_{0 j}(\sigma) \right) \times \exp \left( -\frac{\eta^{ab} T_{ab}}{k_j} \frac{1}{f_j} \frac{1}{f_j(\sigma)} \sum_{j=1}^{f_j(\sigma) - 1} \sum_{m \geq 0} \dot{J}_{j j}(m + \frac{j}{f_j(\sigma)}) z^{-(m + j_{f_j(\sigma)})} m + \frac{j}{f_j(\sigma)} t_{j j}(\sigma) \right) \right) (3.10b) \]

\[ [\dot{q}^{a j}(\sigma), \dot{J}_{b l}(m + \frac{j}{f_j(\sigma)})] = i \delta_l^j \delta_{m}^0 \delta_{m+n}^0, \quad [\dot{q}^{a j}(\sigma), \dot{q}^{b l}(\sigma)] = 0 \quad (3.11a) \]

\[ [\dot{J}_{j j}(m + \frac{j}{f_j(\sigma)}), \dot{J}_{b l}(n + \frac{j}{f_j(\sigma)})] = \delta_{j l} k f_j(\sigma) \eta_{a b}(m + \frac{j}{f_j(\sigma)})(\sigma)^{m+n+j_{f_j(\sigma)}}, 0 \quad (3.11b) \]

\[ \dot{J}_{j j}(m + \frac{j}{f_j(\sigma)} \geq 0) |0\rangle_{\sigma} = (m + \frac{j}{f_j(\sigma)} \leq 0) = 0 \quad (3.11c) \]

\[ [T_a, T_b] = 0, \quad \frac{(\eta^{ab} T_{ab})_{\alpha \beta}}{2k} = \Delta(T) \delta_{\alpha \beta}, \quad \forall T \text{ for each } \sigma = 0, \ldots, N_c - 1 \quad (3.11d) \]

\[ a, b = 1, \ldots, \dim(\text{Cartan } g), \quad j = 0, \ldots, f_j(\sigma) - 1. \quad (3.11e) \]

Here (3.11b) is the abelian analogue of the general orbifold affine algebra (2.30a), and \( \hat{\Gamma}(T, \dot{J}_0(0), \sigma) \) in (3.10a) is the Klein transformation - which is a so-far undetermined function in this development.

As far as block structure of \( \hat{g}_+ \) is concerned, the only subtlety here is in the Klein transformation \( \hat{\Gamma} \). We know that the Klein transformation has a diagonal block structure, \( \hat{\Gamma} = \bigoplus_j \hat{\Gamma}_j \), due to its dependence on the matrices \( \{T_{j j}(\sigma)\} \) in \( T(T, \sigma) \). We will however also assume that \( \hat{\Gamma} \) is a function only of \( \{T_{0 a j}\} \) and moreover that \( \hat{\Gamma}_j \) is a function only of \( \dot{J}_{0 a j}(0) \), that is

\[ \hat{\Gamma}(T, \dot{J}_0(0), \sigma) \longrightarrow \hat{\Gamma}(T_0, \dot{J}_0(0), \sigma)_{j j}^{i i} = \delta_{i j} \hat{\Gamma}_j(T_{0 j}(j, \sigma), \dot{J}_{0 j}(0), \sigma)_{j j} \quad (3.12) \]
where the matrices $\tau_j^{\hat{o}}(j, \sigma)$ are defined in (3.2f). These assumptions are natural because the $\hat{q}$-dependent factor of the twisted vertex operator excites only the zero modes $\hat{J}_{0j}(0)$ of the twisted currents

$$\left(\hat{J}_{0j}(m + \frac{\hat{m}}{f_j(\sigma)} \geq 0) - \delta_{jl} \delta_{m,0} \delta_{,0} T_{0j}(T, \sigma)\right)\{e^{i \sum_k \hat{q}^k T_{0k}(T, \sigma)}|0\rangle_{\sigma}\} = 0 \quad (3.13)$$

by the amount $T_{0j}(T, \sigma)$.

Then we find with (3.12) that the twisted left-mover vertex operators (3.10) exhibit the same block structure and shift invariance

$$\hat{g}_j^+(T, z, \sigma)_{jj}^{\hat{u}} = \delta_{j}^{\hat{u}} \hat{g}_j^+(T, z, \sigma)_j^i \quad (3.14a)$$

$$\hat{g}_j^+(T, z, \sigma) = \hat{1}_j(T \tau_0(\sigma, \sigma), \hat{J}_{0j}(0), \sigma) \hat{V}_j(T \tau(j, \sigma), z, \sigma) \quad (3.14b)$$

$$\hat{V}_j(T \tau(j, \sigma), z, \sigma) \equiv \hat{V}_j(T t(\sigma), z, \sigma)|_{t(j, \sigma) \rightarrow -t(j, \sigma)} \quad (3.14c)$$

$$\hat{g}_j^+(T, z, \sigma)_{j+m}^{i+m} = \hat{g}_j^+(T, z, \sigma)_{j}^{i}, \quad \forall \hat{j}, \hat{l}, \hat{m} \quad (3.14d)$$

which were found for the group orbifold elements above. The same block structure and shift invariance are found for the twisted right-mover vertex operators $\hat{g}_-(T, \bar{z}, \sigma)$ given in Ref. [13], as well as for the twisted non-chiral vertex operator $\hat{g} = \hat{g}_- \hat{g}_+$.

The new feature here is the chiral block-diagonal structure, which will also generalize to the twisted affine primary fields of the full WZW permutation orbifolds. (Chiral block-diagonal structure can also be argued à la Witten [26] at the action level by considering the classical equations of motion of the group orbifold elements.)

Having completed our presentation of the classical and abelian evidence, we now give a derivation of reducibility in the general WZW permutation orbifold, using the principle of local isomorphisms.

**Derivation by Local Isomorphisms**

For this discussion we return (see Subsec. 2.1) to the untwisted permutation-invariant theory, where the untwisted affine primary fields are block-diagonal [12]

$$g(T, \bar{z}, z)_{\alpha l}^{\beta J} = \delta_l^J g_l(T, \bar{z}, z)_{\alpha}^{\beta} \quad (3.15a)$$

$$J_{aJ}(z)g_{aJ}(T, \bar{w}, w) = \delta_{\hat{a}J} g_{aJ}(T, \bar{w}, w)_{z-w} T_{a} + O(z - w)^0 \quad (3.15b)$$
\[ T_I(z)g_I(T, \bar{w}, w) = \delta_{IJ} \left( \frac{g_I(T, \bar{w}, w) \Delta_{g}(T)}{(z-w)^2} + \frac{\partial_w g_I(T, \bar{w}, w)}{z-w} \right) + O(z-w)^0 \quad (3.15c) \]

because the representation \( T \) is reducible (see Eq. (2.1c)). The blocks \( g_I \) are the affine primary fields of each simple affine \( g^I \). In parallel with the earlier relabellings in (2.19a), the blocks \( g_I \) are relabelled as \( \{\tilde{g}_{j}\} \) in the \( \sigma \)-dependent cycle basis [13]. Furthermore, it is known [25] that the untwisted affine primary fields of each copy of simple \( g \) can be factorized into left- and right-moving chiral components:

\[ g_I(T, \bar{z}, z, \sigma)_{\alpha}{}^{\beta} \longrightarrow \tilde{g}_{jj}(T, \bar{z}, z, \sigma)_{\alpha}{}^{\beta} \quad (3.16a) \]

\[ \tilde{g}_{jj}(T, \bar{z}, z, \sigma)_{\alpha}{}^{\beta} = g_{jj}^{-}(T, \bar{z}, z, \sigma)_{\alpha} \cdot g_{jj}^{+}(T, z, \sigma)_{\beta}, \quad \alpha, \beta = 1, \ldots, \text{dim} \, T. \quad (3.16b) \]

Here the dot commutes multiplication in the quantum group space, whose indices are suppressed; in what follows, we will often suppress the Lie indices \( \alpha, \beta \) as well.

We are now prepared to look for extra structure in the affine eigenprimary fields \( g_j \), defined earlier in Eq. (2.12c). Using (3.15a) in (2.12c) gives

\[ g(T, \bar{z}, z, \sigma)_{jj}^{\hat{\imath}} = U(\sigma)_{jj}^{\hat{\imath} \hat{\imath}} g_I(T, \bar{z}, z) U(\sigma)_{\hat{\imath} \hat{\imath}} \]

\[ = \sum_{m=0}^{f_{\alpha}(\sigma)} U(\sigma)_{jj}^{\hat{\imath} \hat{\imath}} g_m g_{\hat{\imath} \hat{\imath}} (T, \bar{z}, z, \sigma). \quad (3.17) \]

Then using the explicit form of \( U(\sigma) \) in (2.19d) we find after some algebra that the affine eigenprimary fields can be factorized into left- and right-mover chiral blocks:

\[ g_j(T, \bar{z}, z, \sigma)_{jj}^{\hat{\imath}} = \delta_{j}^{\hat{\imath}} g_j(T, \bar{z}, z, \sigma)_{j}^{\hat{\imath}} \quad (3.18a) \]

\[ g_j(T, \bar{z}, z, \sigma)_{j}^{\hat{\imath}} = \frac{1}{f_{\hat{\imath}}(\sigma)} \sum_{j=0}^{f_{\hat{\imath}}(\sigma)-1} e^{2\pi i j \hat{\imath}/f_{\hat{\imath}}(\sigma)} g_{jj}^{-}(T, \bar{z}, z, \sigma) \cdot (e^{-2\pi i j \hat{\imath}/f_{\hat{\imath}}(\sigma)} g_{jj}^{+}(T, z, \sigma)) \delta_{j - j \mod f_{\hat{\imath}}(\sigma)} \]

\[ = \sum_{\hat{\imath}=0}^{f_{\hat{\imath}}(\sigma)-1} g_{\hat{\imath}}^{-}(T, \bar{z}, z, \sigma)_{j}^{\hat{\imath}} \cdot g_{\hat{\imath}}^{+}(T, z, \sigma)_{\hat{\imath}} \quad (3.18b) \]

\[ g_j^{+}(T, z, \sigma)_{\bar{m}}^{\hat{\imath}} = \frac{1}{f_{\hat{\imath}}(\sigma)} \sum_{j=0}^{f_{\hat{\imath}}(\sigma)-1} e^{2\pi i (\bar{m} - j \hat{\imath})/f_{\hat{\imath}}(\sigma)} g_{\bar{m} + j}^{+}(T, z, \sigma) = g_{\hat{\imath}}^{+}(T, z, \sigma)_{j}^{\bar{m} + \hat{\imath}} \]

\[ g_j^{-}(T, \bar{z}, z, \sigma)_{\hat{\imath}} = \frac{1}{f_{\hat{\imath}}(\sigma)} \sum_{\hat{\imath}=0}^{f_{\hat{\imath}}(\sigma)-1} e^{2\pi i (\hat{\imath} - \bar{m} \hat{\imath})/f_{\hat{\imath}}(\sigma)} g_{\hat{\imath}}^{-}(T, \bar{z}, z, \sigma) = g_{\hat{\imath}}^{-}(T, \bar{z}, z, \sigma)_{\hat{\imath}}^{\bar{m} + \hat{\imath}}. \quad (3.18c) \]
Note that each of these blocks, $g_j$ as well as $g_j^\pm$, is a function only of the indicated difference variable.

Next, we apply the principle of local isomorphisms

$$g_j(T, \bar{z}, z, \sigma) \rightarrow \hat{g}_j(T, \bar{z}, z, \sigma)$$

$$g_j^\pm(T, z, \sigma) \rightarrow \hat{g}_j^\pm(T, z, \sigma), \quad g_j^-(T, \bar{z}, \sigma) \rightarrow \hat{g}_j^-(T, \bar{z}, \sigma)$$

which promotes the properties (3.18) of the affine eigenprimary fields to isomorphic properties of the twisted affine primary fields $\hat{g}, \hat{g}_j$ and $\hat{g}_j^\pm$:

$$\hat{g}_j(T, \bar{z}, z, \sigma)_{\alpha_j}^{\beta l} = \delta_{j}^{\beta} \hat{g}_j(T, \bar{z}, z, \sigma)_{\alpha_j}^{\beta l}, \quad \sigma = 0, \ldots, N_c - 1$$

$$\hat{g}_j^\pm(T, z, \sigma)_{\alpha_j}^{\beta l} = \sum_{\hat{m}=0}^{f_j(\sigma)-1} \hat{g}_j^-(T, \bar{z}, z, \sigma)_{\alpha_j}^{\hat{m}} \cdot \hat{g}_j^+(T, z, \sigma)_{\alpha_j}^{\hat{m}}$$

$$\hat{g}_j(T, \bar{z}, z, \sigma)_{\alpha_j}^{\beta_{\hat{m}+j}} = \hat{g}_j^+(T, z, \sigma)_{\alpha_j}^{\beta_{\hat{m}}}, \quad \hat{g}_j^-(T, \bar{z}, z, \sigma)_{\alpha_j}^{\hat{m}+l} = \hat{g}_j^-(T, \bar{z}, \sigma)_{\alpha_j}^{\hat{m}}$$

$$\hat{g}_j(T, \bar{z}, z, \sigma)_{\alpha_{j+\hat{m}}}^{\beta_{\hat{m}+l}} = \hat{g}_j(T, \bar{z}, z, \sigma)_{\alpha_j}^{\beta l}, \quad \bar{z}, \tilde{z}, \hat{m} = 0, \ldots, f_j(\sigma) - 1.$$ (3.20c)

We emphasize that the left-right orbifold factorization (3.20b) followed directly from left-right factorization in the symmetric theory and the principle of local isomorphisms, whereas only the consistency of such a factorization was checked in Ref. [12]. Note also that the quantum group space, denoted by the dot, is the same quantum group space we started with in the untwisted factorization (3.16b). We remind the reader that, in addition to the shift invariance (3.20c), all quantities are also periodic $\hat{j} \rightarrow \hat{j} \pm f_j(\sigma)$ in any spectral index.

Finally, we remark that, although the twisted affine primary fields of the WZW permutation orbifolds are generically reducible as above, there are special cases which are irreducible. This happens in sector $\sigma$ of the orbifold whenever $h_\sigma \in H(\text{permutation})$ is composed of a single cycle ($j = 0$ and $f_o(\sigma) = K$); for example, this phenomenon occurs in all twisted sectors of the $\mathbb{Z}_\lambda$ cyclic permutation orbifolds when $\lambda = \text{prime}$.

### 3.2 Reduction of the Extended Operator Algebra

The factorized and block-diagonal form (3.20) is consistent with the operator development in Sec.2, and will lead us to a simpler reduced dynamics.

To begin, we will substitute the block structure (3.20) into the OPEs and mode algebras of the WZW permutation orbifolds, and we will do this in two stages. First, we can use
(3.20a) to express all of our equations in terms of the unfactorized blocks \( \hat{g}_j(T, \bar{z}, z, \sigma) \) of the twisted affine primary fields. For example, the reduced results

\[
[\hat{J}_{ja_0}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l(T, \bar{z}, z, \sigma)] = \delta_{jl} \hat{g}_j(T, \bar{z}, z, \sigma) T_a \tau_j(j, \sigma) z^{(m + \hat{j}) \tau_j(\sigma)}
\] (3.21a)

\[
[\hat{J}_{ja_0}^s(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l(T, \bar{z}, z, \sigma)] = -\delta_{jl} \hat{z}^{(m + \hat{j}) \tau_j(\sigma)} T_a \tau_{-j}(j, \sigma) \hat{g}_j(T, \bar{z}, z, \sigma)
\] (3.21b)

\[
\partial \hat{g}_j(T, \bar{z}, z, \sigma) = \frac{2\eta^{ab}}{2k + Q_g f_j(\sigma)} \frac{1}{f_j(\sigma)^{-1}} \sum_{j=0}^{f_j(\sigma)-1} : \hat{J}_{ja_0}(z) \hat{g}_j(T, \bar{z}, z, \sigma) : M T_b \tau_j(j, \sigma) - \frac{\Delta_g(T)}{z} (1 - \frac{1}{f_j(\sigma)}) \hat{g}_j(T, \bar{z}, z, \sigma)
\] (3.21c)

follow from their unreduced counterparts in (2.30d), (2.55a) and (2.28). We note that all the reduced relations of this type can be easily obtained by the mnemonic

\[
\hat{g} \longrightarrow \hat{g}_j, \quad \hat{g}_\pm \longrightarrow \hat{g}_j^\pm, \quad t_{j_1}(\sigma) \longrightarrow \tau_j(j, \sigma), \quad t_{0j}(\sigma) \longrightarrow \tau_0(j, \sigma) = \mathbb{1}
\] (3.22)

from their unreduced counterparts. The matrices \( \{\tau_j(j, \sigma)\} \) are defined in (3.2f).

Next, we can use (3.20b) to factorize all the reduced OPEs and mode algebras into relations on the left- and right-mover blocks \( \hat{g}_j^\pm \) of the twisted affine primary fields. We give here only the reduced and factorized form of the extended operator algebra:

\[
[\hat{J}_{ja_0}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l^+(T, z, \sigma)] = \delta_{jl} \hat{g}_j^+(T, z, \sigma) T_a \tau_j(j, \sigma) z^{m + \hat{j} \tau_j(\sigma)}
\] (3.23a)

\[
[\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l^+(T, z, \sigma)] =
= \delta_{jj} \hat{g}_j^+(T, z, \sigma) \left( \frac{\hat{j}}{f_j(\sigma)} z + (m + \frac{\hat{j}}{f_j(\sigma)} + 1) \Delta_g(T) \right) \tau_j(j, \sigma) z^{m + \hat{j} \tau_j(\sigma)}
\] (3.23b)

\[
[L_\sigma(m), \hat{g}_j^+(T, z, \sigma)] = \hat{g}_j^+(T, z, \sigma) \left( \frac{\hat{j}}{f_j(\sigma)} z + (m + 1) \Delta_g(T) \right) z^m
\] (3.23c)

\[
[\hat{J}_{ja_0}^s(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l^-(T, \bar{z}, \sigma)] = -\delta_{jl} \hat{z}^{m + \hat{j} \tau_j(\sigma)} T_a \tau_{-j}(j, \sigma) \hat{g}_j^-(T, \bar{z}, \sigma)
\] (3.23d)
\[
\left[ \hat{L}^\pm_j(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}^- (T, \bar{z}, \sigma) \right] = \\
= \delta_{jl} \bar{z}^{m+\frac{\hat{j}}{f_j(\sigma)}} \left( \bar{z} \partial + (m + \frac{\hat{j}}{f_j(\sigma)} + 1) \Delta_\theta(T) \right) \tau_{-j}(j, \sigma) \hat{g}^- (T, \bar{z}, \sigma) \tag{3.23e}
\]

\[
\left[ \hat{L}_\sigma(m), \hat{g}^- (T, \bar{z}, \sigma) \right] = \bar{z}^{m} (\bar{z} \partial + (m + 1) \Delta_\theta(T)) \hat{g}^- (T, \bar{z}, \sigma) \tag{3.23f}
\]

\[
\left[ \hat{L}, \hat{g}^- \right] = [\hat{J}, \hat{g}^-] = 0, \quad [\hat{J}^\#, \hat{g}^-] = [\hat{J}^\#, \hat{g}^+] = 0. \tag{3.23g}
\]

Similarly, we obtain the reduced twisted vertex operator equations

\[
\partial \hat{g}^+_j(T, z, \sigma) = \frac{2 \eta^{ab}}{2k + Q_\theta f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} : \hat{J}_{j,0} (z) \hat{g}^+_j(T, z, \sigma) : M \quad T_b \tau_{-j}(j, \sigma)
\]

\[
- \frac{\Delta_\theta(T)}{z} (1 - \frac{1}{f_j(\sigma)}) \hat{g}^+_j(T, z, \sigma) \tag{3.24a}
\]

\[
\bar{\partial} \hat{g}^-_j(T, \bar{z}, \sigma) = - \frac{2 \eta^{ab}}{2k + Q_\theta f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} T_b \tau_j(j, \sigma) : \hat{J}_{j,0} (\bar{z}) \hat{g}^-_j(T, \bar{z}, \sigma) : M
\]

\[
- \frac{\Delta_\theta(T)}{z} (1 - \frac{1}{f_j(\sigma)}) \hat{g}^-_j(T, \bar{z}, \sigma) \tag{3.24b}
\]

for the left- and right-mover blocks of the twisted affine primary fields.

### 3.3 The Single-Cycle Form of the Twisted KZ System

Using Eq. (3.20) and the twisted KZ system (2.46), (2.59), the same procedure leads after some algebra to the reduced and factorized form of the twisted KZ system:

\[
\partial_\mu \hat{A}_+(\vec{j}; T, z, \sigma) = \hat{A}_+(\vec{j}; T, z, \sigma) \hat{W}_\mu(\vec{j}; T, z, \sigma) \tag{3.25a}
\]

\[
\bar{\partial}_\mu \hat{A}_-(\vec{j}; T, \bar{z}, \sigma) = \hat{W}_\mu(\vec{j}; T, \bar{z}, \sigma) \hat{A}_-(\vec{j}; T, \bar{z}, \sigma) \tag{3.25b}
\]

\[
\hat{W}_\mu(\vec{j}; T, z, \sigma) \equiv \frac{2}{2k + Q_\theta} \sum_{\nu \neq \mu} \eta^{ab} T^{(\mu)}_a T^{(\nu)}_b \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{z_\nu}{z_\mu} \right) f_j^{\sigma(\nu)}(\vec{j}, \sigma) \tau^{(\mu)}_j(j, \sigma) \tau^{(\nu)}_{-j}(j, \sigma) \delta_{j,0} \delta_\nu \]

\[
- \frac{\Delta_\theta(T^{(\mu)})}{z_\mu} (1 - \frac{1}{f_j(\sigma)}) \tag{3.25c}
\]

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\[
\hat{W}_\mu (\vec{g}; T, \vec{z}, \sigma) = \frac{2}{2k + Q_\theta} \sum_{\nu \neq \mu} \eta^{ab} T_a^{(\nu)} T_b^{(\mu)} f_{j_\nu} (\sigma) z_{\mu\nu} \sum_{j=0}^N \left( \frac{z_{\nu}}{\bar{z}_{\mu}} \right) T_j^{(\sigma)} (j_\mu, \sigma) T_j^{(\nu)} (j_\nu, \sigma) \delta_{j_\mu, j_\nu} - \Delta_\theta (T^{(\nu)}) (1 - \frac{1}{f_{j_\nu} (\sigma)})
\]

(3.25d)

\[
\hat{A}_+(\vec{g}; T, z, \sigma) \left( \sum_{\mu=1}^N T^{(\mu)}_a \delta_{j_\mu} \right) = \left( \sum_{\mu=0}^N T^{(\mu)}_a \delta_{j_\mu} \right) \hat{A}_-(\vec{g}; T, \bar{z}, \sigma) = 0, \quad \forall j.
\]

(3.25e)

Here the reduced left- and right-mover correlators \( \hat{A}_\pm \) are defined as

\[
\hat{A}_+(\vec{g}; T, z, \sigma) \equiv \langle 0 | \hat{g}^{+}_1 (T^{(1)}, z_1, \sigma), \ldots \hat{g}^{+}_N (T^{(N)}, z_N, \sigma) | 0 \rangle_{\sigma}
\]

\[
\hat{A}_-(\vec{g}; T, \bar{z}, \sigma) \equiv \langle 0 | \hat{g}^{-}_1 (T^{(1)}, \bar{z}_1, \sigma), \ldots \hat{g}^{-}_N (T^{(N)}, \bar{z}_N, \sigma) | 0 \rangle_{\sigma}
\]

(3.26a)

\[
\hat{A}(\vec{g}; T, \bar{z}, z, \sigma) \equiv \langle \hat{g}_1 (T^{(1)}, \bar{z}_1, z_1, \sigma), \ldots \hat{g}_N (T^{(N)}, \bar{z}_N, z_N, \sigma) \rangle_{\sigma}
\]

\[
\hat{A}(\vec{g}; T, \bar{z}, z, \sigma) = \hat{A}_-(\vec{g}; T, \bar{z}, \sigma) \cdot \hat{A}_+(\vec{g}; T, z, \sigma)
\]

(3.26b)

\[
\hat{A}(T, \bar{z}, z, \sigma)^{\beta_{1} l_{1} \ldots ; \beta_{N} l_{N}}_{\alpha_{1} j_{1} ; \ldots ; \alpha_{N} j_{N}} \equiv \delta^{\beta_{1} l_{1} \ldots ; \beta_{N} l_{N}}_{\alpha_{1} j_{1} ; \ldots ; \alpha_{N} j_{N}} \hat{A}(\vec{g}; T, \bar{z}, z, \sigma)
\]

(3.26c)

where \( \hat{A}(T, \bar{z}, z, \sigma) \) in (3.26c) is the full, non-chiral unreduced correlator defined earlier in (2.42).

The Kronecker factors \( \delta_{j_\mu, j_\nu} \) in the reduced connections \( \hat{W}_\mu (\vec{g}), \hat{W}_\mu (\vec{j}) \) and the reduced global Ward identity (3.25e) tell us that the reduced affine primary fields \( \hat{g}_j \) as well as \( \hat{g}_j^\pm \) decouple for distinct values of \( j \). This allows us to choose the ‘natural’ solution in which the correlators factorize according to distinct cycles \( j \), as in the functional integral formulation above. As examples, we note that the factorized forms

\[
\langle \hat{g}^+_j (T^{(1)}, z_1, \sigma) \hat{g}^+_l (T^{(2)}, z_2, \sigma) \rangle_{\sigma} = \langle \hat{g}^+_j (T^{(1)}, z_1, \sigma) \rangle_{\sigma} \langle \hat{g}^+_l (T^{(2)}, z_2, \sigma) \rangle_{\sigma}, \quad j \neq l
\]

(3.27a)

\[
\langle \hat{g}^+_j (T^{(1)}, z_1, \sigma) \hat{g}^+_l (T^{(2)}, z_2, \sigma) \hat{g}^+_j (T^{(3)}, z_3, \sigma) \hat{g}^+_l (T^{(4)}, z_4, \sigma) \rangle_{\sigma} = \langle \hat{g}^+_j (T^{(1)}, z_1, \sigma) \hat{g}^+_l (T^{(2)}, z_2, \sigma) \hat{g}^+_j (T^{(3)}, z_3, \sigma) \rangle_{\sigma} \times \langle \hat{g}^+_l (T^{(4)}, z_4, \sigma) \rangle_{\sigma}, \quad j \neq l
\]

(3.27b)

are solutions of the reduced twisted KZ system - in parallel with (3.5). Such relations hold as well for \( \hat{g}^-_j \rightarrow \hat{g}^-_j \) and \( \hat{g}^+_j \rightarrow \hat{g}^-_j \). Moreover, it is known [12] that all the one-point correlators vanish for non-trivial \( T \) so that, e.g.

\[
\langle \hat{g} (T, \bar{z}, z, \sigma) \rangle_{\sigma} = \langle \hat{g}_j (T, \bar{z}, z, \sigma) \rangle_{\sigma} = 0, \quad T \neq 0
\]

(3.28a)
\[ \langle \hat{g}_j (T^{(1)}), \hat{g}_l (T^{(2)}), \bar{z}_1, z_1, \sigma \rangle = \delta_{jl} \langle \hat{g}_j (T^{(1)}), \hat{g}_j (T^{(2)}), \bar{z}_2, z_2, \sigma \rangle \]

where (3.28b) follows from (3.27a) and (3.28a). Similarly, one finds that \( \langle \hat{g}_j \hat{g}_k \hat{g}_l \rangle \) vanishes unless \( j = k = l \).

In the natural solution we therefore need only consider the single-cycle correlators at fixed values of \( j \)

\[ \hat{A}_+(j; T (T, \sigma), z, \sigma) \equiv \langle \hat{g}_j^+ (T^{(1)}), z_1, \sigma \rangle \ldots \hat{g}_j^+ (T^{(N)}), z_N, \sigma \rangle \]

\[ \hat{A}_-(j; T (T, \sigma), \bar{z}, \sigma) \equiv \langle \hat{g}_j^- (T^{(1)}), \bar{z}_1, \sigma \rangle \ldots \hat{g}_j^- (T^{(N)}), \bar{z}_N, \sigma \rangle \]

\[ \forall j, \forall T \text{ for each } \sigma = 0, \ldots, N_c - 1 \]

from which all other correlators can be constructed, as in (3.27). Moreover, we find from (3.25) that the single-cycle correlators satisfy the single-cycle twisted KZ system

\[ \partial_\mu \hat{A}_+(j; T (T, \sigma), z, \sigma) = \hat{A}_+(j; T (T, \sigma), z, \sigma) \hat{W}_\mu (j; T (T, \sigma), z, \sigma) \]

\[ \partial_\bar{\sigma} \hat{A}_-(j; T (T, \sigma), \bar{z}, \sigma) = \hat{W}_\mu (j; T (T, \sigma), \bar{z}, \sigma) \hat{A}_-(j; T (T, \sigma), \bar{z}, \sigma) \]

\[ \hat{W}_\mu (j; T (T, \sigma), z, \sigma) = \frac{2}{2k + Q_0} \sum_{\nu \neq \mu} \eta_{ab} T_a^{(\mu)} T_b^{(\nu)} \sum_{j=0}^{f_j (\sigma)-1} \left( \frac{z_\nu}{z_\mu} \right)^{\frac{1}{2}} \tau_j^{(\nu)} (j, \sigma) \tau_{-j}^{(\mu)} (j, \sigma) \]

\[ -\frac{\Delta_q (T^{(\mu)})}{z_\mu} (1 - \frac{1}{f_j (\sigma)}) \]

\[ \hat{W}_\mu (j; T (T, \sigma), \bar{z}, \sigma) = \frac{2}{2k + Q_0} \sum_{\nu \neq \mu} \eta_{ab} T_a^{(\nu)} T_b^{(\mu)} \sum_{j=0}^{f_j (\sigma)-1} \left( \frac{\bar{z}_\nu}{\bar{z}_\mu} \right)^{\frac{1}{2}} \tau_j^{(\sigma)} \tau_j^{(\nu)} (j, \sigma) \tau_{-j}^{(\mu)} (j, \sigma) \]

\[ -\frac{\Delta_q (T^{(\mu)})}{\bar{z}_\mu} (1 - \frac{1}{f_j (\sigma)}) \]

\[ \hat{A}_+(j; T (T, \sigma), z, \sigma) \left( \sum_{\mu=1}^{N} T_a^{(\mu)} \right) = \left( \sum_{\mu=1}^{N} T_a^{(\mu)} \right) \hat{A}_-(j; T (T, \sigma), \bar{z}, \sigma) = 0, \forall j \]

which operate entirely within each disjoint cycle. Note in particular that the single-cycle global Ward identity in (3.30d) has the same form as the usual global Ward identity [23] found in untwisted KZ systems.

The single-cycle twisted KZ system (3.30), (3.20c) is one of the central results of this paper.
As a first application of (3.30), we note that the single-cycle forms of the orbifold \( \mathfrak{sl}(2) \) Ward identities

\[
\hat{A}_+(j; T, \bar{z}, z, \sigma) \sum_{\mu=1}^{N} \left( \partial_\mu z_\mu + \Delta_\sigma(T^{(\mu)}) \right) = 0 \tag{3.31a}
\]

\[
\hat{A}_+(j; T, \bar{z}, z, \sigma) \sum_{\mu=1}^{N} \left[ \left( \partial_\mu z_\mu + (1 + \frac{1}{f_j(\sigma)}) \Delta_\sigma(T^{(\mu)}) \right) \tau_1^{(j, \sigma)}(j, \sigma) z_\mu \right] = 0 \tag{3.31b}
\]

\[
\hat{A}_+(j; T, \bar{z}, z, \sigma) \sum_{\mu=1}^{N} \left[ \left( \partial_\mu z_\mu + (1 - \frac{1}{f_j(\sigma)}) \Delta_\sigma(T^{(\mu)}) \right) \tau_{-1}^{(j, \sigma)}(j, \sigma) z_\mu \right] = 0 \tag{3.31c}
\]

\[
\sum_{\mu=1}^{N} \left( \bar{z}_\mu \partial_\mu \Delta_\sigma(T^{(\mu)}) \right) \hat{A}_-(j; T, \bar{z}, z, \sigma) = 0 \tag{3.32a}
\]

\[
\sum_{\mu=1}^{N} \left( \bar{z}_\mu \partial_\mu \Delta_\sigma(T^{(\mu)}) \right) \hat{A}_-(j; T, \bar{z}, z, \sigma) = 0 \tag{3.32b}
\]

\[
\sum_{\mu=1}^{N} \left( \bar{z}_\mu \partial_\mu \Delta_\sigma(T^{(\mu)}) \right) \hat{A}_-(j; T, \bar{z}, z, \sigma) = 0 \tag{3.32c}
\]

are also satisfied by any solution of the single-cycle twisted KZ system.

### 3.4 First Form of the Single-Cycle Two-Point Correlator

The unreduced left-mover two-point correlator was given for the case of the WZW permutation orbifolds in Refs. [12, 13]. Similarly, using the shift condition (3.20c) and the global Ward identity (3.30d), we have worked out the single-cycle non-chiral two-point correlator

\[
\hat{A}(j; 1, 2) = \hat{A}_-(j; 1, 2) \cdot \hat{A}_+(j; 1, 2) = \langle \hat{g}_j(T^{(1)}(T^{(1)}), \bar{z}_1, z_1, \sigma) \hat{g}_j(T^{(2)}(T^{(2)}, \sigma), \bar{z}_2, z_2, \sigma) \rangle_{\sigma}
\]

\[
= \mathcal{C}(j; T, \sigma) |z_1|^{-2\Delta_\sigma(T^{(1)})} |z_2|^{-2\Delta_\sigma(T^{(2)})} \times \exp \left\{ \frac{2}{2k + Q_\sigma} F(j; 1, 2) \right\} \tag{3.33a}
\]
\[ F(j;1,2) \equiv \frac{T_a^{(2)} \eta^{ab} T_b^{(1)}}{f_j(\sigma)} \sum_{j=1}^{f_j(\sigma)-1} \{ \tau_j^{(1)}(j,\sigma) \tau_{-j}^{(2)}(j,\sigma) I_{\frac{j}{f_j(\sigma)}}(\frac{z_1}{z_2},\infty) \} \]

\[ + \tau_j^{(2)}(j,\sigma) \tau_{-j}^{(1)}(j,\sigma) I_{\frac{j}{f_j(\sigma)}}(\frac{z_1}{z_2},\infty) \} \]  

\[ (3.33b) \]

\[ C(j;\mathcal{T},\sigma) = C_-(j;\mathcal{T},\sigma) \cdot C_+(j;\mathcal{T},\sigma), \quad I_{\frac{j}{f_j(\sigma)}}(y,\infty) \equiv \int_y^{\infty} \frac{dx}{x-1} x^{-\frac{j}{f_j(\sigma)}} \]  

\[ (3.33c) \]

as the solution to the single-cycle twisted KZ equations. After some algebra, we have verified that this solution satisfies the orbifold \(\mathfrak{sl}(2)\) Ward identities \((3.31)\) and \((3.32)\), as it must.

We explain some of the steps used in obtaining \((3.33)\), emphasizing properties of the constant matrix \(\mathcal{C}(j;\mathcal{T},\sigma)\). In the first place, the blocks \(\hat{g}_j\) are functions of difference variables, and hence the constant matrices \(\mathcal{C}_\pm, \mathcal{C}\) are also functions of their respective difference variables

\[ O(j;\mathcal{T},\sigma)_{j_1;j_2}^{l_1+m_1;l_2} = O(j;\mathcal{T},\sigma)_{j_1+j_2;m_1;l_2} = O(j;\mathcal{T},\sigma)_{j_1;j_2}^{l_1;l_2} \]  

\[ (3.34a) \]

\[ O(j;\mathcal{T},\sigma) = \mathcal{C}_\pm(j;\mathcal{T},\sigma) \text{ or } \mathcal{C}(j;\mathcal{T},\sigma) \]  

\[ (3.34b) \]

where we have suppressed Lie indices and quantum group indices. Moreover, the global Ward identity places further constraints on the matrix \(\mathcal{C}\), namely

\[ [T_c^{(2)} \eta^{cd} T_d^{(1)}, (T_a^{(1)} + T_a^{(2)})] = 0 \]  

\[ (3.35a) \]

\[ \mathcal{C}(j;\mathcal{T},\sigma)(T_a^{(1)} + T_a^{(2)}) = (T_a^{(1)} + T_a^{(2)})\mathcal{C}(j;\mathcal{T},\sigma) = 0. \]  

\[ (3.35b) \]

Taken together, the conditions \((3.34)\) and \((3.35b)\) imply that

\[ [\mathcal{C}(j;\mathcal{T},\sigma), T_a^{(2)} \eta^{ab} T_b^{(1)}] = 0, \quad [\mathcal{C}(j;\mathcal{T},\sigma), F(j;1,2)] = 0 \]  

\[ (3.36a) \]

\[ \exp \left\{ \frac{2}{2k + Q_\theta} \frac{T_a^{(2)} \eta^{ab} T_b^{(1)}}{f_j(\sigma)} \sum_{j=1}^{f_j(\sigma)-1} \{ \tau_j^{(1)}(j,\sigma) \tau_{-j}^{(2)}(j,\sigma) I_{\frac{j}{f_j(\sigma)}}(\frac{z_1}{z_2},\infty) \} \times \right\} \]

\[ \times \mathcal{C}(j;\mathcal{T},\sigma) \exp \left\{ \frac{2}{2k + Q_\theta} \frac{T_a^{(2)} \eta^{ab} T_b^{(1)}}{f_j(\sigma)} \sum_{j=1}^{f_j(\sigma)-1} \{ \tau_j^{(2)}(j,\sigma) \tau_{-j}^{(1)}(j,\sigma) I_{\frac{j}{f_j(\sigma)}}(\frac{z_1}{z_2},\infty) \} \right\} \]

\[ = \mathcal{C}(j;\mathcal{T},\sigma) \exp \left\{ \frac{2}{2k + Q_\theta} F(j;1,2) \right\} \]  

\[ (3.36b) \]
which allowed us to pull the matrix $C$ to the left in the result (3.33).

Furthermore, the complete solution of the global Ward identity (3.35b) gives a form which is equivalent to the standard Haar integration over two Lie group elements, and we find that the the matrix $C(j;\,T,\,\sigma)$ has the following form

$$C(j;\,T,\,\sigma)_{\alpha_1\beta_1;\alpha_2\beta_2}^{i_1:\beta_1 i_2:\beta_2} = \delta_{T(2),\bar{T}(1)}\delta_{\alpha_1\alpha_2}\delta_{\beta_1\beta_2}D(j;\,T,\,\sigma)^i_{i_1 i_2}$$  \hspace{1cm} (3.37a)

$$D(j;\,T,\,\sigma)^i_{i_1 i_2} = D(j;\,T,\,\sigma)^{i_1 i_2}_{i_1 i_2} = D(j;\,T,\,\sigma)^{i_1 i_2}_{i_1 i_2}$$  \hspace{1cm} (3.37b)

in terms of an as-yet-undetermined constant matrix $D$. Finally, the single-cycle two-point correlator is non-vanishing only when $T \equiv T^{(2)} = \bar{T}^{(1)}$, so the untwisted conformal weights in (3.33a) are equal. This gives the first form of the single-cycle two-point correlator

$$\hat{A}(j;\,1,\,2) = C(j;\,T,\,\sigma)|z_1 z_2\rangle^{-2\Delta_g(T(1) - 1)}|z_{12}|^{-4\Delta_g((T) / f_j(\sigma))} \exp\left\{\frac{2}{2k + Q_g} F(j;\,1,\,2)\right\}$$  \hspace{1cm} (3.38a)

$$C(j;\,T,\,\sigma) = \delta_{T(2),\bar{T}(1)}D(j;\,T,\,\sigma), \quad \Delta_g(T) \equiv \frac{\eta^{ab}T_a T_b}{2k + Q_g} = \Delta_g(T^{(1)}) = \Delta_g(T^{(2)})$$  \hspace{1cm} (3.38b)

where $F(j;\,1,\,2)$ is defined in (3.33). We will return in Subsec. 4.4 to fix the multiplicative constant matrix $D$ in (3.38b).

4 Twisted Affine Primary and Principal Primary States

4.1 The Twisted Affine Primary States of Cycle $j$

The object of this subsection is to obtain the orbifold-modified asymptotic formulae [1] for the in- and out-states created by the reduced twisted affine primary fields, and to show that in fact these states are twisted affine primary states, as one might expect. The twisted affine primary states are also seen to be primary under the orbifold Virasoro algebra.

To obtain some information about excited states in the orbifold, we first consider the following limits of the single-cycle two-point correlator in (3.38),

$$I_{\frac{f_j(\sigma)}{f_j(\sigma)}}(y, \infty) = O(y^{-\frac{1}{f_j(\sigma)}}) \quad \text{for } y >> 1, \, \hat{j} = 1, \ldots, f_j(\sigma) - 1$$  \hspace{1cm} (4.1)

$$\lim_{|z_2| \rightarrow 0} |z_2|^{2\Delta_g(T(1) - 1 - \frac{1}{f_j(\sigma)}) \sigma} \langle 0| \hat{g}_j(T^{(1)}, \bar{z}_1, z_1, \sigma) \hat{g}_j(T^{(2)}, \bar{z}_2, z_2, \sigma)|0\rangle_\sigma = C(j;\,T,\,\sigma)|z_1|^{-2\Delta_g(T(1) - 1 + \frac{1}{f_j(\sigma)})}$$  \hspace{1cm} (4.2a)
where the form of the constant matrix \( C(\sigma) \) is given in (3.37a). These limits define the ‘in’ and ‘out’ states

\[
\lim_{|z_1| \to \infty} \lim_{|z_2| \to 0} |z_1|^2 \Delta g(T(1+ \alpha^\tau_j(\sigma))) |z_2|^2 \Delta g(T(1- \alpha^\tau_j(\sigma))) \langle 0 | \hat{g}_j^{(1)}(T, z_1, \sigma) \hat{g}_j^{(2)}(T, z_2, \sigma) |0 \rangle = C(j; T, \sigma) \tag{4.2b}
\]

created by the twisted affine primary field \( \hat{g}_j(T, z, \sigma) \) on the ground state (scalar twist-field state) of sector \( \sigma \). In fact, these states are matrix states

\[
(A(j; T)|_\sigma)_{\alpha_j}^{\beta_i}, \quad (\sigma A(j; T))_{\alpha_j}^{\beta_i}
\]

but we will generally suppress their indices. The relations in (4.2b) and (4.3) tell us that the constant matrix \( C(j; T, \sigma) \) is the inner product of these states

\[
\langle A(j; T^{(1)}) | A(j; T^{(2)}) \rangle = C(j; T, \sigma)
\]

and, according to (3.37a), both sides of this relation are proportional to \( \delta_{T^{(2)}, T^{(1)}} \).

We can similarly define chiral and antichiral states, created by the left- and right-mover twisted affine primary fields \( \hat{g}_j^\pm \):

\[
| A^+(j; T) \rangle = \lim_{|z| \to 0} z^{\Delta g(T)(1+ \alpha^\tau_j(\sigma))} \hat{g}_j^+(T, z, \sigma) |0 \rangle \tag{4.6a}
\]

\[
\sigma A^+(j; T) \equiv \lim_{|z| \to \infty} \sigma |0 \rangle z^{\Delta g(T)(1+ \alpha^\tau_j(\sigma))} \hat{g}_j^+(T, z, \sigma) \tag{4.6b}
\]

\[
| A^-(j; T) \rangle = \lim_{|z| \to 0} z^{\Delta g(T)(1- \alpha^\tau_j(\sigma))} \hat{g}_j^-(T, z, \sigma) |0 \rangle \tag{4.6c}
\]

\[
\sigma A^-(j; T) \equiv \lim_{|z| \to \infty} \sigma |0 \rangle z^{\Delta g(T)(1- \alpha^\tau_j(\sigma))} \hat{g}_j^-(T, z, \sigma) \tag{4.6d}
\]

\[
| A(j; T) \rangle = | A^-(j; T) \rangle \sigma \cdot | A^+(j; T) \rangle \sigma, \quad \sigma | A(j; T) \rangle = \sigma A^-(j; T) \cdot \sigma A^+(j; T). \tag{4.6e}
\]

Note that the orbifold-modified asymptotic formulae (4.6) reduce to the conventional asymptotic formulae for Virasoro primary fields and states only when the cycle size \( f_j(\sigma) = 1 \),
which includes the untwisted sector $\sigma = 0$. These orbifold modifications can be traced back to the $z_\mu^{-1}$ terms in the single-cycle twisted KZ connections (3.30), and these terms in turn reflect the translation non-invariance of the scalar twist-field state. Such modified asymptotic formulae were first seen in Ref. [1].

In what follows, we discuss some important properties of the states in (4.6). For brevity, we focus on the left-mover ‘in’ state in (4.6a), but similar properties are easily established for all the states in (4.6).

To simplify the following arguments, we will define rescaled affine primary fields

$$\hat{\phi}_j^+(T(T,\sigma), z, \sigma) \equiv z^{\Delta_\sigma(T)}(1 - \frac{1}{f_j(\sigma)}) \hat{\phi}_j^+(T(T,\sigma), z, \sigma), \quad |A^+(j; T)_\sigma = \lim_{z \to 0} \hat{\phi}_j^+(T, z, \sigma)|0\rangle_\sigma \quad (4.7)$$

and we observe that rescaling does not remove all orbifold modification of the asymptotic formulae; for example, the formula

$$\sigma \langle A^+(j; T) | = \lim_{z \to \infty} \sigma \langle 0| z^{2\Delta_\sigma(T)/f_j(\sigma)} \hat{\phi}_j^+(T, z, \sigma) \quad (4.8)$$

agrees with the conventional form only when $f_j(\sigma) = 1$.

Our next observation is that the asymptotic state $|A^+(j; T)_\sigma$ created by the left-mover twisted affine primary field is indeed a twisted affine primary state\(^{11}\):

$$[\hat{J}_{a\sigma}(m + \frac{j}{f_j(\sigma)}), \hat{\phi}_l^+(T, z, \sigma)] = \delta_{jl} z^{(m+\frac{j}{f_j(\sigma)})} \hat{\phi}_l^+(T, z, \sigma)T_a \tau_j(j, \sigma) \quad (4.9a)$$

$$\hat{J}_{a\sigma}(m + \frac{j}{f_j(\sigma)} \geq 0) |A^+(l; T)_\sigma = \lim_{z \to 0} \hat{J}_{a\sigma}(m + \frac{j}{f_j(\sigma)} \geq 0) \hat{\phi}_l^+(T, z, \sigma)|0\rangle_\sigma$$

$$= \lim_{z \to 0} |J_{a\sigma}(m + \frac{j}{f_j(\sigma)} \geq 0), \hat{\phi}_l^+(T, z, \sigma)|0\rangle_\sigma$$

$$= \lim_{z \to 0} \delta_{jl} z^{(m+\frac{j}{f_j(\sigma)})} \hat{\phi}_l^+(T, z, \sigma)T_a \tau_j(j, \sigma)|0\rangle_\sigma$$

$$= \delta_{jl} \delta_{m+\frac{j}{f_j(\sigma)}0} |A^+(j; T)_\sigma T_a. \quad (4.9b)$$

To obtain this result, we used the ground state conditions (2.37a) and the algebra (3.23a).

Moreover, these twisted affine primary states are primary under the orbifold Virasoro

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\(^{11}\)Note that the scalar twist-field state $|0\rangle_\sigma$ can also be considered as a twisted affine primary state with $T = 0$. 

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algebra

\[ [\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{g}_l(T, z, \sigma)] = \delta_{jl} \hat{g}_l(T, z, \sigma) \left( -\partial z + (m + \frac{\hat{j} + 1}{f_j(\sigma)}) \Delta_g(T) \right) T_j(j, \sigma) z^{m + \frac{\hat{j}}{f_j(\sigma)}} \]  

(4.10a)

\[ \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0) A^+(l; T) \rangle_\sigma = \lim_{z \to 0} \hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0) \hat{g}_l(T, z, \sigma) |0\rangle_\sigma = \]

\[ \lim_{z \to 0} \left( [\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0), \hat{g}_l(T, z, \sigma)] + \delta_m \delta_f \hat{g}_l(T, z, \sigma) \right) |0\rangle_\sigma = \]

\[ = \lim_{z \to 0} \hat{g}_l(T, z, \sigma) \left( \delta_m \delta_f (\partial z + (m + \frac{\hat{j} + 1}{f_j(\sigma)}) \Delta_g(T)) T_j(j, \sigma) z^{m + \frac{\hat{j}}{f_j(\sigma)}} + \delta_m \delta_f \hat{g}_l(T, z, \sigma) \right) |0\rangle_\sigma = \]

\[ = \delta_m \delta_f \hat{g}_l(T, z, \sigma) + \delta_m \delta_f \frac{\Delta_g(T)}{f_j(\sigma)} |A^+(l; T)\rangle_\sigma \]  

(4.10b)

where we have used the ground state relation (2.39b) and the commutator (3.23b). The partial conformal weight \( \hat{\Delta}_0(\sigma) \) is given in (2.39d).

Finally, Eq. (4.10b) tells us that the twisted affine primary state is also a Virasoro primary state under the full Virasoro generators

\[ L_0(m \geq 0) |A^+(j; T)\rangle_\sigma = \delta_m \delta_f (\hat{\Delta}_0(\sigma) + \frac{\Delta_g(T)}{f_j(\sigma)}) |A^+(j; T)\rangle_\sigma \]  

(4.11)

where the ground state conformal weight \( \hat{\Delta}_0(\sigma) \) is given in (2.38d).

The corresponding relations on the left-mover ‘out’ state

\[ \langle A^+(l; T) \rangle |\hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)} \leq 0) = \delta_{ml} \delta_{f_j(\sigma)} |\langle A^+(j; T) \rangle T_a \]  

(4.12a)

\[ = \delta_m \delta_f \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)} \leq 0) (\hat{\Delta}_0(\sigma) + \frac{\Delta_g(T)}{f_j(\sigma)}) \langle A^+(l; T) \rangle \]  

(4.12b)

are the adjoint of the relations above. Similarly, the right-mover ‘in’ state in (4.6c) is primary under the rectified right-mover currents and the rectified right-mover orbifold Virasoro generators

\[ \hat{J}_{ja}(m + \frac{\hat{j}}{f_j(\sigma)} \geq 0) |A^-(j; T)\rangle_\sigma = -\delta_m \delta_{f_j(\sigma)} |\langle A^-(j; T) \rangle T_a \]  

(4.13a)
\[
\hat{L}_{jj}^\dagger (m + \frac{\hat{j}}{f_j(\sigma)}) \geq 0 |A^-(j; T)\rangle_\sigma = \delta_{m+\frac{\hat{j}}{f_j(\sigma)},0} (\hat{\Delta}_{0j}(\sigma) + \delta_{jl} \frac{\Delta_g(T)}{f_j(\sigma)}) |A^-(j; T)\rangle_\sigma
\]  
(4.13b)

where we have used the corresponding right-mover ground state conditions (2.56) and right-mover relations (3.19d,e). The adjoint of this result is easily obtained for the right-mover ‘out’ state (4.6d).

### 4.2 The Principal Primary States of Twisted Rep \( T \) and Cycle \( j \)

In this subsection, we will use the twisted affine primary state \(| A^+(j; T)\rangle_\sigma \) to construct the following particular set of principal primary states

\[
|\hat{j}, j; T(\sigma)\rangle_\sigma, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1, \quad \forall T \text{ for each } \sigma = 0, \ldots, N_c - 1
\]  
(4.14a)

\[
|0, j; T\rangle_\sigma \equiv |A^+(j; T)\rangle_\sigma
\]  
(4.14b)

which are examples of the principal primary states of Ref. [1]. More precisely, the states (4.14a) are the principal primary states of twisted representation \( T \) in block \( j \) of sector \( \sigma \). A brief description of more general sets of principal primary states is given in App. B, including as another set of examples the principal primary states associated to the twisted currents.

Up to constants of normalization (\( \cong \)), the set of principal primary states (4.14) are defined as follows

\[
|\hat{j}, j; T\rangle_\sigma \cong (\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^\hat{j}|0, j; T\rangle_\sigma, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1
\]  
(4.15)

where the twisted affine primary state \(|0, j; T\rangle_\sigma \) serves as the base state for the set. Using the properties (4.9b) and (4.10b) of the base state, together with the orbifold Virasoro algebra (2.32b), we verify the following additional properties of the principal primary states:

\[
\hat{L}_{\hat{j},j}(m + \frac{\hat{j}'}{f_j(\sigma)}) |\hat{j}, j; T\rangle_\sigma = 0, \quad \text{if } (m + \frac{\hat{j}'}{f_j(\sigma)}) > \frac{\hat{j}}{f_j(\sigma)}
\]  
(4.16a)

\[
\hat{L}_{\hat{j}l}(m \geq 0) |\hat{j}, j; T\rangle_\sigma = \delta_{m,0} (\hat{\Delta}_{0l}(\sigma) + \delta_{jl} \frac{\Delta_g(T)}{f_j(\sigma)}) |\hat{j}, j; T\rangle_\sigma
\]  
(4.16b)

\[
L_{\sigma}(m \geq 0) |\hat{j}, j; T\rangle_\sigma = \delta_{m,0} (\hat{\Delta}_{0\sigma}(\sigma) + \frac{\Delta_g(T)}{f_j(\sigma)}) |\hat{j}, j; T\rangle_\sigma.
\]  
(4.16c)
According to (4.16b), all these principal primary states are primary under the semisimple integral Virasoro subalgebra (2.34). On the other hand, the base state (4.14b) with \( \hat{j} = 0 \) is the only principal primary state in the set which is both a) twisted affine primary and b) primary under the full orbifold Virasoro algebra. The other principal primary states (with \( \hat{j} \neq 0 \)) are not primary under either the orbifold affine algebra or the orbifold Virasoro algebra. (The definition (4.15) can be extended to the range \( \hat{j} \geq f_j(\sigma) \), but such states are no longer primary under the integral Virasoro subalgebra).

From the definition (4.15), we can derive the following \textit{first asymptotic formula} for the principal primary states

\[
|\hat{j}, j; T\rangle_\sigma \simeq (\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^{\hat{j}}|0, j; T\rangle_\sigma
\]

\[
= \lim_{z \to 0}(\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^{\hat{j}}\hat{g}_j(T, z, \sigma)|0\rangle_\sigma
\]

\[
= \lim_{z \to 0}((\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^{\hat{j}}\hat{g}_j(T, z, \sigma)|0\rangle_\sigma
\]

\[
= \lim_{z \to 0}(z^{1-\frac{1}{f_j(\sigma)}}\partial)^{\hat{j}}\hat{g}_j(T, z, \sigma)\tau_{-\hat{j}}(j, \sigma)|0\rangle_\sigma
\]  

(4.17)

where we have also used (4.10a) and the ground state property (2.39b). When \( \hat{j} = 0 \), this result reduces to the asymptotic formula (4.7) for the twisted affine primary state.

Combining the left-mover result in Eq. (4.17) with the analogous right-mover result gives the following non-chiral form

\[
|0, 0, j; T\rangle_\sigma \equiv |A(j; T)\rangle_\sigma = (|A^+(j; T)\rangle_\sigma \cdot |A^-(j; T)\rangle_\sigma) = (|0_{-}, j; T\rangle_\sigma \cdot |0_{+}, j; T\rangle_\sigma)
\]  

(4.18a)

\[
|\hat{j}_{-}, \hat{j}_{+}, j; T\rangle_\sigma \simeq \hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^{\hat{j}_{-}}\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)}))^{\hat{j}_{+}}|0, 0, j; T\rangle_\sigma \simeq |\hat{j}_{-}, j; T\rangle_\sigma \cdot |\hat{j}_{+}, j; T\rangle_\sigma
\]

\[
= \lim_{|z| \to 0}(z^{1-\frac{1}{f_j(\sigma)}}\partial)^{\hat{j}_{-}}(z^{1-\frac{1}{f_j(\sigma)}}\partial)^{\hat{j}_{+}}|z|^{2\Delta g(T)(1-\frac{1}{f_j(\sigma)})}\tau_{-\hat{j}_{-}}(j, \sigma)\hat{g}_j(T, z, \sigma)\tau_{-\hat{j}_{+}}(j, \sigma)|0\rangle_\sigma
\]  

(4.18b)

where we have used Eqs. (4.6e), (4.14b) and the algebra (3.23).

\section{The Twisted Affine Primary Fields are Principal Primary Fields}

It is explained in Ref. [1] that every set of principal primary states can be created by a corresponding set of fields called the \textit{principal primary fields}. A brief description of general principal primary fields is given in App. B, where the twisted currents themselves are discussed as a simple example. As a more intricate example, we will establish here that
the matrix components (see (3.20c)) of the twisted affine primary fields are themselves the principal primary fields for the set of principal primary states (4.14).

To begin this discussion, we note that it is consistent to require as a boundary condition that the matrix component \( \hat{g}_j^+ (T, z, \sigma)_j^i \) with \( \hat{j} = \hat{l} \mod f_j(\sigma) \) is the most singular\(^{12} \) matrix element of \( \hat{g}_j^+ \), and hence the one that creates the twisted affine primary or base state (4.14b):

\[
\lim_{z \to 0} \hat{g}_j^+ (T, z, \sigma)_j^i |0\rangle_\sigma = \lim_{z \to 0} \hat{g}_j^+ (T, z, \sigma)_0^0 |0\rangle_\sigma = |0, j; T\rangle_\sigma.
\] (4.19)

As we shall see below, this specification corresponds to choosing a particular solution \( \hat{g}_j^+ \) of the twisted vertex operator equation or, equivalently, to fixing the multiplicative constant matrices (such as \( C(j; T, \sigma) \) in (3.33a)) in the solutions of the twisted KZ system.

Given the boundary condition (4.19), we will demonstrate below that all the principal primary states (4.15) can be obtained from the matrix elements of \( \hat{g}_j^+ \) by the second asymptotic formula

\[
|\hat{j}, j; T\rangle_\sigma = \lim_{z \to 0} z^{-\hat{j}} \hat{g}_j^+ (T, z, \sigma)_0^j |0\rangle_\sigma, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1, \quad \sigma = 0, \ldots, N_c - 1
\] (4.20a)

which identifies the matrix components of the twisted affine primary field as principal primary fields\(^[1] \). To compare (4.20) to the asymptotic formula given for principal primary fields in Eq. (3.15) of Ref. [1], it is useful to make the following redefinition:

\[
\hat{\phi}_{\Delta g(T)}^{(j)}(z) \equiv \hat{g}_j^+ (T, z, \sigma)_0^{-f_j(\sigma) - 1 - \hat{j}}, \quad \hat{j} = 0, \ldots, f_j(\sigma) - 1
\] (4.21a)

\[
[\hat{L}_{\hat{j}j}(m + \hat{j} f_j(\sigma)), \hat{g}_j^+ (T, z, \sigma)_0^\hat{l} \hat{j}] = \hat{g}_j^+ (T, z, \sigma)_0^{-\hat{j}} (\partial z + (m + \frac{\hat{j}}{f_j(\sigma)} + 1) \Delta_g(T)) z^{m + \frac{\hat{j}}{f_j(\sigma)}}
\] (4.21b)

\[
[\hat{L}_{\hat{j}j}(m + \hat{j} f_j(\sigma)), \hat{\phi}_{\Delta g(T)}^{(j)}(z)] = \hat{\phi}_{\Delta g(T)}^{(j+\hat{l},j)}(z) (\partial z + (m + \frac{\hat{j}}{f_j(\sigma)} + 1) \Delta_g(T)) z^{m + \frac{\hat{j}}{f_j(\sigma)}}. \] (4.21c)

This is essentially the notation

\[
\hat{\phi}_{\Delta}^{(r)}(z) = \hat{\phi}_{\Delta}^{(j=0)}(z)|_{j=r}, \quad r = 0, \ldots, \lambda - 1
\] (4.22)

\(^{12}\)One can also require that some other matrix element of \( \hat{g} \) is the most singular, but we expect that other choices of this type are equivalent to a relabelling of the principal primary fields below.
given for the principal primary fields of the $\mathbb{Z}_\lambda$ cyclic permutation orbifolds with $\lambda = \text{prime}$ in Ref. [1]. In fact, we chose an equality rather than a proportionality in (4.20) in order to have an exact match with the asymptotic formula of Ref. [1]. This choice sets the relative normalization of the twisted affine primary fields versus the principal primary states, and we will continue with this convention below.

We turn finally to the proof of the second asymptotic formula (4.20): One may combine (4.17) with the boundary condition (4.19) to obtain a matrix form of the first asymptotic formula

\[
\hat{\mathcal{J}}^+_{j,\sigma} \simeq \lim_{z \to 0} \left( \hat{L}_{j-1,j} \left( \frac{1}{f_j(\sigma)} \right) \right) \hat{g}^+_{j}(T, z, \sigma) \tau^{\sigma}_j(0) \sigma.
\]

The solution of (4.23) is

\[
\hat{g}^+_{i}(T, z, \sigma) \sigma = \mathcal{O}(z^{1/2}) \text{ as } z \to 0, \quad i = 0, \ldots, f_j(\sigma) - 1
\]

and, up to constants of normalization, this is equivalent to the second asymptotic formula.

There is an alternate proof of the second asymptotic formula which uses the twisted vertex operator equation for $\hat{\mathcal{J}}^+_{j}$

\[
\partial \hat{g}^+_{j}(T, z, \sigma) \sigma \sigma = \mathcal{O}(z^{1/2}) \text{ as } z \to 0, \quad i = 0, \ldots, f_j(\sigma) - 1
\]

which follows from (4.7) and (3.24a). In particular, we need the most singular terms of this equation when acting on the ground state of sector $\sigma$:

\[
\partial \hat{g}^+_{j}(T, z, \sigma) \sigma \sigma \sigma = \mathcal{O}(z^{1/2}) \text{ as } z \to 0, \quad i = 0, \ldots, f_j(\sigma) - 1
\]

Assuming the boundary condition (4.19), a careful analysis of leading powers of $z$ in this equation leads to the same conclusion (4.20). As a bonus, Eq. (4.26) can also be used to establish that

\[
\lim_{z \to 0} z \partial \hat{g}^+_{j}(T, z, \sigma) \sigma \sigma = 0
\]
which was in fact assumed to obtain (4.10b) and (4.11).

Similarly, one may establish the second asymptotic formula for the right-mover principal primary states

\[ \hat{g}_j^-(T(T, \sigma), \bar{z}, \sigma) \equiv z^{\Delta_g(T)(1 - \frac{1}{f_j(\sigma)})} \hat{g}_j^-(T(T, \sigma), \bar{z}, \sigma) \]  

\[ |\hat{j}_-, j; T\rangle_{\sigma} = \lim_{z \to 0} z^{-\frac{\hat{j}^2}{f_j(\sigma)}} \hat{g}_j^- (T, \bar{z}, \sigma)^{j_0} |0\rangle_{\sigma}, \quad \hat{j}_- = 0, \ldots, f_j(\sigma) - 1 \]  

and corresponding results are easily obtained for the left- and right-mover ‘out’ states.

### 4.4 Final Form of the Single-Cycle Two-Point Correlator

In (3.38) we presented the form of the single-cycle two-point correlator up to a multiplicative constant matrix \( D \). In this subsection, we use some of the information above about the twisted affine primary fields and states to determine this constant matrix.

We begin by writing the following proportionalities

\[ \lim_{z \to 0} z^{\Delta_g(T)(1 - \frac{1}{f_j(\sigma)})} \hat{g}_j^+ (T, z, \sigma)^{i_0} |0\rangle_{\sigma} \propto \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \]  

\[ \lim_{z \to 0} z^{\Delta_g(T)(1 - \frac{1}{f_j(\sigma)})} \hat{g}_j^- (T, \bar{z}, \sigma)^{j_0} |0\rangle_{\sigma} \propto \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \]  

\[ \lim_{z \to \infty} z^{\Delta_g(T)(1 + \frac{1}{f_j(\sigma)})} |0\rangle_{\sigma} \langle \hat{g}_j^+ (T, z, \sigma)^{i} \rangle_{\sigma} \propto \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \]  

\[ \lim_{b \to -\infty} z^{\Delta_g(T)(1 + \frac{1}{f_j(\sigma)})} |0\rangle_{\sigma} \langle \hat{g}_j^- (T, \bar{z}, \sigma)^{j} \rangle_{\sigma} \propto \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \]

which hold because the most singular matrix element of each operator is \( \hat{j} = \hat{l} \mod f_j(\sigma) \).

A more precise form of (4.29) is the following list of relations:

\[ \lim_{z \to 0} z^{\Delta_g(T)(1 - \frac{1}{f_j(\sigma)})} \hat{g}_j^+ (T, z, \sigma)^{i_0} |0\rangle_{\sigma} = (|A^+(j; T)\rangle_{\sigma})^i_j = \delta_{j - l, 0 \text{ mod } f_j(\sigma)} |0, j; T\rangle_{\sigma} \]  

\[ \lim_{z \to 0} z^{\Delta_g(T)(1 - \frac{1}{f_j(\sigma)})} \hat{g}_j^- (T, \bar{z}, \sigma)^{j_0} |0\rangle_{\sigma} = (|A^-(j; T)\rangle_{\sigma})^i_j = \delta_{j - l, 0 \text{ mod } f_j(\sigma)} |0, j; T\rangle_{\sigma} \]  

\[ \lim_{z \to \infty} z^{\Delta_g(T)(1 + \frac{1}{f_j(\sigma)})} |0\rangle_{\sigma} \langle \hat{g}_j^+ (T, z, \sigma)^{i} \rangle_{\sigma} = (\langle \sigma | A^+(j; T) \rangle_{\sigma})^i_j = \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \langle 0, j; T \rangle_{\sigma} \]  

\[ \lim_{z \to -\infty} z^{\Delta_g(T)(1 + \frac{1}{f_j(\sigma)})} |0\rangle_{\sigma} \langle \hat{g}_j^- (T, \bar{z}, \sigma)^{j} \rangle_{\sigma} = (\langle \sigma | A^-(j; T) \rangle_{\sigma})^i_j = \delta_{j - l, 0 \text{ mod } f_j(\sigma)} \langle 0, j; T \rangle_{\sigma} \]
\[
\lim_{z \to \infty} z^{\Delta g(T)(1 + \frac{1}{\Delta_j J(z)})} \langle 0 | \hat{g}_j (T, \tilde{z}, \sigma) \rangle_j \hat{i} = (\sigma \langle A^{-}(j; T) | )_j \hat{i} = \delta_{j-l,0 \text{mod } f_j(\sigma)} \langle 0, j; T | \tag{4.30d}
\]

\[
|0, 0, j; T\rangle_{\sigma} = |0, -, j; T\rangle_{\sigma} \cdot |0, j; T\rangle_{\sigma}, \quad \sigma \langle 0, 0, j; T | = \sigma \langle 0, -, j; T | \cdot \sigma \langle 0, j; T |. \tag{4.30e}
\]

Note in particular that (4.30a) is the explicit form of Eq. (4.14b), now including all \( \hat{j}, \hat{l} \) matrix indices. Together, Eqs. (4.30) and (4.15) tell us that the principal primary states \( |\hat{j}, j; T\rangle_{\sigma} \) do not have \( \hat{j}, \hat{l} \) matrix indices (and similarly for their right-mover analogues).

Abelian analogues of all the relations in (4.30) can be obtained with the left- and right-mover twisted vertex operators of Ref. [13]. In particular, the left-mover relations

\[
\lim_{z \to 0} z^{\Delta g(T)(1 - \frac{1}{\Delta_j J(z)})} \hat{g}_j^+ (T, z, \sigma) |0\rangle_{\sigma} = \hat{\Gamma}_j (T \tau_0 (j, \sigma), \tilde{J}_{0j}(0, \sigma)) |\tilde{J}_{0aj}(0) = T_a \rangle_{\sigma} \tag{4.31a}
\]

\[
\hat{\Gamma}_j (T \tau_0 (j, \sigma), \tilde{J}_{0j}(0, \sigma), \sigma) \hat{i} \propto \delta_{j-l,0 \text{mod } f_j(\sigma)}, \quad |\tilde{J}_{0aj}(0) = T_a \rangle_{\sigma} \equiv e^{i\tilde{q}^j_{\pm}(\sigma) T_a} |0\rangle_{\sigma} \tag{4.31b}
\]

are obtained from (3.14b). This tells us that the natural assumption (3.12) is the abelian analogue of our boundary condition (4.19).

We now apply the asymptotic relations (4.30) to the single-cycle two-point correlator (3.38). In particular, these relations and the limit (4.2b) give us further information about the multiplicative constant matrix \( C(j; T, \sigma) \):

\[
\lim_{|z_1| \to \infty} \lim_{|z_2| \to 0} \left| z_1 \right|^{2\Delta g(T)(1 + \frac{1}{\Delta_j J(z)})} \left| z_2 \right|^{2\Delta g(T)(1 - \frac{1}{\Delta_j J(z)})} \langle 0 | \hat{g}_j (T^{(1)}, \tilde{z}_1, z_1, \sigma) \hat{i} \hat{g}_j (T^{(2)}, \tilde{z}_2, z_2, \sigma) \hat{i} |0\rangle_{\sigma} = C(j; T, \sigma)_{j_1,j_2} \hat{i} \hat{i} = \delta_{j_1-l_1,0 \text{mod } f_j(\sigma)} \delta_{j_2-l_2,0 \text{mod } f_j(\sigma)} \langle 0, 0, j; T^{(1)} |0, 0, j; T^{(2)} \rangle_{\sigma}. \tag{4.32}
\]

Combining this relation with the form of \( C(j; T, \sigma) \) in (3.37a), we find that:

\[
(\sigma \langle 0, 0, j; T |)_{\alpha_1}^{\beta_1} (|0, 0, j; T\rangle_{\sigma})^{\beta_2} = c \delta_{T^{(2)}, T^{(1)}} \delta_{\alpha_1 \alpha_2} \delta_{\beta_1 \beta_2} \tag{4.33a}
\]

\[
D(j; T, \sigma)_{j_1,j_2} \hat{i} \hat{i} = c \delta_{j_1-l_1,0 \text{mod } f_j(\sigma)} \delta_{j_2-l_2,0 \text{mod } f_j(\sigma)} \tag{4.33b}
\]

\[
C(j; T, \sigma)_{\alpha_1,j_1,j_2}^{\beta_1,j_1,j_2} = c \delta_{T^{(2)}, T^{(1)}} \delta_{\alpha_1 \alpha_2} \delta_{\beta_1 \beta_2} \delta_{j_1-l_1,0 \text{mod } f_j(\sigma)} \delta_{j_2-l_2,0 \text{mod } f_j(\sigma)}. \tag{4.33c}
\]

We will also choose the overall normalization constant \( c = 1 \), which gives the final, completely-determined form of the single-cycle two-point correlator:

\[
\hat{A}(j, 1, 2) \equiv \langle \hat{g}_j (T^{(1)} \tilde{z}_1, z_1, \sigma) \hat{g}_j (T^{(2)} \tilde{z}_2, z_2, \sigma) \rangle_{\sigma} = \delta_{T^{(2)}, T^{(1)}} |z_1 z_2|^{-2\Delta g(T)(1 - \frac{1}{\Delta_j J(z)})} |z_1 z_2|^{-4\Delta g(T)/f_j(\sigma)} \exp \left\{ \frac{2}{2k + Q_g} F(j; 1, 2) \right\} \tag{4.34a}
\]
∀j, ∀T^{(1)}, T^{(2)} for each σ = 0, \ldots, N_c - 1. \quad (4.34b)

Here Δ_0(T) is the untwisted conformal weight in (3.38b) and F(j; 1, 2) is defined in (3.33).

Our relative normalization (4.20) and our orthonormality condition ((4.33a) with c=1)

\[ \sigma \langle 0, 0, j; T^{(1)} \mid 0, 0, j; T^{(2)} \rangle_\sigma = \delta_{T^{(2)}, T^{(1)}} \quad (4.35) \]

can be used to compute the constants in all the proportionality (~) above.

We remark that the single-cycle two-point correlator (4.34) is symmetric under 1 ↔ 2 exchange

\[ \hat{A}(j; 1, 2) = \hat{A}(j; 2, 1) \quad (4.36) \]

as expected. To establish this one needs to verify the symmetry relation

\[ F(j; 1, 2) = F(j; 2, 1) \quad (4.37) \]

which follows by steps which are essentially identical to those given in Eq. (9.32) of Ref. [12] and App. C of Ref. [13].

Finally, we record the completely-determined form of the original unreduced two-point correlator

\[ \langle \hat{g}(T^{(1)}, \bar{z}_1, z_1, \sigma)_{a_1j_1l_1} \beta_{j_1l_1} \hat{g}(T^{(2)}, \bar{z}_2, z_2, \sigma)_{a_2j_2l_2} \beta_{j_2l_2} \rangle_\sigma = \delta_{j_1j_2} \delta_{l_1l_2} \delta_{a_1a_2} \delta_{\beta_{j_1} \beta_{j_2}} \times \]

\[ \times |z_1z_2|^{-2\Delta_g(T)(1-1/j_1j_2)} |z_{12}|^{-4\Delta_g(T)/f_j(\sigma)} \exp \left\{ \frac{2}{2k + Q_g} F(j; 1, 2) \right\} \hat{J}_{j_1l_1} \hat{J}_{j_2l_2} \quad (4.38) \]

which follows immediately from (3.28b) and (4.34).

5 General WZW Orbifolds

In this section, we consider the following topics

- asymptotic formulae
- conformal weights
- reducibility

for the twisted affine primary fields and states of general WZW orbifolds.

We begin by reminding the reader that twisted KZ equations are known [12, 13] for the correlators in the scalar twist-field state\footnote{For the inner-automorphic WZW orbifolds, a different set of twisted KZ equations [12] was given for the correlators in the untwisted affine vacuum state.}

\[ \hat{J}_{n(r)\mu}(m + \frac{n(r)}{\rho(\sigma)} \geq 0) |0\rangle_\sigma = \sigma |0\rangle \hat{J}_{n(r)\mu}(m + \frac{n(r)}{\rho(\sigma)} \leq 0) = 0 \quad (5.1) \]
in each sector of any WZW orbifold, where \( \hat{J}_{n(r)\mu} (m + \frac{n(r)}{\rho(\sigma)}) \) are the modes of the twisted current algebra \([3, 5, 12]\) of that sector. Such scalar twist-field states exist for all sectors of all WZW orbifolds. The reason is easy to understand in the equivalent form

\[
\hat{J}_{n(r)\mu} (m + \frac{n(r)}{\rho(\sigma)}) > 0) |0\rangle_\sigma = 0, \quad \hat{J}_0(0)|0\rangle_\sigma = 0.
\] (5.2)

The first condition holds for any primary state of any (infinite-dimensional) Lie algebra and the second condition restricts our attention to the “s-wave” or trivial representation of the untwisted residual symmetry algebra of the sector.

In further detail, the general twisted KZ system which describes the correlators of the twisted left-mover affine primary fields \( \hat{g}_+ (T, z, \sigma) \) in the scalar twist-field states reads as follows \([12, 13]\):

\[
\hat{A}_+ (T, z, \sigma) = \hat{g}_+ (T^{(1)}, z_1, \sigma) \hat{g}_+ (T^{(2)}, z_2, \sigma) \cdots \hat{g}_+ (T^{(N)}, z_N, \sigma) |0\rangle_\sigma \quad (5.3a)
\]

\[
\partial_\kappa \hat{A}_+ (T, z, \sigma) = \hat{A}_+ (T, z, \sigma) \hat{W}_\kappa (T, z, \sigma), \quad \kappa = 1 \ldots N, \quad \sigma = 0, \ldots, N_c - 1 \quad (5.3b)
\]

\[
\hat{W}_\kappa (T, z, \sigma) = 2 \mathcal{L}_{\hat{g}_+ (T^{(\rho)})}^{n(r)\mu_1 - n(r)\nu_1} (\sigma) \left[ \sum_{\rho \neq \kappa} \left( \frac{z_\rho}{z_\kappa} \right) \frac{\hat{n}(r)}{z_\kappa} \frac{1}{\rho(\sigma)} \mathcal{T}^{(\kappa)}_{n(r)\mu} \mathcal{T}^{(\kappa)}_{n(r)\mu} - \hat{n}(r) \frac{1}{\rho(\sigma)} \frac{1}{z_\kappa} \mathcal{T}^{(\kappa)}_{n(r)\mu} \mathcal{T}^{(\kappa)}_{n(r)\mu} \right] \quad (5.3c)
\]

\[
\hat{A}_+ (T, z, \sigma) \left( \sum_{\rho=1}^N \mathcal{T}^{(\rho)}_{0\mu} \right) = 0, \quad \forall \mu. \quad (5.3d)
\]

Here \( \rho(\sigma) \) is the order of \( h_\sigma \in H, n(r) \) is determined from the appropriate \( H \)-eigenvalue problem, and \( \hat{n}(r) \in \{0, \ldots, \rho(\sigma) - 1\} \) is the pullback of \( n(r) \) to the fundamental range. General formulae for the twisted inverse inertia tensor \( \mathcal{L}_{\hat{g}_+ (T^{(\rho)})} (\sigma) \) and the twisted representation matrices \( \mathcal{T}_{n(r)\mu} (T, \sigma) \) are given in Refs. \([12, 13]\). For the special case of the WZW permutation orbifolds the general twisted KZ system (5.3) reduces

\[
n(r)\mu \rightarrow n(r)a_j \rightarrow \hat{j}aj \quad (5.4)
\]
to the twisted KZ system (2.46), and the explicit form of the system (5.3) is also given for the (outer-automorphic) charge conjugation orbifold on \( \mathfrak{su}(n \geq 3) \) in Ref. \([13]\).

Using the general twisted KZ system (5.3), and in particular the last term of (5.3c), we find the following asymptotic formulae for the twisted affine primary fields

\[
\hat{g}_+ (T, z, \sigma) \equiv \hat{g}_+ (T, z, \sigma) z^{\gamma(T, \sigma)} \quad (5.5a)
\]
\[ \gamma(T, \sigma) \equiv 2 L^{n(r)\mu - n(r),\nu}_\delta(\sigma) \frac{\bar{n}(r)}{\rho(\sigma)} T_{n(r)\mu}(T, \sigma) T_{-n(r),\nu}(T, \sigma) \]  

(5.5b)

\[ |A^+(T)\rangle_\sigma \equiv \lim_{z \to 0} \hat{g}_+(T, z, \sigma)|0\rangle_\sigma \]  

(5.5c)

where \( \gamma(T; \sigma) \) is called the matrix exponent of the twisted affine primary field \( \hat{g}_+(T, z, \sigma) \). We also find as expected that the states \( |A^+(T)\rangle_\sigma \) are twisted affine primary states:

\[ \hat{J}_{n(r)\mu}(m + \frac{n(r)}{\rho(\sigma)} \geq 0) |A^+(T)\rangle_\sigma = \delta_{m, n(r)} |A^+(T)\rangle_\sigma T_{n(r)\mu}(T, \sigma). \]  

(5.6)

This relation follows from the commutation relations [12, 13] of the general twisted current modes \( \hat{J}_{n(r)\mu}(m + \frac{n(r)}{\rho(\sigma)}) \) with the twisted affine primary field.

Moreover, the twisted affine primary state is Virasoro primary under the full Virasoro generators \( L_\sigma(m) \) of sector \( \sigma \), and we can evaluate the total conformal weight matrix \( \hat{\Delta}(T, \sigma) \) of the twisted affine primary state as follows:

\[ L_\sigma(m \geq 0) |A^+(T)\rangle_\sigma = \delta_{m, 0} |A^+(T)\rangle_\sigma \hat{\Delta}(T, \sigma) \]  

(5.7a)

\[ \hat{\Delta}(T, \sigma) \equiv \hat{\Delta}_0(\sigma) \mathbb{I} + D_{\hat{g}(\sigma)}(T) - \gamma(T, \sigma) \]  

(5.7b)

\[ \hat{\Delta}_0(\sigma) \equiv L^{n(r)\mu - n(r),\nu}_\delta(\sigma) \frac{\bar{n}(r)}{2\rho(\sigma)} \left( 1 - \frac{\bar{n}(r)}{\rho(\sigma)} \right) G_{n(r)\mu - n(r),\nu}(\sigma) \]  

(5.7c)

\[ D_{\hat{g}(\sigma)}(T) \equiv L^{n(r)\mu - n(r),\nu}_\delta(\sigma) T_{n(r)\mu}(T, \sigma) T_{-n(r),\nu}(T, \sigma). \]  

(5.7d)

Here \( \hat{\Delta}_0(\sigma) \) and \( D_{\hat{g}(\sigma)}(T) \) are respectively the conformal weight of the scalar twist-field state and the twisted conformal weight matrix, which appears in the \( \hat{T}_\sigma \hat{g}_+ \) OPE [12]:

\[ \hat{T}_\sigma(z) \hat{g}_+(T, w, \sigma) = \hat{g}_+(T, w, \sigma) \left( \frac{D_{\hat{g}(\sigma)}(T)}{(z - w)^2} + \frac{\hat{g}_+(T)}{z - w} \right) + O(z - w)^0. \]  

(5.8)

For the WZW permutation orbifolds, the explicit forms of \( \hat{\Delta}_0(\sigma) \) and \( D_{\hat{g}(\sigma)}(T) \) are given in Eqs. (2.38d) and (2.17f).

With these tools in hand, we turn next to the question of the reducibility of general twisted affine primary states and fields.

In the case of the WZW permutation orbifolds we have shown above that the twisted affine primary fields \( \hat{g}_+ \) are generically reducible into blocks \( \hat{g}_j^+ \), and hence the unreduced
fields create generically reducible states. As a simpler analogue, we note first for the abelian permutation orbifolds that

$$\gamma(T(T, \sigma), \sigma) = \Delta(T) \sum_j (1 - \frac{1}{f_j(\sigma)})t_{0j}(\sigma) \quad (5.9a)$$

$$|A^+(T)\rangle_\sigma \equiv \lim_{z \to 0} z^{\Delta(T) \sum_j (1 - \frac{1}{f_j(\sigma)})t_{0j}(\sigma)} \hat{g}_+(T, z, \sigma)|0\rangle_\sigma = \hat{\Gamma}(T, \hat{J}_0(0), \sigma) \{ \otimes_j |\hat{J}_{0aj}(0) = T_a\rangle_\sigma \} \quad (5.9b)$$

where \( \hat{g}_+(T, z, \sigma) \) is the twisted left-mover vertex operator [13] given in (3.10). The results for the unreduced twisted affine primary fields of the WZW permutation orbifolds are quite similar

$$\gamma(T(T, \sigma), \sigma) = \Delta_g(T) \sum_j (1 - \frac{1}{f_j(\sigma)})t_{0j}(\sigma) \quad (5.10a)$$

$$|A^+(T)\rangle_\sigma \equiv \lim_{z \to 0} z^{\Delta_g(T) \sum_j (1 - \frac{1}{f_j(\sigma)})t_{0j}(\sigma)} \hat{g}_+(T, z, \sigma)|0\rangle_\sigma = \otimes_j |A^+(j; T)\rangle_\sigma \quad (5.10b)$$

where \( |A^+(j; T)\rangle_\sigma \) is the base state (4.14b) of twisted representation \( T \) in block \( j \) of sector \( \sigma \). In this case we have computed the total conformal weight matrix

$$\hat{\Delta}_0(\sigma) = \frac{c_g}{24}(K - \sum_j \frac{1}{f_j(\sigma)}), \quad D_{\hat{g}(\sigma)}(T(T, \sigma)) = \Delta_g(T) \mathbb{1} \quad (5.11a)$$

$$\hat{\Delta}(T(T, \sigma), \sigma) = \hat{\Delta}_0(\sigma) \mathbb{1} + \Delta_g(T) \sum_j t_{0j}(\sigma) \frac{1}{f_j(\sigma)} \quad (5.11b)$$

$$\hat{\Delta}(T(T, \sigma), \sigma)_{\alpha \beta \gamma \delta} = \delta_\gamma^\delta \delta_\beta^\alpha \delta_\alpha^\gamma \delta_j \eta \mod f_j(\sigma) \frac{1}{f_j(\sigma)} \text{ for } n = 3 \quad (5.11c)$$

which shows reducibility with respect to \( j \), as expected.

As another set of examples, consider the single twisted sector \( \sigma = 1 \) of the charge conjugation orbifold [13] on \( \mathfrak{su}(n \geq 3) \). Ref. [13] explains that charge conjugation has a simple diagonal action in the standard Cartesian basis, with the irregularly-embedded \( \mathfrak{so}(n) \) subalgebra

$$\mathfrak{so}(n)_{2\tau} \subset \mathfrak{su}(n), \quad x = \frac{2k}{\psi_{\mathfrak{su}(n)}}^2, \quad \tau = \begin{cases} 2 & \text{for } n = 3 \\ 1 & \text{for } n \geq 4 \end{cases} \quad (5.12)$$
as an invariant subalgebra. Moreover, we read from this reference that

\[
\hat{\Delta}_0 = \frac{(n - 1)(n + 2)x}{32(x + n)} \quad (5.13a)
\]

\[
\gamma(T) = \frac{1}{2k + Q_g} \sum_I T_{1I}(T) T_{1I}(T) \quad (5.13b)
\]

\[
(D_{\hat{g}(\sigma)}(T) - \gamma(T)) = \frac{1}{2k + Q_g} \sum_A T_{0A}(T) T_{0A}(T) \quad (5.13c)
\]

\[
A \in \mathfrak{so}(n), \quad I \in \frac{\mathfrak{su}(n)}{\mathfrak{so}(n)} \quad (5.13d)
\]

for any twisted representation \( T(T) \).

Further details depend on whether the untwisted representation \( T \) is complex or real. In the case of the (complex) fundamental representation \( T = T^{(n)} \) of \( \mathfrak{su}(n) \), the matrix exponent and the total conformal weight matrix are

\[
\gamma(T^{(n)})) = \frac{1}{4n(x + n)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{I}_n \quad (5.14a)
\]

\[
\hat{\Delta}(T^{(n)})) = \frac{n - 1}{4(x + n)} \left( \frac{(n + 2)x}{8} + 1 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{I}_n. \quad (5.14b)
\]

These matrices show a reducibility in the 2x2 space, which has in fact already been discussed [13] for all complex representations. Next, we have computed the matrix elements of the matrices \( \gamma \) and \( \hat{\Delta} \) in the case of the (real) adjoint representation \( T^{\text{adj}} \) of \( \mathfrak{su}(n) \):

\[
\gamma(T^{\text{adj}}))_{AB} = \delta_{AB} \frac{n + 2}{2(x + n)} , \quad \gamma(T^{\text{adj}}))_{IJ} = \delta_{IJ} \frac{n}{2(x + n)} \quad (5.15a)
\]

\[
\gamma(T^{\text{adj}}))_{AI} = \gamma(T^{\text{adj}}))_{IA} = 0 \quad (5.15b)
\]

\[
\hat{\Delta}(T^{\text{adj}}))_{AB} = \delta_{AB} \frac{(n - 1)(n + 2)x}{32} + \frac{n - 2}{2} \quad (5.15c)
\]

\[
\hat{\Delta}(T^{\text{adj}}))_{IJ} = \delta_{IJ} \frac{(n - 1)(n + 2)x}{32} + \frac{n}{2} \quad (5.15d)
\]
\[ \Delta(T(T_{\text{adj}}))_{AI} = \Delta(T(T_{\text{adj}}))_{IA} = 0. \] (5.15e)

These matrices show reducibility of the adjoint of \( \mathfrak{su}(n) \) into representations of \( \mathfrak{so}(n) \)

\[ (n^2 - 1) = \left( \frac{n(n - 1)}{2} \right) \oplus \left( \frac{n(n + 1)}{2} - 1 \right) \] (5.16)

which are identified in this case as an adjoint of \( \mathfrak{so}(n) \) plus a second rank symmetric traceless \( \mathfrak{so}(n) \) tensor. No such \( \mathfrak{su}(n) \to \mathfrak{so}(n) \) splitting is observed in the total conformal weight matrix (5.14b) associated to the fundamental representation because, in this situation, both the \( n \) and the \( \bar{n} \) of \( \mathfrak{su}(n) \) are interpreted as \( n \)'s of \( \mathfrak{so}(n) \).

Using Eq. (5.3) and the data of Refs. [5, 12], one finds the results for the inner-automorphic WZW orbifolds

\[ \hat{W}_\kappa(T, z, \sigma) = \frac{2}{2k + Q_g} \left[ \sum_{\rho \neq \kappa} \frac{1}{z_{\rho \kappa}} \left( \sum_A T_A^{(\rho)} T_A^{(\kappa)} + \sum_\alpha \left( \frac{z_{\rho \kappa}}{z_{\kappa}} \bar{n}_\alpha \right) T_\alpha^{(\rho)} T_{-\alpha}^{(\kappa)} \right) - \sum_\alpha \frac{\bar{n}_\alpha}{\rho(\sigma)} T_\alpha^{(\kappa)} T_{-\alpha}^{(\rho)} \right] \] (5.17a)

\[ \hat{A}_+(T, z, \sigma) \left( \sum_{\kappa=1}^N T_A^{(\kappa)} \right) = 0, \quad \frac{n_\alpha}{\rho(\sigma)} = -\sigma \alpha \cdot d, \quad \frac{\bar{n}_\alpha}{\rho(\sigma)} = -\sigma \alpha \cdot d - \lfloor -\sigma \alpha \cdot d \rfloor \] (5.17b)

\[ \hat{\Delta}_0(\sigma) = \frac{x/4}{x + h_g} \sum_\alpha \frac{\bar{n}_\alpha}{\rho(\sigma)} (1 - \frac{\bar{n}_\alpha}{\rho(\sigma)}), \quad D_{\bar{g}(\sigma)}(T(T, \sigma)) = \Delta_g(T) \mathbb{I} \] (5.18a)

\[ \gamma(T(T, \sigma), \sigma) = \frac{2}{2k + Q_g} \sum_\alpha \frac{\bar{n}_\alpha}{\rho(\sigma)} T_\alpha T_{-\alpha} \] (5.18b)

where \( d \) is the shift vector [27], \( (T_A, T_\alpha) \) is irrep \( T \) of \( g \) in the Cartan-Weyl basis, and \( \lfloor x \rfloor \) is the floor of \( x \). These twisted affine primary fields are also seen to be reducible due to symmetry breaking, as expected.

On the basis of these examples, one expects that the twisted affine primary fields of general WZW orbifolds are generically reducible. We remind the reader that there are special cases in which the twisted affine primary fields are irreducible, including all the twisted affine primary fields of the single-cycle sectors of the WZW permutation orbifolds.

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A  About the Orbifold $\mathfrak{sl}(2)$ Ward Identities

As stated in the text, the Ward identities (2.44) associated to the centrally-extended $\mathfrak{sl}(2)$ algebra are in fact implied by the twisted KZ system (2.46). The proof of this statement is non-trivial, and the reader may find helpful the following hints.

We begin with the $\hat{L}_{0j}(0)$ Ward identity in (2.44a), for which we list the intermediate steps:

$$2\eta_{ab} \sum_{\mu,\nu \neq \mu} \sum_{j=1} f_j^{(\sigma)} \left( \frac{z_\nu}{z_\mu} \right)^{\frac{j}{f_j^{(\sigma)}}} \frac{z_\mu}{z_{\mu\nu}} T_{ab}^{(\nu)} T_{b}^{(\mu)} t_{j}^{(\nu)} (\sigma) t_{-j,\nu}^{(\mu)} (\sigma) t_{0j}^{(\mu)} (\sigma) = 0 \quad (A.1a)$$

$$2\eta_{ab} \sum_{\mu,\nu \neq \nu} \frac{z_\mu}{z_{\mu\nu}} z_{\nu} \frac{1}{T_{0j}^{(\mu)} (\sigma) t_{0j}^{(\mu)} (\sigma)} \eta_{ab} \sum_{\mu,\nu \neq \nu} T_{b}^{(\mu)} t_{0j}^{(\nu)} (\sigma) t_{0j}^{(\mu)} (\sigma) = 0 \quad (A.1b)$$

$$\hat{A}_+ (\sigma) \sum_{\mu} \left( \tilde{\partial}_\mu z_\mu + \Delta_{\phi} (T^{(\mu)}) \right) t_{0j}^{(\mu)} (\sigma) = 0 \quad (A.1c)$$

$$= \hat{A}_+ (\sigma) \left( \sum_{\mu} z_\mu \tilde{W}_\mu (\sigma) + \frac{\Delta_{\phi} (T^{(\mu)})}{2} \right) t_{0j}^{(\mu)} (\sigma) \quad (A.1d)$$

The first relation in (A.1) holds for each $(\mu \nu) + (\nu \mu)$ pair separately, by the variable change $\hat{j} \rightarrow f_j^{(\sigma)} - \hat{j}$ in the second term. Eqs. (A.1a,b) and the form of the twisted KZ connection in (2.46) are used to obtain (A.1c), and then (A.1d) follows from the global Ward identity (2.46c). The reader will note that the steps given for this case are quite similar to those followed in the proof of the $L_\sigma(0)$ Ward identity in Ref. [12].

Similarly, we can use the form of the twisted KZ connection to simplify the $\hat{L}_{1j} \left( \frac{1}{f_j^{(\sigma)}} \right)$ Ward identity (2.44b), as follows:

$$2\eta_{ab} \sum_{\mu,\nu \neq \nu} \sum_{j=2} f_j^{(\sigma)} \left( \frac{z_\nu}{z_\mu} \right)^{\frac{j}{f_j^{(\sigma)}}} \frac{z_\mu}{z_{\mu\nu}} T_{a}^{(\nu)} T_{b}^{(\mu)} t_{j}^{(\nu)} (\sigma) t_{-j,\nu}^{(\mu)} (\sigma) t_{1j}^{(\mu)} (\sigma) = 0 \quad (A.2a)$$

$$\eta_{ab} \sum_{\mu,\nu \neq \nu} \left[ \frac{z_\mu}{z_{\mu\nu}} T_{a}^{(\nu)} T_{b}^{(\mu)} t_{0j}^{(\nu)} (\sigma) t_{1j}^{(\mu)} (\sigma) + \left( \frac{z_\nu}{z_\mu} \right)^{\frac{j}{f_j^{(\sigma)}}} \frac{\mu}{z_{\mu\nu}} T_{a}^{(\nu)} T_{b}^{(\mu)} t_{0j}^{(\nu)} (\sigma) t_{1j}^{(\mu)} (\sigma) \right]$$

$$= \eta_{ab} \sum_{\mu,\nu \neq \nu} \frac{z_\mu}{z_{\mu\nu}} T_{a}^{(\nu)} T_{b}^{(\mu)} t_{0j}^{(\nu)} (\sigma) t_{1j}^{(\mu)} (\sigma) \quad (A.2b)$$
\[ \hat{A}_+ (\sigma) \sum_\mu (\partial_\mu z_\mu + \left(1 + \frac{1}{f_j(\sigma)}\right) \Delta_g(T^{(\mu)}) t_{ij}^{(\mu)}(\sigma) z_\mu^{-\frac{1}{\hat{j}(\sigma)}}) \]

\[ = \hat{A}_+ (\sigma) \sum_\mu (z_\mu \hat{W}_\mu(\sigma) + \left(1 + \frac{1}{f_j(\sigma)}\right) \Delta_g(T^{(\mu)}) t_{ij}^{(\mu)}(\sigma) z_\mu^{-\frac{1}{\hat{j}(\sigma)}}) \]

\[ = \hat{A}_+ (\sigma) \left( \sum_\nu T^{(\nu)}_a t_{0j}^{(\nu)}(\sigma) \right) \frac{2\eta_{ab}}{2k + Q_a f_j(\sigma)} \left( \sum_\mu z_\mu^{-\frac{1}{\hat{j}(\sigma)}} T^{(\mu)}_b t_{ij}^{(\mu)}(\sigma) \right) \]

\[ = 0. \quad (A.2c) \]

Again, the first equation in (A.2) holds for each (\(\mu\nu\)) + (\(\nu\mu\)) pair separately, by taking \(\hat{j} \rightarrow (f_j(\sigma) + 1 - \hat{j})\) in the second term. Eq. (A.2c) follows from (A.2a,b), and the result (A.2d) then follows from the global Ward identity.

Finally for the \(\hat{L}_{-1,j}(-\frac{1}{f_j(\sigma)})\) Ward identity in (2.44c), follow the steps:

\[ 2\eta_{ab} \sum_{\mu,\nu \mu \neq \nu} \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{z_\nu}{z_\mu} \right) \frac{1}{f_j^{(\sigma)}} T^{(\nu)}_a T^{(\mu)}_b \left( t_{\hat{j},j}^{(\nu)}(\sigma) t_{-\hat{j},j}^{(\mu)}(\sigma) t_{-1,j}^{(\mu)}(\sigma) \right) = 0 \quad (A.3a) \]

\[ \sum_\mu z_\mu^{-\frac{1}{f_j(\sigma)}} \hat{W}_\mu(\sigma) t_{-1,j}^{(\mu)}(\sigma) = -(1 - \frac{1}{f_j(\sigma)}) \sum_\mu z_\mu^{-\frac{1}{f_j(\sigma)}} \Delta_g(T^{(\mu)}) t_{-1,j}^{(\mu)}(\sigma) \quad (A.3b) \]

\[ \hat{A}_+ (\sigma) \sum_\mu (\partial_\mu z_\mu + \left(1 - \frac{1}{f_j(\sigma)}\right) \Delta_g(T^{(\mu)}) t_{-1,j}^{(\mu)}(\sigma) z_\mu^{-\frac{1}{\hat{j}(\sigma)}}) \]

\[ = \hat{A}_+ (\sigma) \sum_\mu (z_\mu \hat{W}_\mu(\sigma) + \left(1 - \frac{1}{f_j(\sigma)}\right) \Delta_g(T^{(\mu)}) t_{-1,j}^{(\mu)}(\sigma) z_\mu^{-\frac{1}{\hat{j}(\sigma)}}) = 0. \quad (A.3c) \]

Here the first equation in (A.3) holds for each (\(\mu\nu\)) + (\(\nu\mu\)) pair separately, by taking \(\hat{j} \rightarrow (f_j(\sigma) - 1 - \hat{j})\) in the second term. Eq. (A.3b) is obtained from (A.3a) and the form of the twisted KZ connection, and the result (A.3c) is implied immediately.

**B More General Sets of Principal Primary States and Fields**

We know from the orbi-fold induction procedure of Ref. [1] that each Virasoro primary state (with conformal weight \(\Delta\) in the untwisted symmetric theory) gives us a set of principal primary states \(|\Delta, \tilde{j}, j, \sigma\rangle\) in each sector of the corresponding permutation orbifold. Following Ref. [1], we may choose the base state of cycle \(j\) as the state with \(\tilde{j} = f_j(\sigma) - 1\) (although this is a relabelling relative to that used in Subsec. 4.2). The base state is always primary.
under the orbifold Virasoro algebra and each of the principal primary states is primary under the semisimple integral Virasoro subalgebra. Each set of principal primary states is created asymptotically by its corresponding set of principal primary fields $\hat{\phi}^{(ij)}(z)$,

$$[\hat{L}_{jj}(m + \frac{\hat{j}}{f_j(\sigma)}), \hat{\phi}^{(ij)}(z)] = \delta_{jj} z^{m + \frac{\hat{j}}{f_j(\sigma)}} (z\partial + \Delta(m + \frac{\hat{j}}{f_j(\sigma)} + 1)) \hat{\phi}^{(j+l,j)}(z) \tag{B.1a}$$

whose correlators obey orbifold $sl(2)$ Ward identities as discussed in Ref. [1] and Subsec. 2.6.

In the text (see Eq. (4.21)) we discussed the principal primary states

$$|\Delta, \bar{\hat{j}}, j\rangle_{\sigma} = \lim_{z \to 0} z^{(1-\frac{\hat{j}}{f_j(\sigma)})-\frac{\hat{j}}{f_j(\sigma)}} \hat{\phi}^{(ij)}(z)|0\rangle_{\sigma}, \quad \hat{\phi}^{(ij)}(z) = \hat{\phi}^{(ij)}(z) \tag{B.1b}$$

associated to each block of each twisted affine primary field. As a simpler example, we mention here (see also Ref. [1]) the set of principal primary states associated to the twisted currents

$$|\Delta = 1, \bar{\hat{j}}, j\rangle_{\sigma} = \lim_{z \to 0} z^{(1-\frac{\hat{j}}{f_j(\sigma)})-\frac{\hat{j}}{f_j(\sigma)}} J_{j,\bar{\hat{j}}} (z)|0\rangle_{\sigma} = J_{j,\bar{\hat{j}}}(\Delta = 1, \bar{\hat{j}}, j\rangle_{\sigma} \tag{B.3a}$$

$$L_{\sigma}(0)|\Delta = 1, \bar{\hat{j}}, j\rangle_{\sigma} = \left(\hat{\Delta}_0(\sigma) + \frac{\hat{j}}{f_j(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} - 1 + \Delta \right) \to \hat{\Delta}^{(j,\bar{j})}(\sigma) \tag{B.3b}$$

where the ground state conformal weight $\hat{\Delta}_0(\sigma)$ is given in Eq. (2.38d). More generally, one finds that

$$\hat{\Delta}^{(j,\bar{j})}(\sigma) = \left(\hat{\Delta}_0(\sigma) + \frac{\hat{j}}{f_j(\sigma)} - \frac{\hat{j}}{f_j(\sigma)} - 1 + \Delta \right) \tag{B.4}$$

is the conformal weight of the state $|\Delta, \bar{\hat{j}}, j\rangle_{\sigma}$.

References


